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Direct Coupling of Natural Boundary Element and Finite Element on Elastic Plane Problems in Unbounded Domains

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Abstract: The advantages of coupling of a natural boundary element method and a finite element method are introduced. Then we discuss the principle of the direct coupling of NBEM and FEM and its implementation. The comparison of the results between the direct coupling method and FEM proves that the direct coupling method is simple, feasible and valid in practice.

Keywords: FEM, NBEM, coupling method, elastic plane problem in unbounded domain

1 Introduction

The finite element method (FEM) is one of the most widely used methods in computational mechanics. It is a convenient way to solve problems of nonlinear equations or asymmetric media with high precision [**Wang** and **Shao** (2001)]. But the FEM method is not directly designed to account for problems in unbounded domains, which are usually solved in bounded domains, but only approximately, causing large errors. In order to overcome this shortcoming, some scholar developed a type of coupling between the boundary element method (BEM), which can be used to solve problems in unbounded domains and the FEM method. This coupling method of BEM and FEM was studied to calculate wave forces [**Liu** and **Zai** (2004)]. The Navier-Stokes equation was solved with the coupling method [**He** (2002)] and linear exterior boundary value problems were solved with a domain decomposition method based on BEM and FEM [**Gatica**, **Hsiao** and **Mellado** (2001)]. It is very difficult to construct a required rigidity matrix and the results from coupling are not very ideal, since several good properties cannot be retained during boundary reduction. A new type of boundary element method, the natural boundary element method (NBEM)[**Yu** (1993)], was developed. The NBEM has

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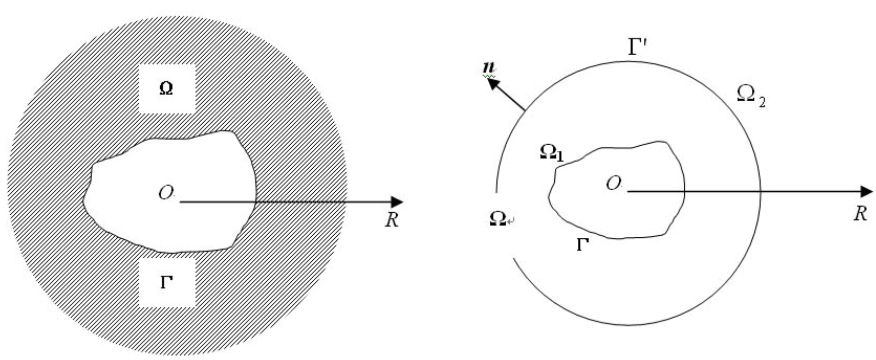
not only the advantage of solving problems in exterior domains, but has additional advantages such as direct derivation, a unique form of equations, small calculation requirements and an energy function which remains unchanged before and after the boundary reduction. Especially NBEM and FEM are based on the same variational principle, which can lead to a direct and natural coupling. The coupling method was applied to study a torsion problem of a square cross-section bar with cracks [**Zhao**, **Dong** and **Cao** (2000)]. A parabolic equation was studied with natural boundary reduction [**Du** (2000)]. Outside China, the NBEM is also referred to as the Dirichlet-to-Neumann (DtN) mapping method. Variational formulations of transmission problems were studied via FEM, BEM and DtN mappings [**Gatica** (2000)]. The DtN mapping method was uesd to solve three-dimensional elastic waves [**Gächter** and **Grote** (2003)]. Exterior problems of wave propagation were solved by an iterative variation of local DtN operators [**Miroslav** and **Igor** (2004)]. In our study, we focused on solving an elastic plane problem in an unbounded domain by the direct coupling of NBEM and FEM.

2 Principle of Coupling NBEM and FEM

Consider the following boundary problem in the domain denoted in Figure 1(a):

$$\begin{cases} \mu\Delta\vec{u} + (\lambda+\mu)\text{grad div}\vec{u} = 0 & \text{in} \quad \Omega \\ \sum\limits_{j=1}^{2} \sigma_{ij} n_j \vec{g} & \text{on} \quad \Gamma \end{cases} \tag{1}$$



(a) Unbounded domain with hole (b) Decomposed domain

Figure 1: Problem solving domain

Let

$$D(\vec{u},\vec{v}) = \iint\limits_{\Omega} \sum_{i,j=1}^{2} \sigma_{ij}(\vec{u})\varepsilon_{ij}(\vec{v})\,dp$$

$$F(v) = \int_{\Gamma} \vec{g}\cdot\vec{v}ds$$

Then the boundary problem (1) is equivalent to the following variational problem:

$$\begin{cases} \text{Find} \quad \vec{u}\in W_0^1(\Omega)^2 \quad \text{such} \quad \text{that} \\ D(\vec{u},\vec{v}) = F(\vec{v}), \quad \forall\vec{v}\in W_0^1(\Omega)^2 \end{cases} \tag{2}$$

where

$$W_0^1(\Omega)^2 = \left\{ \frac{u}{\sqrt{1+r^2}\ln(2+r^2)} \in L^2(\Omega),\ \frac{\partial u}{\partial x_i} \in L^2(\Omega),\ i=1,2,\ r=\sqrt{x_1^2+x_2^2} \right\}$$

For an elastic plane problem in an unbounded domain, the zero-strain state contains only rigid translational displacement. Let

$$\Re = \{(C_1,\, C_2)\,|C_1,\, C_2 \in R\}$$

Then the variational problem (2) has a unique solution in quotient space $W_0^1(\Omega)^2/\Re$. A circle Γ' is drawn with radius R to divide the domain Ω into two parts, Ω_1 and Ω_2. Sub-domain Ω_2 shown in Figure 1(b) is an exterior circular domain. At the same time the acting domain of a bilinear form $D(\vec{u},\vec{v})$ is decomposed into Ω_1 and Ω_2. Hence, new bilinear forms $D_i(\vec{u},\vec{v})$, $i=1,2$ are obtained, where:

$$D(\vec{u},\vec{v}) = D_1(\vec{u},\vec{v}) + D_2(\vec{u},\vec{v}) \tag{3}$$

The FEM method can be used directly in domain Ω_1 to construct a rigidity matrix while the NBEM method is applied to the exterior domain Ω_2. Let K be a natural integral operator of the elastic plane problem in an exterior domain outside the circle with radius R and obtain:

$$D_2(\vec{u},\vec{v}) = \int_{\Gamma'} \vec{v}_0 \cdot K\vec{u}_0 ds$$

Then the variational problem (2) is equivalent to the following variational problem:

$$\begin{cases} \text{Find} \quad \vec{u}\in W_0^1(\Omega)^2, \quad \text{such} \quad \text{that} \\ D(\vec{u},\vec{v}) = F(\vec{v}), \quad \forall\vec{v}\in W_0^1(\Omega)^2 \end{cases} \tag{4}$$

3 Implementing Direct Coupling of NBEM and FEM

The rigidity matrix from FEM can be obtained by discretizing D_1 in domain Ω_1, which we will not describe in detail. Our focus is on implementing the natural boundary reduction in domain Ω_2 for D_2.

Divide the artificial boundary Γ' into N equal parts. The piecewise linear basis function can then be expressed as:

$$Li(\theta)\begin{cases} N(\theta-\theta_{i-1})/2\pi, & \theta_{i-1}\le\theta\le\theta_i, \\ N(\theta_{i+1}-\theta)/2\pi, & \theta_i\le\theta\le\theta_{i+1} \\ 0, & \text{other} \end{cases} \tag{5}$$

Let

$$u_{r0}^h(\theta)=\sum_{j=1}^N U_jL_j(\theta), \quad u_{\theta 0}^h(\theta)=\sum_{j=1}^N V_jL_j(\theta), \tag{6}$$

where U_j and V_j (j=1,2,.....,N) are undetermined coefficients. Then the rigidity matrix of NBEM in an exterior circle domain is:

$$Q=\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \tag{7}$$

where

$$Q_{lm}=\left[q_{ij}^{(lm)}\right]_{i,j=1,\dots,N,} \quad l,\ m=1,\ 2$$

$$q_{ij}^{(11)}=\hat{D}(L_j,\quad 0;\quad L_i,\quad 0),\quad q_{ij}^{(12)}=\hat{D}(0,\quad L_j;\quad L_i,\quad 0),$$

$$q_{ij}^{(21)}=\hat{D}(L_j,\quad 0;\quad 0,\quad L_i),\quad q_{ij}^{(22)}=\hat{D}(0,\quad L_j;\quad L_i\quad 0),$$

i, j=1, 2, ..., N.

Using the method of a series of integral kernels and the following formula:

$$-\frac{1}{4\sin^2\frac{\theta}{2}}=\sum_{n=1}^{\infty} n\cos n\theta, \quad \frac{1}{2}\text{ctg}\frac{\theta}{2}=\sum_{n=1}^{\infty}\sin n\theta,$$

the matrix Q can be calculated as follows:

$$Q_{11} = Q_{22} =$$
$$\frac{2ab}{a+b}((a_0, a_1, ..., a_{N-1})) + \frac{2\pi b^2}{3N(a+b)}((4,1,0,...,0,1)) + \frac{4\pi ab}{N^2(a+b)}((1,\ ...,\ 1))$$

$$Q_{12} = -Q_{21} = \frac{2ab}{a+b}((0,\ d_1\ ...,\ d_{N-1})) + \frac{b^2}{a+b}((0,\ 1,\ 0,\ ...,\ -1)),$$

where

$$a_k = \frac{4N^2}{\pi^3}\sum_{j=1}^{\infty}\frac{1}{j^3}\sin^4\frac{j\pi}{N}\cos\frac{jk}{N}2\pi,$$

$$d_k = \frac{4N^2}{\pi^3}\sum_{j=1}^{\infty}\frac{1}{j^3}\sin^4\frac{j\pi}{N}\sin\frac{jk}{N}2\pi,$$

k=0, 1, 2, ..., N-1.

Clearly, Q_{11} and Q_{22} are symmetric circulant matrices, while Q_{12} and Q_{21} are anti-symmetric circulant matrices and Q is a semi-positive defined symmetric matrix.

The so-called direct coupling of NBEM and FEM has the property that the domain in which NBEM is applied, is regarded as a special element of FEM while coupling. So the total rigidity matrix can be constructed by direct addition of the rigidity matrix from FEM and that from NBEM. In the end, the linear algebraic equations can be solved.

4 Examples

Example 1 An elastic plane problem in unbounded domain with a square hole is shown in Fig. 2. Let the modulus of elasticity E=40 GPa and the Poisson's ratio μ=0.3. A uniformly distributed load q=100 kN/m is acting on the edges of the square hole.

In order to guarantee the uniqueness of the solution and the symmetry of the constraint, some conditions are generally added. We assumed the displacements at point (-2,0) and point (2,0) to be zero. Hence, we can only take one quarter of the model for our study because of the symmetry of its structure, constraints and loads.

Several different values of R were taken to solve numerically the problem by the coupling of NBEM and FEM. We first obtained the displacement of domain Ω_1 and compare these results with those obtained by FEM. The programs of the coupling

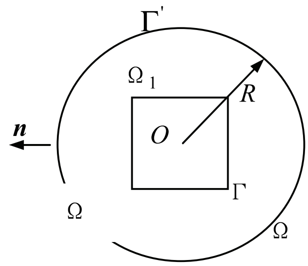


Figure 2: Elastic plane problem in unbounded domain with a square hole

method and FEM are coded by the MATHACAD software. Their meshing styles are the same and their mesh density similar. The two following tables show the difference of these results at point (2,2), obtained by the coupling method and FEM respectively.

Table 1: Displacement (10^{-8}m) along x axis at point (2, 2)

Radius/m	5	15	30	50	100
Coupling Method	1.4273	1.3354	1.3234	1.3175	1.3126
FEM	1.1101	1.3452	1.3268	1.3187	1.3126

Table 2: Displacement (10^{-8}m) along y axis at point (2, 2)

Radius/m	5	15	30	50	100
Coupling Method	4.0836	4.5311	4.4490	4.4185	4.3964
FEM	2.2245	4.1284	4.3462	4.3815	4.3965

From Tables 1 and 2, it can be seen that the results from the coupling of NBEM and FEM can easily approximate the convergence value with a small R. The FEM can also approximate this convergence value, but would require more computational complexity with a larger R. The results from these two methods are about the same when R=100 m. This shows that results with sufficient precision can be obtained with a small solution domain, which is the actual significance of the coupling method.

Example 2 Given the same model as in example 1, we assumed a concentrated force $\boldsymbol{F}$ =100 kN acting at the midpoint of the top edge.

The load is not symmetrical. Can we produce ideal results by the coupling method? Let us see the results. The results at point (2,2) are chosen for a comparison in the following tables.

Table 3: Displacement (10^{-7}m) along x axis at point (2,2)

Radius/m	5	15	30	50	100
Coupling Method	-6.0862	-5.4047	-5.1977	-5.1559	-5.1352
FEM	-2.2684	-4.1022	-4.3859	-4.4824	-4.5540

Table 4: Displacement (10^{-6}m) along y axis at point (2,2)

Radius/m	5	15	30	50	100
Coupling Method	-6.0862	-5.4047	-5.1977	-5.1559	-5.1352
FEM	-2.2684	-4.1022	-4.3859	-4.4824	-4.5540

The results in Tables 3 and 4 show that the tendency of convergence of the coupling method is the same with FEM. But these two methods do not converge to the same value. What is the reason? We want to know which result is better. Hence, the same problem is studied with a different solution domain and a different meshing style by the ANSYS software.

(1) Let R, the radius of the solution domain, be 50 m. We used the item 'Smartsize' in Meshtool with a mesh density up to grade 1. Given these conditions, the value of the displacement at point (2,2) along the x axis is -4.9989$\times 10^{-7}$m, which approximates -5.1352$\times 10^{-7}$ m, the value from the coupling method. The value, 1.6987$\times 10^{-6}$m, of the displacement at point (2,2) along the y axis is approximately 1.7723$\times 10^{-8}$ m, the value from the coupling method.

(2) Let R be 50 m while using 'Smartsize' in Meshtool with a mesh density up to grade 1. We refined the elements near the square hole with a refining grade up to 3. Given these conditions, the value of the displacement at point (2,2) along the x axis is -5.0169$\times 10^{-7}$m The difference between this value and that from the coupling method has become smaller.

From these two examples it can be seen that the stability of the coupling method is better than that of the FEM and that the precision of the coupling method is also higher. The FEM would need more computational complexity to reach a similar value.

5 Conclusions

Based on our study, we draw the following conclusions:

(1)The procedure for the coupling of NBEM and FEM is simple and direct. The sub-domain in which the NBEM is applied can be regarded as a special element of the FEM which is based on the same variational principle. The total rigidity matrix can be easily constructed.

(2) The results from the coupling method can approximate values with ideal precision when R is very small. It will save a large number of finite elements. So the computational efficiency of the coupling method is higher than that of the FEM.

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