

Using a Lie-Group Adaptive Method for the Identification of a Nonhomogeneous Conductivity Function and Unknown Boundary Data

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Abstract: Only the left-boundary data of temperature and heat flux are used to estimate an unknown parameter function $\alpha(x)$ in $T_t(x,t) = \partial(\alpha(x)T_x)/\partial x + h(x,t)$, as well as to recover the right-boundary data. When $\alpha(x)$ is given the above problem is a well-known inverse heat conduction problem (IHCP). This paper solves a mixed-type inverse problem as a combination of the IHCP and the problem of parameter identification, without needing to assume a function form of $\alpha(x)$ a priori, and without measuring extra data as those used by other methods. We use the one-step Lie-Group Adaptive Method (LGAM) for the semi-discretizations of heat conduction equation, respectively, in time domain and spatial domain to derive algebraic equations, which are used to solve $\alpha(x)$ through a few iterations. To test the stability of the present LGAM we also add a random noise in the initial data. When $\alpha(x)$ is identified, a sideways approach is employed to recover the unknown boundary data. The convergence speed and accuracy are examined by numerical examples.

Keywords: Inverse problem, Mixed-type inverse problem, Parameter identification, Inverse heat conduction problem, Lie-group adaptive method, Spatial-dependence heat conductivity

1 Introduction

For evolutionary partial differential equations (PDEs) the inverse problems are classified into five types: unknown boundary conditions, unknown initial conditions, unknown parameters, unknown sources and unknown domains. Recently, there has a great interest to study the mixed-type inverse problems of PDEs. We consider an inverse problem of finding an unknown parameter $\alpha(x)$ as well as unknown boundary conditions in a one-dimensional heat conduction equation, of which one needs

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to find the temperature distribution $T(x,t)$ as well as the heat conductivity function $\alpha(x)$ that simultaneously satisfy

$$\frac{\partial}{\partial x} \left(\alpha(x) \frac{\partial T(x,t)}{\partial x} \right) = \frac{\partial T(x,t)}{\partial t} - h(x,t), \quad 0 < x < \ell, \quad 0 < t \leq t_f, \quad (1)$$

$$T(0,t) = F_0(t), \quad T_x(0,t) = q_0(t), \quad (2)$$

$$T(x,0) = f(x). \quad (3)$$

In many physical problems the measurements of temperature and heat flux can experience a practical difficulty for the inaccessible part of boundaries, which include the measurements of temperature and heat flux at a highly heated hostile boundary, and the difficulty in the determination of surface temperature of a reentry vehicle. Therefore, in the present study, we left the right-boundary condition unspecified, which is inaccessible to measure.

When $\alpha(x)$ is given the above problem is a well-known inverse heat conduction problem (IHCP). In principle, under the conditions in Eq. (2) and (3) the IHCP can be solved as a Cauchy ill-posed problem [Chang, Liu and Chang (2005)]. But the present situation is quite complex, because both $\alpha(x)$ and other side boundary conditions are unknown, and Eqs. (1)-(3) indeed form an underdetermined non-linear ill-posed system. Usually, $\alpha(x)$ can be estimated, provided that an extra measurement of data is available. There have been many methods to estimate the spatial-dependence heat conductivity, for example, Yeung and Lam (1996), Keung and Zou (1998), Lin, Chen and Yang (2001), Chang and Chang (2006), Jia and Wang (2004), Liu, Liu and Hong (2007), and Liu (2008a). Most of the studies are mainly relied on an iterative optimization formulation, and all these methods without exception required that an extra temperature or heat flux be measured internally or at the boundary [Chen and Liu (2006)]. The inverse problems are usually ill-posed. In order to overcome this problem, there have been many studies, for example, Yeung and Lam (1996), Keung and Zou (1998), Lin, Chen and Yang (2001), Chang and Chang (2006), Engl and Zou (2000), Ben-yu and Zou (2001), Jia and Wang (2004), and references therein. Most of the studies applied the least squares method to estimate the heat conductivity in the inverse heat conduction problems. Usually, the function form of the unknown heat conductivity is assumed, and the inverse problem is solved through some iteration processes.

Ito and Kunisch (1990, 1996) have proposed a very stable and efficient Lagrangian method for the identification of $\alpha(x)$ under a steady-state condition and with a smooth assumption on $\alpha(x)$. Then, Chen and Zou (1999) extended that method to a non-smooth case in the steady-state elliptic system. Lam and Yeung (1995) have employed a first-order finite difference method to determine the heat conductivity

in a one-dimensional heat conduction equation. Yeung and Lam (1996) extended that result by using a second-order finite difference technique. Chang and Chang (2006) derived linear equations by using a finite volume method to determine the unknown heat conductivity without needing of iterations.

In the present paper, we will develop a novel method to estimate the unknown heat conductivity $\alpha(x)$ and also the unknown right-boundary data of the above inverse problem, which merely requires the boundary conditions and initial condition given by Eqs. (2) and (3). To the author's best knowledge, in the open literature of estimation of unknown spatial-dependence parameter, there has no researcher to discuss such an inverse problem of a combination of the IHCP and the problem of parameter identification, without needing the help from an overspecified data. A novel Lie-group adaptive method (LGAM) is developed for this purpose of parameter identification and boundary condition recovery governed by Eqs. (1)-(3), which is for a possible application in heat conduction engineering by considering nonhomogeneous materials used in a hostile environment.

The group preserving scheme (GPS) developed previously by Liu (2001) for ODEs has being extended by Liu (2006a, 2006b, 2006c) to solve the boundary value problems (BVPs). In the construction of the Lie group method for the solutions of BVPs, Liu (2006a) has introduced the idea of one-step GPS by utilizing the closure property of the Lie group, and hence, the new shooting method has been named the Lie-group shooting method (LGSM).

After that, Liu (2006d) has used this concept to establish a one-step estimation method to estimate the temperature-dependent heat conductivity, and then extended the Lie-group method to estimate the thermophysical properties of heat conductivity and heat capacity [Liu, Liu and Hong (2007); Liu (2006e); Liu (2007)]. The Lie-group method possesses a great advantage than other numerical methods due to its Lie-group structure, and it is a powerful technique to solve the inverse problem of parameter identification.

Because the problem in Eqs. (1)-(3) has an unknown function $\alpha(x)$, it cannot be solved directly to find $T(x,t)$. This point is drastically different from the direct problem. When $\alpha(x)$ is given, the solution of $T(x,t)$ can be found numerically by a sideways approach [Carraso (1992); Elden (1995)]. This process is also employed here by using the GPS, when we suppose an initial guess of $\alpha(x)$.

In order to estimate $\alpha(x)$, Liu, Liu and Hong (2007) and Liu (2008a) required an overspecified final temperature

$$T(x, t_f) = F_m(x). \quad (4)$$

Liu, Liu and Hong (2007) have developed a highly accurate Lie-group estimation method, but required to know the boundary values of α a priori, and the measur-

ing time t_f should be close to the initial time. Then, Liu (2008a) obtained very accurate estimated results by using the LGSM. Here, we do not need the data in Eq. (4). For the problem governed by Eqs. (1)-(3), some estimated results are reported in this paper by using an LGAM. Recently, Liu and Atluri (2010) have made a breakthrough for solving the Calderón's inverse problem by an effective combination of the Lie-Group Adaptive Method (LGAM) and the finite-strip technique. The LGAM views the Lie-group shooting equation developed in the LGSM as a two-point Lie-group equation, describing a nonlinear relation between the state quantities defined at two different times or at two different positions of 1-D space. In this view of LGAM we do not have a real target in the problem, and thus we can freely use the Lie-group equation as a supplemented equation, which is inherent in the ODEs, and thus we can solve many inverse problems by an iteration process. It is interesting that Liu (2010a) has applied the Lie-group adaptive method (LGAM) to identify the rigidity function of wave propagation problems without resorting on other data, besides those needed for the direct wave problem, and Liu (2011) has identified unknown initial condition and heat source by using the LGAM.

2 The numerical procedures

We are going to solve the present inverse problem of a combination of the IHCP and parameter identification through several steps. First, we solve the heat conduction problem in the spatial interval of $0 < x < \ell$ by subjecting it to initial condition and boundary conditions. For this purpose we let $-S(x, t) = -\alpha(x)\partial T(x, t)/\partial x$ be the heat flux. Then, Eq. (1) is equivalent to

$$\frac{\partial T(x, t)}{\partial x} = a(x)S(x, t), \quad (5)$$

$$\frac{\partial S(x, t)}{\partial x} = \frac{\partial T(x, t)}{\partial t} - h(x, t), \quad (6)$$

where

$$a(x) = \frac{1}{\alpha(x)}. \quad (7)$$

Let us use a semi-discretization method to discretize the quantities of $T(x, t)$ and $S(x, t)$ in the time domain, and then we can obtain a system of ODEs for T and S

with x as an independent variable:

$$\frac{\partial T^i(x)}{\partial x} = a(x)S^i(x), \quad i = 1, \dots, n, \quad (8)$$

$$\frac{\partial S^i(x)}{\partial x} = \frac{T^{i+1}(x) - T^{i-1}(x)}{2\Delta t} - h^i(x), \quad i = 1, \dots, n-1, \quad (9)$$

$$\frac{\partial S^n(x)}{\partial x} = \frac{3T^n(x) - 4T^{n-1}(x) + T^{n-2}(x)}{2\Delta t} - h^n(x), \quad (10)$$

where $\Delta t = t_f/n$ is a uniform time increment, and $t_i = i\Delta t$ is the discretized time. $T^i(x) = T(x, t_i)$, $S^i(x) = S(x, t_i)$ and $h^i(x) = h(x, t_i)$ are the discretized quantities at the discretized points of time.

When $i = 1$, the term $T^0(x)$ appeared in Eq. (9) is determined by the initial condition (3). While the central difference is used in Eq. (9), we have used the backward difference in Eq. (10) at the last time point to keep the same second-order accuracy.

The two boundary conditions are obtained from Eq. (2) by discretizations:

$$T^i(0) = F_0(t_i), \quad i = 1, \dots, n, \quad (11)$$

$$S^i(0) = \alpha(0)q_0(t_i), \quad i = 1, \dots, n, \quad (12)$$

Here, we assume that $\alpha(0)$ is given.

In addition Eqs. (8)-(10), the Lie-group shooting method as first developed by Liu (2006a) will be extended and applied to another spatial-domain discretized equation in Section 6. After giving necessary mathematical backgrounds of the LGSM in next section, we will derive linear equations in Section 6 to determine the unknown parameter $\alpha(x)$.

3 Mathematical backgrounds

In order to explore our new method self-content and for the integrity of this paper, we first briefly sketch the group-preserving scheme (GPS) for ODEs and the one-step GPS in this section.

3.1 The GPS

Let us write Eqs. (8)-(10) in a vector form:

$$\mathbf{y}' = \mathbf{f}(x, \mathbf{y}), \quad (13)$$

where the prime denotes the differential with respect to x , and

$$\mathbf{y} := \begin{bmatrix} \mathbf{T} \\ \mathbf{S} \end{bmatrix}, \quad \mathbf{f} := \begin{bmatrix} \mathbf{f}_1(x, \mathbf{S}) \\ \mathbf{f}_2(x, \mathbf{T}) \end{bmatrix}, \quad (14)$$

in which $\mathbf{T} = (T^1, \dots, T^n)^t$ and $\mathbf{S} = (S^1, \dots, S^n)^t$. The components of \mathbf{f}_1 and \mathbf{f}_2 represent the right-hand sides of Eqs. (8)-(10). While the dependence of \mathbf{f}_1 on x is due to the dependence of heat conductivity $\alpha(x)$ on x , the dependence of \mathbf{f}_2 on x is due to the dependence of initial condition (3) on x as well as the heat generation term $h^i(x)$ on x .

When both the vector \mathbf{y} and its magnitude $\|\mathbf{y}\| := \sqrt{\mathbf{y}^t \mathbf{y}} = \sqrt{\mathbf{y} \cdot \mathbf{y}}$ are combined into a single augmented vector

$$\mathbf{X} = \begin{bmatrix} \mathbf{y} \\ \|\mathbf{y}\| \end{bmatrix}, \quad (15)$$

Liu (2001) has transformed Eq. (13) into an augmented system:

$$\mathbf{X}' = \mathbf{A}\mathbf{X} := \begin{bmatrix} \mathbf{0}_{2n \times 2n} & \frac{\mathbf{f}(x, \mathbf{y})}{\|\mathbf{y}\|} \\ \frac{\mathbf{f}^t(x, \mathbf{y})}{\|\mathbf{y}\|} & 0 \end{bmatrix} \mathbf{X}, \quad (16)$$

where \mathbf{A} is an element of the Lie algebra $so(2n, 1)$ satisfying

$$\mathbf{A}^t \mathbf{g} + \mathbf{g} \mathbf{A} = \mathbf{0}, \quad (17)$$

and

$$\mathbf{g} = \begin{bmatrix} \mathbf{I}_{2n} & \mathbf{0}_{2n \times 1} \\ \mathbf{0}_{1 \times 2n} & -1 \end{bmatrix} \quad (18)$$

is a Minkowski metric. Here, \mathbf{I}_{2n} is the identity matrix, and the superscript t stands for the transpose.

The augmented variable \mathbf{X} can be viewed as a point in the Minkowski space \mathbb{M}^{2n+1} , satisfying the cone condition:

$$\mathbf{X}^t \mathbf{g} \mathbf{X} = \mathbf{y} \cdot \mathbf{y} - \|\mathbf{y}\|^2 = 0. \quad (19)$$

Accordingly, Liu (2001) has developed a group preserving scheme (GPS) to guarantee that each \mathbf{X}_k can locate on the cone:

$$\mathbf{X}_{k+1} = \mathbf{G}(k) \mathbf{X}_k, \quad (20)$$

where \mathbf{X}_k denotes the numerical value of \mathbf{X} at the discrete x_k , and $\mathbf{G}(k) \in SO_o(2n, 1)$ satisfies

$$\mathbf{G}^t \mathbf{g} \mathbf{G} = \mathbf{g}, \quad (21)$$

$$\det \mathbf{G} = 1, \quad (22)$$

$$G_0^0 > 0, \quad (23)$$

where G_0^0 is the 00-th component of \mathbf{G} .

3.2 One-step Lie-group transformation

Throughout this paper we use the superscripted symbol, for example, \mathbf{y}^0 to denote the value of \mathbf{y} at $x = 0$, and \mathbf{y}^ℓ the value of \mathbf{y} at $x = \ell$.

Applying scheme (20) to Eq. (16) with a specified left-boundary condition $\mathbf{X}(0) = \mathbf{X}^0$ we can compute the solution $\mathbf{X}(x)$ by the GPS. Assuming that the spatial mesh size used in the GPS is $\Delta x = \ell/K$, and starting from an augmented left-boundary condition $\mathbf{X}^0 = ((\mathbf{y}^0)^t, \|\mathbf{y}^0\|)^t \neq \mathbf{0}$ we will calculate the value of $\mathbf{X}^\ell = ((\mathbf{y}^\ell)^t, \|\mathbf{y}^\ell\|)^t$ at the right-boundary $x = \ell$.

By applying Eq. (20) step-by-step we can obtain

$$\mathbf{X}^\ell = \mathbf{G}_K \cdots \mathbf{G}_1 \mathbf{X}^0. \quad (24)$$

However, let us recall that each \mathbf{G}_i , $i = 1, \dots, K$, is an element of the Lie group $SO_o(2n, 1)$, and by the closure property of the Lie group, $\mathbf{G}_K \cdots \mathbf{G}_1$ is also a Lie-group element denoted by \mathbf{G} . Hence, from Eq. (24) it follows that

$$\mathbf{X}^\ell = \mathbf{G} \mathbf{X}^0. \quad (25)$$

This is a one-step Lie-group transformation from \mathbf{X}^0 to \mathbf{X}^ℓ .

The remaining problem is how to calculate \mathbf{G} . While an exact solution of \mathbf{G} is not available, we can calculate \mathbf{G} through a numerical method by a generalized mid-point rule, which is obtained from an exponential mapping of \mathbf{A} by taking the values of the argument variables of \mathbf{A} at a generalized mid-point. The Lie-group element generated from such a constant $\mathbf{A} \in so(2n, 1)$ by an exponential mapping is

$$\mathbf{G} = \begin{bmatrix} \mathbf{I}_{2n} + \frac{a-1}{\|\hat{\mathbf{f}}\|^2} \hat{\mathbf{f}}\hat{\mathbf{f}}^t & \frac{b\hat{\mathbf{f}}}{\|\hat{\mathbf{f}}\|} \\ \frac{b\hat{\mathbf{f}}^t}{\|\hat{\mathbf{f}}\|} & a \end{bmatrix}, \quad (26)$$

where

$$\hat{\mathbf{y}} = r\mathbf{y}^0 + (1-r)\mathbf{y}^\ell, \quad (27)$$

$$\hat{\mathbf{f}} = \mathbf{f}(\hat{x}, \hat{\mathbf{y}}), \quad (28)$$

$$a = \cosh\left(\frac{\ell\|\hat{\mathbf{f}}\|}{\|\hat{\mathbf{y}}\|}\right), \quad b = \sinh\left(\frac{\ell\|\hat{\mathbf{f}}\|}{\|\hat{\mathbf{y}}\|}\right). \quad (29)$$

Here, we use the left-side $\mathbf{y}^0 = (\mathbf{T}(0), \mathbf{S}(0))$ and the right-side $\mathbf{y}^\ell = (\mathbf{T}(\ell), \mathbf{S}(\ell))$ through a suitable weighting factor r to calculate \mathbf{G} , where $r \in (0, 1)$ is a parameter to be determined and $\hat{x} = (1-r)\ell$. To stress its dependence on r we denote this \mathbf{G} by $\mathbf{G}(r)$.

3.3 A Lie-group mapping between two points on the cone

Let us define a new vector

$$\mathbf{F} := \frac{\hat{\mathbf{f}}}{\|\hat{\mathbf{y}}\|}, \quad (30)$$

such that Eqs. (26) and (29) can be expressed as

$$\mathbf{G} = \begin{bmatrix} \mathbf{I}_{2n} + \frac{a-1}{\|\mathbf{F}\|^2} \mathbf{F}\mathbf{F}^t & \frac{b\mathbf{F}}{\|\mathbf{F}\|} \\ \frac{b\mathbf{F}^t}{\|\mathbf{F}\|} & a \end{bmatrix}, \quad (31)$$

$$a = \cosh(\ell\|\mathbf{F}\|), \quad b = \sinh(\ell\|\mathbf{F}\|). \quad (32)$$

From Eqs. (15), (25) and (31) it follows that

$$\mathbf{y}^\ell = \mathbf{y}^0 + \eta\mathbf{F}, \quad (33)$$

$$\|\mathbf{y}^\ell\| = a\|\mathbf{y}^0\| + b\frac{\mathbf{F} \cdot \mathbf{y}^0}{\|\mathbf{F}\|}, \quad (34)$$

where

$$\eta := \frac{(a-1)\mathbf{F} \cdot \mathbf{y}^0 + b\|\mathbf{y}^0\|\|\mathbf{F}\|}{\|\mathbf{F}\|^2}. \quad (35)$$

Substituting \mathbf{F} in Eq. (33) written as

$$\mathbf{F} = \frac{1}{\eta}(\mathbf{y}^\ell - \mathbf{y}^0) \quad (36)$$

into Eq. (34) and dividing both the sides by $\|\mathbf{y}^0\|$ by noting $\|\mathbf{y}^0\| > 0$, we obtain

$$\frac{\|\mathbf{y}^\ell\|}{\|\mathbf{y}^0\|} = a + b\frac{(\mathbf{y}^\ell - \mathbf{y}^0) \cdot \mathbf{y}^0}{\|\mathbf{y}^\ell - \mathbf{y}^0\|\|\mathbf{y}^0\|}, \quad (37)$$

where, after inserting Eq. (36) for \mathbf{F} into Eq. (32), a and b are now written as

$$a = \cosh\left(\frac{\ell\|\mathbf{y}^\ell - \mathbf{y}^0\|}{\eta}\right), \quad b = \sinh\left(\frac{\ell\|\mathbf{y}^\ell - \mathbf{y}^0\|}{\eta}\right). \quad (38)$$

Let

$$\cos \theta := \frac{(\mathbf{y}^\ell - \mathbf{y}^0) \cdot \mathbf{y}^0}{\|\mathbf{y}^\ell - \mathbf{y}^0\|\|\mathbf{y}^0\|}, \quad (39)$$

$$\ell_y := \ell\|\mathbf{y}^\ell - \mathbf{y}^0\|, \quad (40)$$

and thus from Eqs. (37) and (38) it follows that

$$\frac{\|\mathbf{y}^\ell\|}{\|\mathbf{y}^0\|} = \cosh\left(\frac{\ell_y}{\eta}\right) + \cos\theta \sinh\left(\frac{\ell_y}{\eta}\right). \quad (41)$$

Upon defining

$$Z := \exp\left(\frac{\ell_y}{\eta}\right), \quad (42)$$

we can derive [Liu (2008b, 2008c, 2010b)]

$$Z = \frac{(\cos\theta - 1)\|\mathbf{y}^0\|}{\cos\theta\|\mathbf{y}^0\| + \|\mathbf{y}^\ell - \mathbf{y}^0\| - \|\mathbf{y}^\ell\|}. \quad (43)$$

From Eqs. (42) and (40) it follows that

$$\eta = \frac{\ell\|\mathbf{y}^\ell - \mathbf{y}^0\|}{\ln Z}. \quad (44)$$

Therefore, we come to an important result that between any two points $(\mathbf{y}^0, \|\mathbf{y}^0\|)$ and $(\mathbf{y}^\ell, \|\mathbf{y}^\ell\|)$ on the cone, there exists a Lie group element $\mathbf{G} \in SO_o(2n, 1)$ mapping $(\mathbf{y}^0, \|\mathbf{y}^0\|)$ onto $(\mathbf{y}^\ell, \|\mathbf{y}^\ell\|)$, which is given by

$$\begin{bmatrix} \mathbf{y}^\ell \\ \|\mathbf{y}^\ell\| \end{bmatrix} = \mathbf{G} \begin{bmatrix} \mathbf{y}^0 \\ \|\mathbf{y}^0\| \end{bmatrix}, \quad (45)$$

where \mathbf{G} is uniquely determined by \mathbf{y}^0 and \mathbf{y}^ℓ through the following equations:

$$\mathbf{G}(\ell) = \begin{bmatrix} \mathbf{I}_{2n} + \frac{a-1}{\|\mathbf{F}\|^2} \mathbf{F}\mathbf{F}^t & \frac{b\mathbf{F}}{\|\mathbf{F}\|} \\ \frac{b\mathbf{F}^t}{\|\mathbf{F}\|} & a \end{bmatrix}, \quad (46)$$

$$a = \cosh(\ell\|\mathbf{F}\|), \quad b = \sinh(\ell\|\mathbf{F}\|), \quad (47)$$

$$\mathbf{F} = \frac{1}{\eta}(\mathbf{y}^\ell - \mathbf{y}^0) = \frac{\ln Z}{\ell} \frac{\mathbf{y}^\ell - \mathbf{y}^0}{\|\mathbf{y}^\ell - \mathbf{y}^0\|}. \quad (48)$$

In view of Eqs. (43), (44) and (39), it can be seen that \mathbf{G} is fully determined by \mathbf{y}^0 and \mathbf{y}^ℓ .

It should be stressed that the above \mathbf{G} is different from the one in Eq. (26). In order to feature its property as being a Lie-group mapping between the quantities spanned a whole length ℓ we write it to be $\mathbf{G}(\ell)$. Conversely, $\mathbf{G}(r)$ is a function of r . However, these two Lie group elements $\mathbf{G}(r)$ and $\mathbf{G}(\ell)$ are both indispensable in our development of the Lie-group method in the next section for the inverse problem of parameter identification.

4 A Lie-group method

From Eqs. (8)-(12) it follows that

$$\mathbf{T}' = a(x)\mathbf{S}, \quad (49)$$

$$\mathbf{S}' = \mathbf{f}_2(x, \mathbf{T}), \quad (50)$$

$$\mathbf{T}(0) = \mathbf{T}^0, \quad \mathbf{T}(\ell) = \mathbf{T}^\ell, \quad (51)$$

$$\mathbf{S}(0) = \mathbf{S}^0, \quad \mathbf{S}(\ell) = \mathbf{S}^\ell. \quad (52)$$

By using Eq. (14) for \mathbf{y} we have

$$\mathbf{y}^0 = \begin{bmatrix} \mathbf{T}^0 \\ \mathbf{S}^0 \end{bmatrix}, \quad \mathbf{y}^\ell = \begin{bmatrix} \mathbf{T}^\ell \\ \mathbf{S}^\ell \end{bmatrix}, \quad (53)$$

and further inserting them into Eq. (36) yields

$$\mathbf{F} := \begin{bmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \end{bmatrix} = \frac{1}{\eta} \begin{bmatrix} \mathbf{T}^\ell - \mathbf{T}^0 \\ \mathbf{S}^\ell - \mathbf{S}^0 \end{bmatrix}. \quad (54)$$

Comparing Eq. (54) with Eq. (30) and by Eqs. (14) and (53), we can obtain

$$\mathbf{T}^\ell = \mathbf{T}^0 + \frac{\eta}{\|\hat{\mathbf{y}}\|} \hat{\mathbf{f}}_1, \quad (55)$$

$$\mathbf{S}^\ell = \mathbf{S}^0 + \frac{\eta}{\|\hat{\mathbf{y}}\|} \hat{\mathbf{f}}_2, \quad (56)$$

where

$$\|\hat{\mathbf{y}}\| = \sqrt{\|\hat{\mathbf{T}}\|^2 + \|\hat{\mathbf{S}}\|^2} = \sqrt{\|r\mathbf{T}^0 + (1-r)\mathbf{T}^\ell\|^2 + \|r\mathbf{S}^0 + (1-r)\mathbf{S}^\ell\|^2}, \quad (57)$$

$$\hat{\mathbf{f}}_1 = \hat{a}\hat{\mathbf{S}} = \hat{a} \begin{bmatrix} \hat{S}^1 \\ \vdots \\ \hat{S}^n \end{bmatrix}, \quad (58)$$

$$\hat{\mathbf{f}}_2 = \begin{bmatrix} \frac{\hat{T}^2 - \hat{T}^0}{2\Delta t} - \hat{h}^1 \\ \vdots \\ \frac{\hat{T}^n - \hat{T}^{n-2}}{2\Delta t} - \hat{h}^{n-1} \\ \frac{3\hat{T}^n - 4\hat{T}^{n-1} + \hat{T}^{n-2}}{2\Delta t} - \hat{h}^n \end{bmatrix}. \quad (59)$$

In above, $\hat{h}^i = h^i(\hat{x})$, $\hat{S}^i = rS^i(0) + (1-r)S^i(\ell)$, $\hat{T}^i = rT^i(0) + (1-r)T^i(\ell) = rF_0(t_i) + (1-r)F_\ell(t_i)$, $i = 1, \dots, n$, $\hat{a} = a(\hat{x})$ and $\hat{T}^0 = f(\hat{x})$. We should stress that $\hat{\mathbf{f}}_1$ is an unknown vector due to the appearance of the unknown coefficient \hat{a} and $S^i(\ell)$.

The above governing equations (55) and (56) are obtained by letting the two \mathbf{F} 's in Eqs. (30) and (48) be equal, which in terms of the Lie group elements $\mathbf{G}(r)$ and $\mathbf{G}(\ell)$ is essentially identical to the specification of $\mathbf{G}(r) = \mathbf{G}(\ell)$.

When both \mathbf{T}^0 and \mathbf{S}^0 are given we can directly skip to Section 6.

5 The Fredholm integral equation

When \mathbf{S}^0 is not given, we require to find it. Suppose that \mathbf{T}^0 and \mathbf{T}^ℓ are given by

$$\mathbf{T}^0 = \begin{bmatrix} F_0(t_1) \\ \vdots \\ F_0(t_n) \end{bmatrix}, \quad \mathbf{T}^\ell := \begin{bmatrix} F_\ell(t_1) \\ \vdots \\ F_\ell(t_n) \end{bmatrix}. \quad (60)$$

Although \mathbf{S}^0 and \mathbf{S}^ℓ are unknowns we can evaluate them as follows. By using $\hat{\mathbf{S}} = r\mathbf{S}^0 + (1-r)\mathbf{S}^\ell$, from Eqs. (55), (56) and (58) we can solve \mathbf{S}^0 and \mathbf{S}^ℓ as follows:

$$\mathbf{S}^0 = \frac{\|\hat{\mathbf{y}}\|}{\hat{a}\eta}(\mathbf{T}^\ell - \mathbf{T}^0) - \frac{(1-r)\eta}{\|\hat{\mathbf{y}}\|}\hat{\mathbf{f}}_2, \quad (61)$$

$$\mathbf{S}^\ell = \frac{\|\hat{\mathbf{y}}\|}{\hat{a}\eta}(\mathbf{T}^\ell - \mathbf{T}^0) + \frac{r\eta}{\|\hat{\mathbf{y}}\|}\hat{\mathbf{f}}_2. \quad (62)$$

For a specified r and the given \hat{a} , \mathbf{T}^0 , \mathbf{T}^ℓ and $\hat{\mathbf{f}}_2$ are all available, and we can use Eqs. (61) and (62), starting from an initial guess of $(\mathbf{S}^0, \mathbf{S}^\ell)$, for example, $(\mathbf{S}^0, \mathbf{S}^\ell) = (\mathbf{0}, \mathbf{0})$, to generate the new $(\mathbf{S}^0, \mathbf{S}^\ell)$, until they are convergent according to a given stopping criterion:

$$\sqrt{\|\mathbf{S}_{k+1}^0 - \mathbf{S}_k^0\|^2 + \|\mathbf{S}_{k+1}^\ell - \mathbf{S}_k^\ell\|^2} \leq \varepsilon, \quad (63)$$

which means that the norm of the difference between the $k+1$ -th and the k -th iterations of $(\mathbf{S}^0, \mathbf{S}^\ell)$ is smaller than ε .

We have derived an LGSM to search the missing left-boundary conditions of $S^i(0)$, $i = 1, \dots, n$. The premise is that we need to know the function of $a(x)$ and that we need to select a suitable weighting factor r in these equations. However, because $a(x)$ is an unknown function, we cannot perform this work of Lie-group shooting. In this section we are going to derive a first-kind Fredholm integral equation to fulfill these requirements.

Let $a(x)$ be discretized in the spatial domain by $a_j = a(x_j)$, where we divide the total length ℓ of heat conducting rod into $m-1$ subintervals, and suppose that in each subinterval the function $a(x)$ is a constant. Here the mesh size is defined

by $\Delta x = \ell / (m - 1)$ and $x_j = (j - 1)\Delta x$. Therefore, we have totally m unknowns a_j , $j = 1, \dots, m$.

Let us return to Eq. (5) and integrate it from $x = 0$ to $x = \ell$, leading to

$$\int_0^\ell S(x,t)a(x)dx = T(\ell,t) - T(0,t), \quad 0 < x < \ell, \quad 0 < t < t_f. \quad (64)$$

This is a first-kind Fredholm integral equation for $a(x)$. Unfortunately, the kernel function $S(x,t)$ is an unknown, even the right-hand side can be determined by the boundary conditions.

Therefore, we turn our attention to the discretized equation (8), and apply the Nyström method with trapezoidal quadrature to the integral terms for each $i = 1, \dots, n$, which results in

$$\Delta x \left[\frac{1}{2}S_1^i a_1 + \sum_{j=2}^{m-1} S_j^i a_j + \frac{1}{2}S_m^i a_m \right] = b^i, \quad i = 1, \dots, n, \quad (65)$$

where $S_j^i := S(x_j, t_i)$ and $b^i := T^i(\ell) - T^i(0)$.

The above system can be rearranged into a matrix form:

$$\mathbf{Ax} = \mathbf{b}, \quad (66)$$

where we denote

$$\mathbf{A} := \Delta x \begin{bmatrix} \frac{1}{2}S_1^1 & S_2^1 & \cdots & S_{m-1}^1 & \frac{1}{2}S_m^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{2}S_1^n & S_2^n & \cdots & S_{m-1}^n & \frac{1}{2}S_m^n \end{bmatrix}, \quad \mathbf{b} := \begin{bmatrix} T^1(\ell) - T^1(0) \\ \vdots \\ T^n(\ell) - T^n(0) \end{bmatrix}, \quad (67)$$

in which $\mathbf{x} = (a_1, \dots, a_m)^t$ is an unknown vector.

Now, we may first guess an initial value of $\mathbf{x} = (a_1, \dots, a_m)^t$, and $\hat{a} = ra_1 + (1-r)a_m$ is available. Then, for each given r we apply the Lie-group shooting method to find the missing left-boundary condition \mathbf{S}^0 , which together with the known left-boundary condition of \mathbf{T}^0 and the guessed coefficients a_j , we can return to Eqs. (8)-(10) and integrate them to obtain $\mathbf{T}(\ell)$ and $\mathbf{S}(\ell)$.

Here, the integrations of Eqs. (8)-(10) can be performed by utilizing the GPS. When the spatially-varying quantities of \mathbf{S} were obtained, we can use Eq. (66) to find a new set of a_j , $j = 1, \dots, m$, until they are convergent under a given stopping criterion. Unfortunately, Eq. (66) is a highly ill-posed linear system. As shown in Fig. 1 for Example 1 given in Section 7.1 the condition number of Eq. (66) is very large, which makes the solution of a_j through these iterations infeasible.

Eq. (64) reveals that the parameter identification problem has an inherently ill-posed property. Below, we will develop a new technology to find the coefficients α_j .

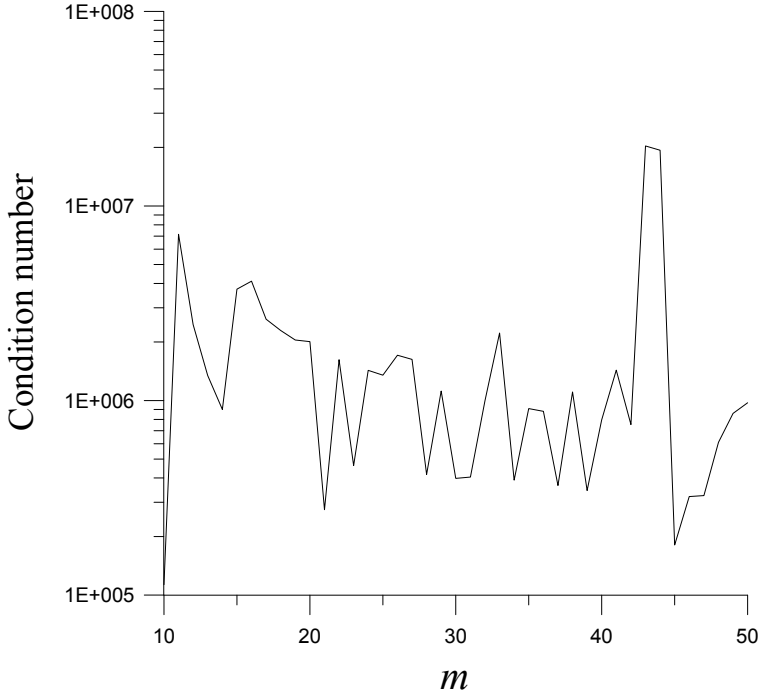


Figure 1: For Example 1 the condition number of discretized Fredholm integral equation is plotted with respect to m .

6 A Lie-group adaptive method

In this section we will derive a more simple linear equations system, which is more easy to handle than Eq. (66) to iteratively solve the coefficients α_i .

Eq. (1) can be written as

$$\frac{\partial T(x,t)}{\partial t} = \alpha'(x) \frac{\partial T(x,t)}{\partial x} + \alpha(x) \frac{\partial^2 T(x,t)}{\partial x^2} + h(x,t), \quad (68)$$

where $\alpha'(x)$ is the derivative of $\alpha(x)$ with respect to x . We adopt the numerical method of line to discretize the above derivatives with respect to x by

$$\left. \frac{\partial T(x,t)}{\partial x} \right|_{x=i\Delta x} = \frac{T_{i+1}(t) - T_i(t)}{\Delta x}, \quad (69)$$

$$\left. \frac{\partial^2 T(x,t)}{\partial x^2} \right|_{x=i\Delta x} = \frac{T_{i+1}(t) - 2T_i(t) + T_{i-1}(t)}{(\Delta x)^2}, \quad (70)$$

where $T_i(t) = T(x_i, t)$, $x_1 = 0$ and $x_m = \ell$. A similar finite difference can be used for $\alpha'(x)$.

In doing so, we can obtain a system of ODEs for T with t as an independent variable:

$$\begin{aligned} \dot{T}_i(t) &= \frac{\alpha_{i+1} - \alpha_i}{\Delta x} \frac{T_{i+1}(t) - T_i(t)}{\Delta x} \\ &+ \alpha_i \frac{T_{i+1}(t) - 2T_i(t) + T_{i-1}(t)}{(\Delta x)^2} + h_i(t), \quad i = 2, \dots, m-1, \end{aligned} \quad (71)$$

where $h_i(t) = h(x_i, t)$ and $\alpha_i = \alpha(x_i)$ are, respectively, the discretized quantities of $h(x, t)$ and $\alpha(x)$ at the nodal point x_i .

When $i = 2$, the term $T_1(t) = F_0(t)$ in Eq. (71) is determined by the first boundary condition in Eq. (2). On the other hand, the terms α_0 and α_m are supposed to be measurable on the boundaries $x = 0$ and $x = \ell$.

The initial condition is given by

$$T_i^0 = f(x_i), \quad i = 2, \dots, m-1, \quad (72)$$

which is obtained from Eq. (3) by a discretization.

Applying the same idea of LGSM to Eq. (71) we can obtain a closed-form formula to estimate α_i [Liu (2008a)]:

$$\alpha_i = \frac{(\Delta x)^2}{\hat{T}_i - \hat{T}_{i-1}} \left[\frac{\hat{T}_{i+1} - \hat{T}_i}{(\Delta x)^2} \alpha_{i+1} + \hat{h}_i - \frac{\|\hat{\mathbf{T}}\|}{\eta} (T_i^f - T_i^0) \right], \quad (73)$$

where

$$\cos \theta := \frac{(\mathbf{T}^f - \mathbf{T}^0) \cdot \mathbf{T}^0}{\|\mathbf{T}^f - \mathbf{T}^0\| \|\mathbf{T}^0\|}, \quad (74)$$

$$Z = \frac{(\cos \theta - 1) \|\mathbf{T}^0\|}{\cos \theta \|\mathbf{T}^0\| + \|\mathbf{T}^f - \mathbf{T}^0\| - \|\mathbf{T}^f\|}, \quad (75)$$

$$\eta = \frac{t_f \|\mathbf{T}^f - \mathbf{T}^0\|}{\ln Z}. \quad (76)$$

The above equation can be used sequentially to find α_i , $i = m-1, \dots, 1$ if we know α_m a priori. Here, α_m is the right-boundary value of α , and is supposed to be a known value for simplicity. In the above the mid-point values of \mathbf{T} and h_i are read as $\hat{\mathbf{T}} = r\mathbf{T}^0 + (1-r)\mathbf{T}^f$ and $\hat{h}_i = h_i(\hat{t})$, where $\hat{t} = (1-r)t_f$.

Now, the numerical procedures for estimating α_i are described as follows. We assume an initial value of α_i , for example, $\alpha_i = 1$. Substituting it into Eqs. (8)-(10), we can obtain the values of T_i^f by a spatial direction integration. Substituting

α_i and T_i^f into Eqs. (61) and (62) to find \mathbf{S}^0 , where the best r can be obtained by satisfying

$$\min_{r \in (0,1)} \sqrt{\|\mathbf{T}(\ell) - \mathbf{T}^\ell\|^2}, \quad (77)$$

such that the right-boundary condition can be fulfilled as best as possible. When \mathbf{S}^0 is available, which together with the given left-boundary condition \mathbf{T}^0 we can apply the GPS to integrate Eqs. (8)-(10) from $x = 0$ to $x = \ell$. Then, we can obtain T_i^f .

In this paper we consider the sideways problem together with the parameter identification problem of α with \mathbf{S}^0 being given, and thus the above process to find \mathbf{S}^0 can be omitted. Substituting T_i^f into Eq. (73) by choosing $r = 1$ we can calculate a new α_i , which is then compared with the old α_i . If the difference of these two sets of α_i is smaller than a given criterion, like as,

$$C_j := \sqrt{\sum_{i=2}^{m-1} (\alpha_i^{j+1} - \alpha_i^j)^2} < \varepsilon, \quad (78)$$

then we stop the iteration and the final α_i is obtained.

In summary, the present Lie-group adaptive method (LGAM) has employed an x -directional sideways approach to calculate the data T_i^f , which is starting from the given conditions in Eq. (2). Then by inserted T_i^f into the t -direction Lie-group shooting equation (73) we can adjust the coefficients α_i until they are convergent. Because the data of T_i^f are acquired through a self-adaptive manner by repeatedly using the governing equations themselves and the Lie-group shooting equation (73), not through a real measurement, this new technology is called a *Lie-group adaptive method* (LGAM).

7 Numerical tests

7.1 Example 1

Let us use the following example to demonstrate the above process. This example is given by

$$\alpha(x) = (x - 3)^2, \quad (79)$$

$$h(x, t) = -7(x - 3)^2 e^{-t}. \quad (80)$$

Under the boundary conditions

$$T(0, t) = 9e^{-t}, \quad T_x(0, t) = -6e^{-t}, \quad (81)$$

and the initial condition

$$T(x, 0) = (x - 3)^2, \quad (82)$$

the exact solution of T is given by

$$T(x, t) = (x - 3)^2 e^{-t}. \quad (83)$$

We first apply the LGAM to this identification of $\alpha(x)$, where we have fixed $\Delta x = 1/40$, $\Delta t = 5 \times 10^{-3}$ and $t_f = 0.2$. Under the stopping criterion with $\varepsilon = 10^{-4}$, the process is convergent within 12 iterations. In Fig. 2 we show the convergence speed, which is exponentially convergent. In Fig. 3(a) we plot the tentative α_i for the third iteration, the fourth iteration and the fifth iteration, the last of which is already very close to the exact solution. The numerical solutions of α_i are rather close to the exact one with the L_2 -norm error about 3.81×10^{-5} and the maximum relative error 2.09×10^{-3} as shown in Fig. 3(b). The convergence speed of the LGAM is very fast; and the accuracy of LGAM is better. Even adding a random noise $s = 0.005$ in the initial data, the relative error as shown in Fig. 3(b) by the dashed line is also smaller than 0.01, which showed that the LGAM is robust against noise.

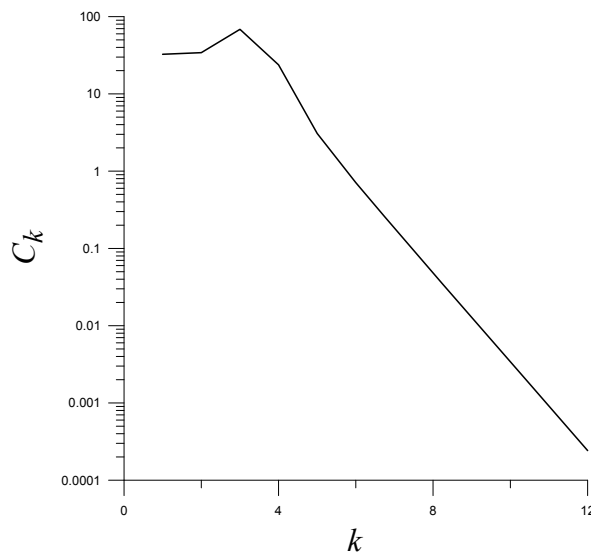


Figure 2: For Example 1 showing the convergence speed.

When $\alpha(x)$ is calculated, it is not difficult to calculate the right-boundary data by a sideways approach. In Fig. 4 we compare the recovered and exact right-boundary

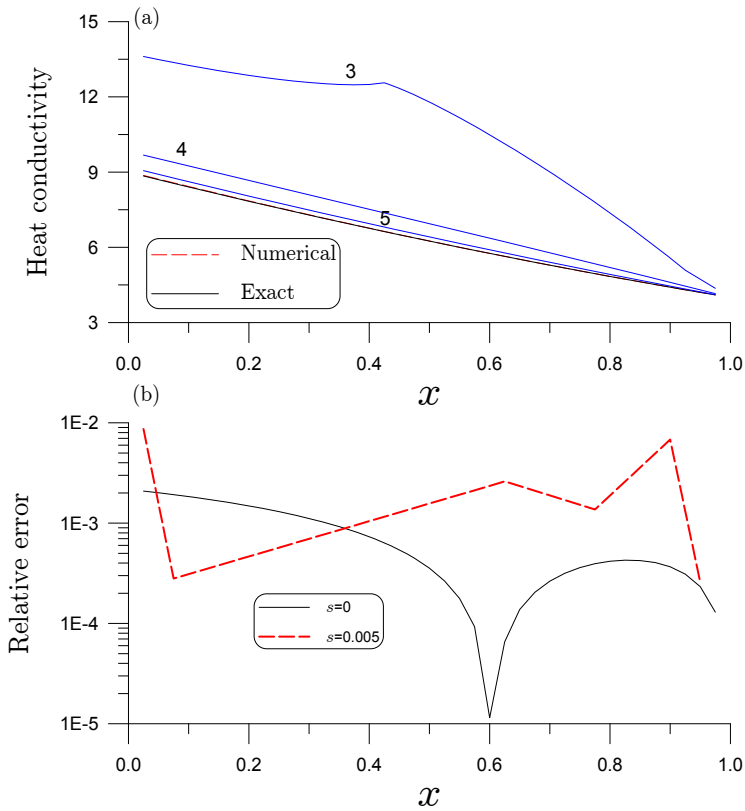


Figure 3: For Example 1 the first few iterative results of heat conductivity are plotted in (a) by using the LGAM, and (b) displaying the relative errors for $s = 0$ and $s = 0.005$.

temperatures in a time interval of $0 \leq t \leq 1$. It can be seen that the numerical solution is rather accurate.

7.2 Example 2

This problem is under the following closed-form solution:

$$T(x, t) = \sin \pi x \exp[\sin \pi t]. \tag{84}$$

The identified function $\alpha(x)$ is oscillatory given as follows:

$$\alpha(x) = 2 + \sin(10\pi x), \tag{85}$$

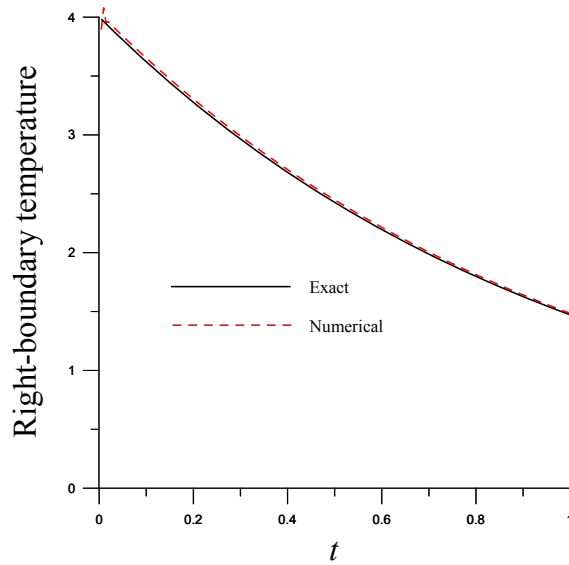


Figure 4: For Example 1 comparing the recovered and exact right-boundary temperatures.

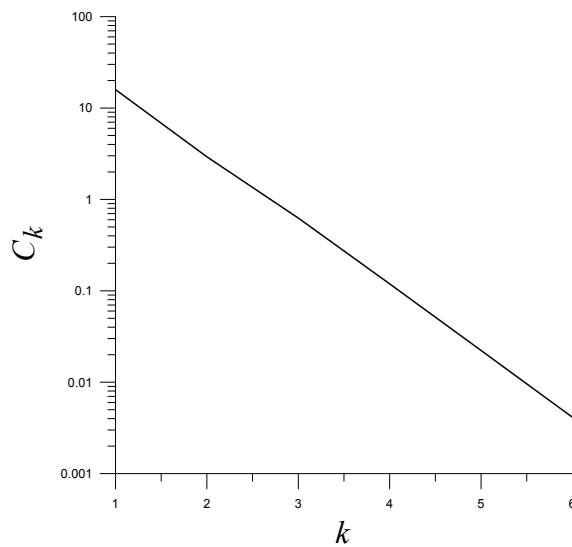


Figure 5: For Example 2 showing the convergence speed.

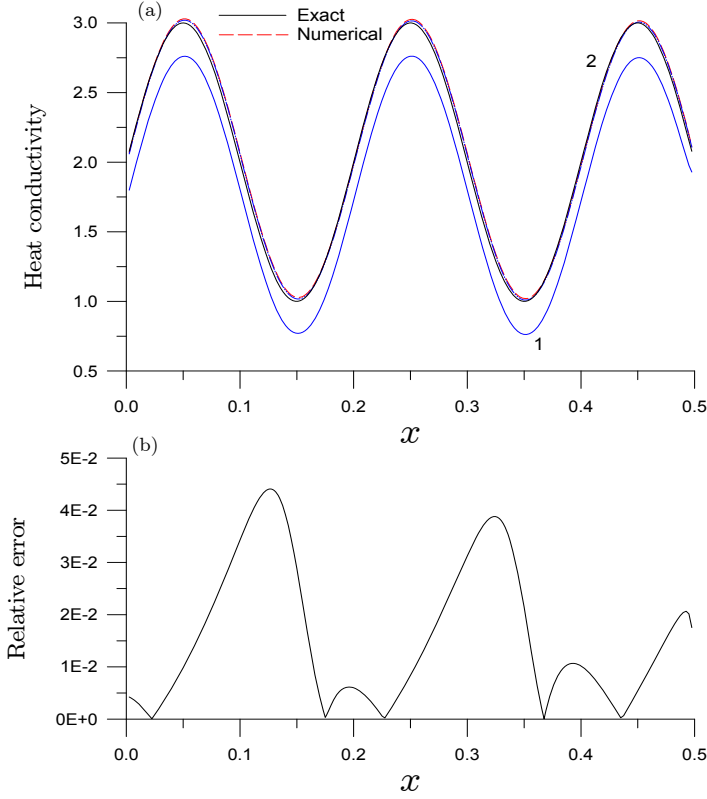


Figure 6: For Example 2 the first two iterative results of heat conductivity are plotted in (a) by using the LGAM, and (b) displaying the relative error.

and the function $h(x, t)$ is given by

$$\begin{aligned}
 h(x, t) &= \{ \pi \cos \pi t + (2 + \sin 10\pi x) \pi^2 \} \exp[\sin \pi t] \sin \pi x \\
 &- 10\pi^2 \cos \pi x \cos 10\pi x \exp[\sin \pi t].
 \end{aligned} \tag{86}$$

In this identification of $\alpha(x)$ we have applied the LGAM and fixed $\Delta x = 0.5/200$, $\Delta t = 5 \times 10^{-3}$ and $t_f = 0.3$. Under the stopping criterion with $\varepsilon = 10^{-3}$ this method is convergent within only four iterations. In Fig. 5 we show the convergence speed, which can be seen being an exponential convergence. In Fig. 6(a) we plot the tentative α_i for the first two iterations, and the final numerical solution marked by the dashed line is very close to the solid line of the exact solution. The numerical solution of α_i with the maximum relative error 4.41×10^{-2} is shown in Fig. 6(b). In

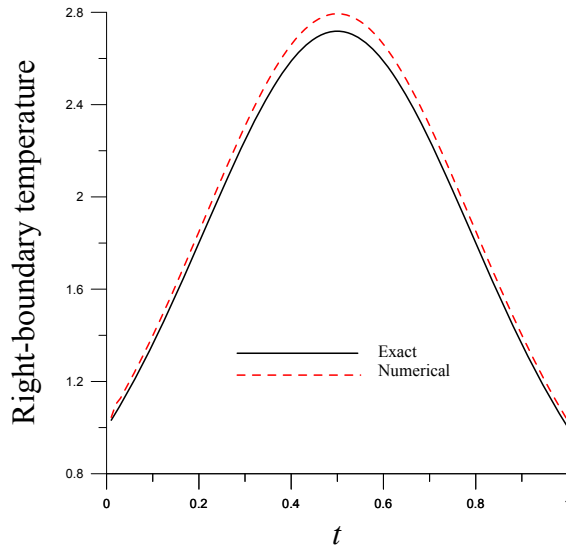


Figure 7: For Example 2 comparing the recovered and exact right-boundary temperatures.

Fig. 7 we compare the recovered and exact right-boundary temperatures in a time interval of $0 \leq t \leq 1$. It can be seen that the numerical solution is close to the exact solution.

Through these identifications of $\alpha(x)$, and the recoveries of unknown boundary data in Examples 1 and 2, it can be seen that the estimations obtained by the present LGAM are rather accurate.

8 Conclusions

A Lie-group adaptive method (LGAM) by using the time-domain discretization and the spatial-domain discretization has been developed for the inverse estimation of spatially-dependent heat conductivity and for the recovery of unknown boundary condition of the IHCP in a one-dimensional rod. Eq. (73) is a critical equation, which plays a pivotal role to adjust the parameter $\alpha(x)$ through iterations. The advantages of the present methods are that no a priori information about the functional form of heat conductivity is necessary, no extra measurement data are required, and the inverse solution can be efficiently solved. After the identification of $\alpha(x)$, we have employed a sideways approach to recover the unknown boundary data. The accuracy and efficiency of the present algorithm are confirmed by comparing the

estimated results with exact solutions. In summary, the present LGAM was employed an x -directional sideways approach to calculate the data T_i^f , and inserted T_i^f into the t -direction Lie-group shooting equation (73) to adjust the coefficients α_i until they are convergent. Because the data of T_i^f are acquired through a self-adaptive manner by a repeated use of the governing equations themselves and the Lie-group shooting equation (73), not through a real measurement, this new technology might be labelled as a *Lie-group adaptive method* (LGAM), which is rather robust against the noisy disturbance and the two-fold ill-posednesses belonging to the IHCP and parameter identification.

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