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3D Analyses of the Stability Loss of the Circular Solid Cylinder Made from Viscoelastic Composite Material

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Abstract: The 3D approach was employed for investigations of the stability loss of the solid circular cylinder made from viscoelastic composite material. This approach is based on investigations of the evolution of the initial infinitesimal imperfections of the cylinder within the scope of 3D geometrically nonlinear field equations of the theory of viscoelasticity for anisotropic bodies. The numerical results of the critical forces and critical time are presented and discussed. To illustrate the importance of the results obtained using the 3D approach, these results are compared with the corresponding ones obtained by employing various approximate beam theories. The viscoelasticity properties of the cylinder's material are described by the fractional-exponential operator. The numerical results and their discussion are presented for the case where the cylinder is made of a uni-directional fibrous viscoelastic composite material. In particular, it is established that the difference between the critical times obtained by employing 3D and third order refined beam theories becomes more non-negligible if the values of the external compressive force are close to the critical compressive force which is obtained at $t = \infty$ (t denotes a time).

Keywords: critical force, critical time, initial imperfection criterion, stability loss, cylinder from viscoelastic composite material.

1 Introduction

Investigations into stability loss problems of the elements of construction from composite materials in many cases require the application of the Three-Dimensional

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Linearized Theory of Stability (TDLTS), the equations and relations of which are obtained from the exact non-linear equations of the deformable solid body mechanics by employing the linearization procedure similarly used by Green et al (1952), Biot (1965), Guz (1999) and others. Some researchers use the notation "General Theory of Stability" instead of TDLTS (see, for instance, Green et al (1952), Biezeno and Hencky (1930) and Southwell (1913)).

Note that from the historical aspect, the equations of TDLTS were first obtained by Southwell (1913) where physical considerations were used and it was assumed that the pre-critical stress state was homogeneous. Later the equations of TDLTS were obtained by Biezeno and Hencky (1930) for the inhomogeneous pre-critical stress state using the physical considerations. Derivation of the equations and relations of the TDLTS from non-linear equations and the relations of the theory of elasticity using a linearization procedure were described in the monographs by Biot (1965) and Guz (1999).

During the second half of the 20^{th} century there were major contributions to the development of the TDLTS and its applications to numerous stability problems of elements of construction from composite materials. A detailed review of the corresponding investigations is given by Babich et al (2001), Babich and Guz (2002) and others. It follows from the research of these and other numerous inquiries that before the beginning of the 21-th century, applications of the TDLTS were mainly related to the stability loss of constructions fabricated from time-independent materials. Furthermore, these investigations were done using the bifurcation (Euler) approach. It is known that, in general, the Euler approach is not suitable for investigations of stability loss problems for elements of constructions from time-dependent materials under static external loading. Therefore in the monograph by Guz (1999) in order to examine these problems, the dynamic investigation method is suggested. Note that under application of this method the sought values are presented with the multiplier $\exp(i\Omega t)$ and the critical parameters of the considered problems are determined from the requirement that $Im\Omega = 0$. However, under application of the dynamic investigation method within the framework of TDLTS other difficulties arise: for the time-dependent material, the coefficients of the equations of the TDLTS depend on t(time) and therefore, in many cases, within the framework of the TDLTS, representation of the sought values with multiplier $\exp(i\Omega t)$ is impossible. To the authors' knowledge, up to the present, no research on stability loss problems on the time-dependent material within the framework of TDLTS has been carried out with the use of the dynamical investigation method. Moreover, in the monograph by Guz (1999) to investigate these problems the critical deformation method by Gerard and Gilbert (1958) is proposed, according to which, it is assumed that the critical deformation of the pure elastic and the corresponding viscoelastic problems are identical. Therefore, using the results of the pure elastic stability problems, the critical time is determined from the corresponding constitutive relations of the viscoelastic body considered. It is evident that the critical deformation method is a very approximate one and can be applied in the case where the pre-critical stress state is homogeneous.

The very reliable and frequently used approach for the investigation of the stability loss of the elements of constructions made from time dependent materials is the approach proposed by Hoff (1954). This is based on the study of the growth of the initial insignificant imperfections of the elements of construction with the flow of time under fixed external static compressive forces. However, before the final years of the 20^{th} century in the framework of the TDLTS, the approach based on the growth of the aforementioned initial infinitesimal imperfections had not been proposed. Such an approach, for the first time, was proposed by Akbarov et al (1997, 1999) for the investigation of the internal stability loss (failure) in the structure of the unidirectional fibrous and layered composites in compression. In these papers, the case for which the initial infinitesimal imperfections given to the fibers or layers starts to increase and grows indefinitely is taken as a criterion for determination of the failure parameter (the values of a critical force or a critical time). Note that the results obtained in the papers by Akbarov et al (1997, 1999) were also detailed in the monograph by Akbarov and Guz (2000). Moreover, in papers by Akbarov and Kosker (2001, 2004) the approach mentioned was employed for the investigation of fiber buckling in a viscoelastic matrix. The study of the three-dimensional surface undulation stability of the viscoelastic half-space covered with a stack of layers in biaxial compression was reported by Akbarov and Tekercioglu (2007). Also, the three-dimensional stability loss of the fiber which is near a convex cylindrical surface was studied in papers by Akbarov and Mamedov (2009, 2011). The material of the cylinder which contains the fiber is taken as a viscoelastic one.

An extension of the foregoing three-dimensional (3D) approach for investigation of the stability loss problems of the elements of constructions made from linear viscoelastic composite materials was the subject of the papers by Akbarov (1998) and Akbarov and Yahnioglu (2001). Note that in these papers, the stability loss of a simply supported (Akbarov (1998)) and clamped (Akbarov and Yahnioglu (2001)) strip of viscoelastic composite materials was studied. In a paper by Akbarov et al (2001) the investigations performed in the latter two above-mentioned papers were developed for the 3D buckling instability of a thick rectangular plate. Two opposite edges of the plate were simply supported, but the other two were clamped.

A rotationally symmetric undulation instability problem for a circular plate of a viscoelastic composite material was investigated in a paper by Kutuk , Yahnioglu and Akbarov (2003). The case of a rectangular plate with all its edges clamped was considered by Selim and Akbarov (2003). An analysis of undulation instability for rotating thick circular and annular discs made from a viscoelastic composite material was carried out by Yahnioglu and Akbarov (2002). Moreover, Kutuk (2009) provided detailed analyses of a simply supported viscoelastic rectangular plate under bi-axial compression in the plate plane.

The foregoing completes a brief review of all investigations carried out up to now within the scope of the TDLTS and related to the elements of constructions made from viscoelastic composite materials. It should be noted that in the papers mentioned above the corresponding results obtained by the approximate plate theories based on the Kirchhoff-Love and Kromm (1955) hypotheses are also presented. A comparison of the corresponding results shows that for pure elastic stability loss problems, the difference between the results obtained in the framework of the 3D approach and that of the third order refined plate theory by Kromm (1955) is not more than 5-6%. However, the difference between the critical times obtained within the framework of the 3D approach and that of the mentioned refined plate theory can be more than a few times. Consequently, application of the 3D approach to the investigations of the stability loss problems of elements of constructions made from time-dependent materials is more necessary than the application of this 3D approach to the investigation of those made from the pure elastic composite materials.

However, as follows from the foregoing discussion, all the studies noted above relate to 3D stability loss of the viscoelastic composite plate. Consequently, there is no investigation related to the three-dimensional stability loss problem of the viscoelastic cylinder which is also used in a lot of elements of constructions. In the present paper, the first attempt in this field is undertaken and the approach proposed by Akbarov (1998) is developed for the study of the three dimensional stability loss of the circular solid cylinder made from viscoelastic composite materials. The same problem is also solved by using approximate beam theories, and the results obtained are compared with those given by the 3D approach which is developed and employed in the present paper.

2 Formulation of the problem

We consider a cylinder which has an initial imperfection in the natural state and determine the position of the points of this cylinder with the Lagrange coordinates in the cylindrical $Or\theta z$ and in the Cartesian $Ox_1x_2x_3$ system of coordinates (Fig. 1). The noted initial imperfection is given through the following equation of the middle line of the cylinder:

$$x_3 = t_3; \quad x_1 = A\sin\left(\frac{\pi}{\ell}t_3\right); \quad x_2 = 0, \tag{1}$$

where t_3 is a parameter and $t_3 \in (0, \ell)$, A is the amplitude of the initial imperfection form.

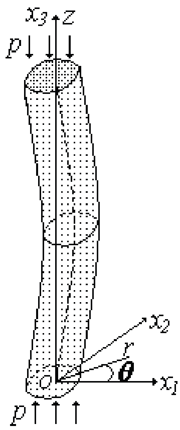

Figure 1: The geometry of the considered cylinder.

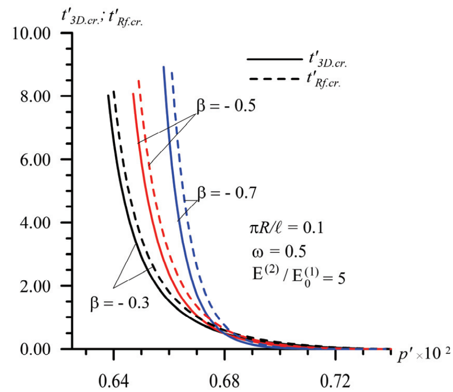

Figure 2: The graphs of the dependencies among $t'_{3D.cr}$, $t'_{Rf.cr.}$ and dimensionless intensity of the compressed force p'.

We assume that the cylinders' cross section which is perpendicular to its middle line tangent vector, is a circle of the constant radius, R. Moreover, as in papers by Akbarov (1998) and Akbarov and Yahnioglu (2001), we assume that $A << l$ and introduce the small parameter:

$$\varepsilon = \frac{A}{\ell}, \quad 0 \leq \varepsilon << 1 \tag{2}$$

We suppose that the material of the cylinder is viscoelastic transversal isotropic, the symmetry axis of which coincides with the $Ox_3(Oz)$ axis. Within the foregoing assumptions, we investigate the evolution of the infinitesimal initial imperfection of the cylinder with time for the case where the cylinder is loaded by uniformly distributed normal compressed forces with intensity p acting on the ends of the cylinder in the direction of the Oz axis.

This investigation is within the scope of the following field equations:

$$\frac{\partial t_{rr}}{\partial r} + \frac{\partial t_{\theta r}}{r\partial \theta} + \frac{\partial t_{zr}}{\partial z} + \frac{1}{r}(t_{rr} - t_{\theta\theta}) = 0, \quad \frac{\partial t_{r\theta}}{\partial r} + \frac{\partial t_{\theta\theta}}{r\partial \theta} + \frac{2}{r}t_{r\theta} + \frac{\partial t_{z\theta}}{\partial z} = 0,$$

$$\frac{\partial t_{rz}}{\partial r} + \frac{\partial t_{\theta z}}{r\partial \theta} + \frac{1}{r}t_{rz} + \frac{\partial t_{zz}}{\partial z} = 0, \tag{3a}$$

$$t_{rr} = \sigma_{rr}\left(1+\frac{\partial u_r}{\partial r}\right)+\sigma_{r\theta}\left(\frac{\partial u_r}{r\partial\theta}-\frac{u_\theta}{r}\right)+\sigma_{rz}\frac{\partial u_r}{\partial z},$$

$$t_{r\theta} = \sigma_{rr}\frac{\partial u_\theta}{\partial r}+\sigma_{r\theta}\left(1+\frac{\partial u_\theta}{r\partial\theta}+\frac{u_r}{r}\right)+\sigma_{rz}\frac{\partial u_\theta}{\partial z},$$

$$t_{rz} = \sigma_{rr}\frac{\partial u_z}{\partial r}+\sigma_{r\theta}\frac{\partial u_z}{r\partial\theta}+\sigma_{rz}\left(1+\frac{\partial u_z}{\partial z}\right),$$

$$t_{\theta r} = \sigma_{\theta r}\left(1+\frac{\partial u_r}{\partial r}\right)+\sigma_{\theta\theta}\left(\frac{\partial u_r}{r\partial\theta}-\frac{u_\theta}{r}\right)+\sigma_{\theta z}\frac{\partial u_r}{\partial z},$$

$$t_{\theta\theta} = \sigma_{\theta r}\frac{\partial u_\theta}{\partial r}+\sigma_{\theta\theta}\left(1+\frac{\partial u_\theta}{r\partial\theta}+\frac{u_r}{r}\right)+\sigma_{\theta z}\frac{\partial u_\theta}{\partial z},$$

$$t_{\theta z} = \sigma_{\theta r}\frac{\partial u_z}{\partial r}+\sigma_{\theta\theta}\frac{\partial u_z}{r\partial\theta}+\sigma_{\theta z}\left(1+\frac{\partial u_z}{\partial z}\right),$$

$$t_{zr} = \sigma_{zr}\left(1+\frac{\partial u_r}{\partial r}\right)+\sigma_{z\theta}\left(\frac{\partial u_r}{r\partial\theta}-\frac{u_\theta}{r}\right)+\sigma_{zz}\frac{\partial u_r}{\partial z},$$

$$t_{z\theta} = \sigma_{zr}\frac{\partial u_\theta}{\partial r}+\sigma_{z\theta}\left(1+\frac{\partial u_\theta}{r\partial\theta}+\frac{u_r}{r}\right)+\sigma_{zz}\frac{\partial u_\theta}{\partial z},$$

$$t_{zz} = \sigma_{zr}\frac{\partial u_z}{\partial r}+\sigma_{z\theta}\frac{\partial u_z}{r\partial\theta}+\sigma_{zz}\left(1+\frac{\partial u_z}{\partial z}\right), \tag{3b}$$

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}+\frac{1}{2}\left\{\left(\frac{\partial u_r}{\partial r}\right)^2+\left(\frac{\partial u_\theta}{\partial r}\right)^2+\left(\frac{\partial u_z}{\partial r}\right)^2\right\},$$

$$\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r}+\frac{\partial u_r}{r\partial\theta}-\frac{u_\theta}{r}\right) + \frac{1}{2}\left\{\frac{\partial u_r}{\partial r}\left(\frac{\partial u_r}{r\partial\theta}-\frac{u_\theta}{r}\right)+\frac{\partial u_\theta}{\partial r}\left(\frac{\partial u_\theta}{r\partial\theta}+\frac{u_r}{r}\right)+\frac{\partial u_z}{\partial r}\frac{\partial u_z}{r\partial\theta}\right\},$$

$$\varepsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right)+\frac{1}{2}\left\{\frac{\partial u_r}{\partial r}\frac{\partial u_r}{\partial z}+\frac{\partial u_\theta}{\partial r}\frac{\partial u_\theta}{\partial z}+\frac{\partial u_z}{\partial r}\frac{\partial u_z}{\partial z}\right\}$$

$$\varepsilon_{\theta\theta} = \frac{\partial u_\theta}{r\partial\theta}+\frac{u_r}{r}+\frac{1}{2}\left\{\left(\frac{\partial u_r}{r\partial\theta}-\frac{u_\theta}{r}\right)^2+\left(\frac{\partial u_\theta}{r\partial\theta}+\frac{u_r}{r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u_z}{\partial\theta}\right)^2\right\},$$

$$\varepsilon_{\theta z} = \frac{1}{2}\left(\frac{\partial u_z}{r\partial\theta}+\frac{\partial u_\theta}{\partial z}\right)+ \frac{1}{2}\left\{\frac{\partial u_r}{\partial z}\left(\frac{\partial u_r}{r\partial\theta}-\frac{u_\theta}{r}\right)+\frac{\partial u_\theta}{\partial z}\left(\frac{\partial u_\theta}{r\partial\theta}+\frac{u_r}{r}\right)+\frac{1}{r}\frac{\partial u_z}{\partial\theta}\frac{\partial u_z}{\partial z}\right\},$$

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z} + \frac{1}{2}\left\{ \left(\frac{\partial u_r}{\partial z}\right)^2 + \left(\frac{\partial u_\theta}{\partial z}\right)^2 + \left(\frac{\partial u_z}{\partial z}\right)^2 \right\}. \tag{3c}$$

Here the equation (3a) is an equilibrium equation in terms of the non-symmetric Kirchhoff stress tensor components t_{rr}, $t_{\theta\theta}$, t_{zz}, $t_{r\theta}$, $t_{\theta r}$, $t_{z\theta}$, $t_{\theta z}$, t_{zr} and t_{rz}, the equation (3b) shows the relation between the components of the non-symmetric Kirchhoff stress tensor and the components of the ordinary stress tensor in the cylindrical system of coordinates and the non-linear relation between the components of the Green strain tensor and the components of the displacement vector is illustrated by the equation (3c).

The constitutive relations for the cylinder materials in the cylindrical system of coordinates are given as follows:

$$\begin{aligned} &\sigma_{rr} = A_{11}^*\varepsilon_{rr} + A_{12}^*\varepsilon_{\theta\theta} + A_{13}^*\varepsilon_{zz};\ \sigma_{\theta\theta} = A_{12}^*\varepsilon_{rr} + A_{11}^*\varepsilon_{\theta\theta} + A_{13}^*\varepsilon_{zz};\\ &\sigma_{zz} = A_{13}^*\varepsilon_{rr} + A_{13}^*\varepsilon_{\theta\theta} + A_{33}^*\varepsilon_{zz};\ \sigma_{r\theta} = (A_{11}^* - A_{22}^*)\,\varepsilon_{r\theta};\\ &\sigma_{rz} = 2G^*\varepsilon_{rz};\ \sigma_{\theta z} = 2G^*\varepsilon_{\theta z}, \end{aligned} \tag{4}$$

where A_{ij}^* and G^* are the following operators:

$$\begin{Bmatrix} A_{ij}^* \\ G^* \end{Bmatrix} \varphi(t) = \begin{Bmatrix} A_{ij0} \\ G_0 \end{Bmatrix} \varphi(t) + \int_0^t \begin{Bmatrix} A_{ij1}(t-\tau) \\ G_1(t-\tau) \end{Bmatrix} \varphi(\tau)\, d\tau \tag{5}$$

Here, A_{ij0} and G_0 are the instantaneous values of elastic constants and $A_{ij1}(t)$ and $G_1(t)$ are the given functions which determine the hereditary properties of the cylinder material.

Assume that on the lateral surface S of the cylinder the following conditions are satisfied:

$$t_{rr}|_S n_r + t_{r\theta}|_S n_\theta + t_{rz}|_S n_z = 0, \quad t_{\theta r}|_S n_r + t_{\theta\theta}|_S n_\theta + t_{\theta z}|_S n_z = 0,$$

$$t_{zr}|_S n_r + t_{z\theta}|_S n_\theta + t_{zz}|_S n_z = 0. \tag{6}$$

In the natural state, the upper and lower ends of the cylinder are on the inclined planes with the following unit normal vectors:

$$\overrightarrow{n_0} = \frac{\overrightarrow{-k} - \varepsilon\pi\,\overrightarrow{i}}{\sqrt{1+\varepsilon^2\pi^2}} \text{ (for the lower end plane)},$$

$$\overrightarrow{n_l} = \frac{\overrightarrow{k} - \varepsilon\pi\,\overrightarrow{i}}{\sqrt{1+\varepsilon^2\pi^2}} \text{ (for the upper end plane)} \tag{7}$$

where i and k are ort-unit vectors in Ox_1 and Ox_3 axes directions respectively.

Denote the upper (lower) end cross section of the cylinder by $S_\ell\,(S_0)$ and the conditions for the forces on these end cross sections we write as follows:

$$t_{zr}|_{S_0} n_{01} + t_{z\theta}|_{S_0} n_{02} + t_{zz}|_{S_0} n_{03} = p, \quad t_{zr}|_{S_\ell} n_{\ell 1} + t_{z\theta}|_{S_\ell} n_{\ell 2} + t_{zz}|_{S_\ell} n_{\ell 3} = -p, \tag{8}$$

where $n_{0j}\ \left(n_{\ell j}\right)$ is a component of the unit normal vectors defined in (7). The end conditions for the displacements will be discussed below.

Thus, formulation of the considered problem has been exhausted and it follows that the evolution of the infinitesimal initial imperfection of the cylinder with time for the fixed value of the initial compressed force p(for the case where the material of the cylinder is viscoelastic) or with initial compressed force p (for the case where the material of the cylinder is pure elastic) will be investigated within the framework of the field equations (3), (4) and (5) and boundary conditions (6) and (8).

3 Method of solution

Now we consider the method of solution of the problem formulated in the previous section. Note that the method employed below can be briefly summarized as follows. By employing the boundary-form perturbation techniques, the considered boundary value problem for the non-linear integro-differential equations (3) – (5) is reduced to the series boundary-value problems for the corresponding system of the linear integro-differential equations. Owing to both the expressions of the operators (5) and the convolution theorem, by using the Laplace transform with respect to time these series problems are reduced to the corresponding series boundary value problems for the linear system of differential equations in the Laplace transform parameter space. For each fixed value of this parameter, the linear problems are solved by employing the variable-separation method and finally, by applying the Schapery (1966) inverse transformation method we determine the sought values. It should be noted that for the case where the material of the cylinder is pure elastic, the operators (5) are replaced by mechanical constants and therefore instead of the integro-differential equations we obtain differential equations and the corresponding problems for these equations are also investigated in the framework of the above procedure but without employing the Laplace transform.

According to the procedure summarized above and the problem statement, first we derive the equation for the lateral surface S of the cylinder. According to the conditions of the cylinder's cross section, we can conclude that the coordinates of this surface must simultaneously satisfy the following equations:

$$\varepsilon f'(t_3)\left(x_{10} - \varepsilon f(t_3)\right) + x_{30} - t_3 = 0,$$

$$x_{20}^2+(x_{30}-t_3)^2+(x_{10}-\varepsilon f(t_3))^2=R^2, \tag{9}$$

where $f(t_3)=\ell\sin(\pi t_3/\ell)$, $f'(t_3)=\pi\cos(\pi t_3/\ell)$; x_{10}, x_{20}, x_{30} are coordinates of the surface S. Note that the first equation in (9) is an equation of the plane perpendicular to the vector which is the tangent vector to the middle line of the cylinder at the point that corresponds to the fixed value of the parameter, t_3; but the second equation in (9) is an equation of the circle which is counter to the cross section of the cylinder which rises on the foregoing plane.

Using the relations $x_{10}=r\cos\theta$ and $x_{20}=r\sin\theta$ we obtain the following equation for the surface S in the cylindrical system of coordinates $Or\theta z$:

$$r^{\pm}=r^{\pm}(\theta,t_3,\varepsilon)=\varepsilon f(t_3)\cos\theta\frac{1+\varepsilon^2(f'(t_3))^2}{1+\varepsilon^2(f'(t_3))^2\cos^2\theta}+$$

$$\left\{\frac{(R^{\pm})^2-\varepsilon^2(f(t_3))^2\left(1+\varepsilon^2(f'(t_3))^2\right)}{1+\varepsilon^2(f'(t_3))^2\cos^2\theta}+\varepsilon^2(f(t_3))^2\cos^2\theta\frac{\left(1+\varepsilon^2(f'(t_3))^2\right)^2}{\left(1+\varepsilon^2(f'(t_3))^2\cos^2\theta\right)^2}\right\}^{\frac{1}{2}}$$

$$z^{\pm}=t_3-\varepsilon f'(t_3)\left(r^{\pm}(\theta,t_3,\varepsilon)-\varepsilon f(t_3)\right),\quad f'(t_3)=\frac{df(t_3)}{dt_3}. \tag{10}$$

Using the assumption (2) and the condition $(\varepsilon f'(t_3))^2<<1$, after some mathematical manipulations, we obtain the following equations:

$$r=R+\varepsilon f(t_3)\cos\theta+O(\varepsilon^2),\quad z=t_3-\varepsilon Rf'(t_3)\cos\theta+O\left(\varepsilon^2\right),$$

$$n_r=\left(1-\varepsilon^2\left(R^{\pm}f''(t_3)-\frac{f(t_3)}{R^{\pm}}\right)^2+O(\varepsilon^3)\right),$$

$$n_\theta=\left(\varepsilon\frac{f(t_3)}{R^{\pm}}\sin\theta+O\left(\varepsilon^2\right)\right),\quad n_z=\left(-\varepsilon f'(t_3)\cos\theta+O\left(\varepsilon^2\right)\right), \tag{11}$$

where n_r, n_θ, n_z are physical components of the unit normal vector to the surface S.

We write the equation of the planes on which lay the lower and upper inclined ends of the cylinder as follows:

$$x_3=-\varepsilon\pi x_1 \text{ (for the lower end) } x_3=\varepsilon\pi x_1+\ell \text{ (for the upper end)} \tag{12}$$

According to Eq. (7), we can also present the expression of the components of the normal vectors to these ends as follows:

$$n_{01}=n_{\ell 1}=-\varepsilon\pi\left(1-\frac{1}{2}(\varepsilon\pi)^2+O\left((\varepsilon\pi)^4\right)\right),$$

$$n_{03}=-\left(1-\frac{1}{2}(\varepsilon\pi)^2+O\left((\varepsilon\pi)^4\right)\right),$$
$$n_{\ell 3}=\left(1-\frac{1}{2}(\varepsilon\pi)^2+O\left((\varepsilon\pi)^4\right)\right). \tag{13}$$

According to the procedures of the boundary perturbation technique, as in the works by Akbarov (1998), Akbarov and Yahnioglu (2001) and many others, we attempt to solve the considered problem by employing the boundary form perturbation method. For this purpose the unknowns are presented in series form in ε (2):

$$\left\{\sigma_{(ij)};\varepsilon_{(ij)};u_{(i)}\right\}=\sum_{q=0}^{\infty}\varepsilon^q\left\{\sigma_{(ij)}^{(q)};\varepsilon_{(ij)}^{(q)};u_{(i)}^{(q)}\right\}, \tag{14}$$

$(ij)=rr;\,r\theta;\,rz;\,\theta z;\,\theta\theta;\,zz,\quad (i)=r;\,\theta;\,z.$

Substituting Eq. (14) into Eq. (3), we obtain set equations for each approximation (14). Using Eq. (11) we expand the values of each approximation (14) in series form in the vicinity of the point $\{r_0=R_0;z_0=t_3\}$. Substituting these last expressions in the boundary conditions in (6) and using the expressions of n_r, n_θ and n_z given in (11), after some mathematical transformations we obtain boundary conditions which are satisfied at $\{r=R;z=t_3\}$ for each approximation in Eq.(14). It is evident that for the zeroth approximation, Eq. (3) is valid and condition (6) is replaced by the same one satisfied at point $\{r_0=R_0;z_0=t_3\}$. We assume that the non-linear parts of the components of the strain tensor are very small and can be neglected with respect to their linear parts. According to this assumption, for the zeroth approximation, we obtain the following system of equations:

$$\frac{\partial\sigma_{rr}^{(0)}}{\partial r}+\frac{\partial\sigma_{r\theta}^{(0)}}{r\partial\theta}+\frac{\partial\sigma_{rz}^{(0)}}{\partial z}+\frac{1}{r}\left(\sigma_{rr}^{(0)}-\sigma_{\theta\theta}^{(0)}\right)=0,$$
$$\frac{\partial\sigma_{\theta r}^{(0)}}{\partial r}+\frac{\partial\sigma_{\theta\theta}^{(0)}}{r\partial\theta}+\frac{2}{r}\sigma_{\theta r}^{(0)}+\frac{\partial\sigma_{\theta z}^{(0)}}{\partial z}=0,$$
$$\frac{\partial\sigma_{rz}^{(0)}}{\partial r}+\frac{\partial\sigma_{\theta z}^{(0)}}{r\partial\theta}+\frac{1}{r}\sigma_{zr}^{(0)}+\frac{\partial\sigma_{zz}^{(0)}}{\partial z}=0,$$
$$\varepsilon_{rr}^{(0)}=\frac{\partial u_r^{(0)}}{\partial r},\quad \varepsilon_{r\theta}^{(0)}=\frac{1}{2}\left(\frac{\partial u_\theta^{(0)}}{\partial r}+\frac{\partial u_r^{(0)}}{r\partial\theta}-\frac{u_\theta^{(0)}}{r}\right),\quad \varepsilon_{rz}^{(0)}=\frac{1}{2}\left(\frac{\partial u_r^{(0)}}{\partial z}+\frac{\partial u_z^{(0)}}{\partial r}\right),$$
$$\varepsilon_{\theta\theta}^{(0)}=\frac{\partial u_\theta^{(0)}}{r\partial\theta}+\frac{u_r^{(0)}}{r},\quad \varepsilon_{\theta z}^{(0)}=\frac{1}{2}\left(\frac{\partial u_z^{(0)}}{r\partial\theta}+\frac{\partial u_\theta^{(0)}}{\partial z}\right),\quad \varepsilon_{zz}^{(0)}=\frac{\partial u_z^{(0)}}{\partial z}, \tag{15}$$

and boundary conditions

$$\sigma_{rr}^{(0)}\Big|_{r=R}=0,\quad \sigma_{r\theta}^{(0)}\Big|_{r=R}=0,\quad \sigma_{rz}^{(0)}\Big|_{r=R}=0. \tag{16}$$

Moreover, we obtain the following end conditions for the zeroth approximation from (7), (8) and (13):

$$\sigma_{zz}^{(0)}(r,\theta,0)=\sigma_{zz}^{(0)}(r,\theta,\ell)=-p, \tag{17}$$

Note that the mathematical procedure, according to which the end condition (17) is obtained, will be given below.

Taking the last assumption into account, for the subsequent approximations we obtain the following system of equations:

$$\frac{\partial\sigma_{rr}^{(q)}}{\partial r}+\frac{\partial\sigma_{r\theta}^{(q)}}{r\partial\theta}+\frac{\partial\sigma_{rz}^{(q)}}{\partial z}+\frac{1}{r}\left(\sigma_{rr}^{(q)}-\sigma_{\theta\theta}^{(q)}\right)+\sigma_{zz}^{(0)}\frac{\partial^2 u_r^{(q)}}{\partial z^2}=$$
$$-\frac{\partial S_{rr}^{(q-1)}}{\partial r}-\frac{\partial S_{r\theta}^{(q-1)}}{r\partial\theta}-\frac{\partial S_{rz}^{(q-1)}}{\partial z}-\frac{1}{r}\left(S_{rr}^{(q-1)}-S_{\theta\theta}^{(q-1)}\right),$$

$$\frac{\partial\sigma_{\theta r}^{(q)}}{\partial r}+\frac{\partial\sigma_{\theta\theta}^{(q)}}{r\partial\theta}+\frac{2}{r}\sigma_{\theta r}^{(q)}+\frac{\partial\sigma_{\theta z}^{(q)}}{\partial z}+\sigma_{zz}^{(0)}\frac{\partial^2 u_\theta^{(q)}}{\partial z^2}=$$
$$-\frac{\partial S_{\theta r}^{(q-1)}}{\partial r}-\frac{\partial S_{\theta\theta}^{(q-1)}}{r\partial\theta}-\frac{2}{r}S_{\theta r}^{(q-1)}-\frac{\partial S_{\theta z}^{(q-1)}}{\partial z},$$

$$\frac{\partial\sigma_{rz}^{(q)}}{\partial r}+\frac{\partial\sigma_{\theta z}^{(q)}}{r\partial\theta}+\frac{1}{r}\sigma_{zr}^{(q)}+\frac{\partial\sigma_{zz}^{(q)}}{\partial z}+\sigma_{zz}^{(0)}\frac{\partial^2 u_z^{(q)}}{\partial z^2}=$$
$$-\frac{\partial S_{rz}^{(q-1)}}{\partial r}-\frac{\partial S_{\theta z}^{(q-1)}}{r\partial\theta}-\frac{1}{r}S_{zr}^{(q-1)}-\frac{\partial S_{zz}^{(q-1)}}{\partial z},$$

$$S_{rr}^{(q-1)}=\sum_{k=1}^{q-1}\left\{\sigma_{rr}^{(k)}\frac{\partial u_r^{(q-k)}}{\partial r}+\sigma_{r\theta}^{(k)}\left(\frac{\partial u_r^{(q-k)}}{r\partial\theta}-\frac{u_\theta^{(q-k)}}{r}\right)+\sigma_{rz}^{(k)}\frac{\partial u_r^{(q-k)}}{\partial z}\right\},$$

$$S_{r\theta}^{(q-1)}=\sum_{k=1}^{q-1}\left\{\sigma_{rr}^{(k)}\frac{\partial u_\theta^{(q-k)}}{\partial r}+\sigma_{r\theta}^{(k)}\left(\frac{\partial u_\theta^{(q-k)}}{r\partial\theta}+\frac{u_r^{(q-k)}}{r}\right)+\sigma_{rz}^{(k)}\frac{\partial u_\theta^{(q-k)}}{\partial z}\right\},$$

$$S_{rz}^{(q-1)}=\sum_{k=1}^{q-1}\left\{\sigma_{rr}^{(k)}\frac{\partial u_z^{(q-k)}}{\partial r}+\sigma_{r\theta}^{(k)}\frac{\partial u_z^{(q-k)}}{r\partial\theta}+\sigma_{rz}^{(k)}\frac{\partial u_z^{(q-k)}}{\partial z}\right\},$$

$$S_{\theta r}^{(q-1)}=\sum_{k=1}^{q-1}\left\{\sigma_{\theta r}^{(k)}\frac{\partial u_r^{(q-k)}}{\partial r}+\sigma_{\theta\theta}^{(k)}\left(\frac{\partial u_r^{(q-k)}}{r\partial\theta}-\frac{u_\theta^{(q-k)}}{r}\right)+\sigma_{\theta z}^{(k)}\frac{\partial u_r^{(q-k)}}{\partial z}\right\},$$

$$S_{\theta\theta}^{(q-1)}=\sum_{k=1}^{q-1}\left\{\sigma_{\theta r}^{(k)}\frac{\partial u_\theta^{(q-k)}}{\partial r}+\sigma_{\theta\theta}^{(k)}\left(\frac{\partial u_\theta^{(q-k)}}{r\partial\theta}+\frac{u_r^{(q-k)}}{r}\right)+\sigma_{\theta z}^{(k)}\frac{\partial u_\theta^{(q-k)}}{\partial z}\right\},$$

$$S_{\theta z}^{(q-1)}=\sum_{k=1}^{q-1}\left\{\sigma_{\theta r}^{(k)}\frac{\partial u_z^{(q-k)}}{\partial r}+\sigma_{\theta\theta}^{(k)}\frac{\partial u_z^{(q-k)}}{r\partial\theta}+\sigma_{\theta z}^{(k)}\frac{\partial u_z^{(q-k)}}{\partial z}\right\},$$

$$S_{zz}^{(q-1)}=\sum_{k=1}^{q-1}\left\{\sigma_{rz}^{(k)}\frac{\partial u_z^{(q-k)}}{\partial r}+\sigma_{z\theta}^{(k)}\frac{\partial u_z^{(q-k)}}{r\partial\theta}+\sigma_{zz}^{(k)}\frac{\partial u_z^{(q-k)}}{\partial z}\right\},$$

$$S_{zr}^{(q-1)}=\sum_{k=1}^{q-1}\left\{\sigma_{zr}^{(k)}\frac{\partial u_r^{(q-k)}}{\partial r}+\sigma_{z\theta}^{(k)}\left(\frac{\partial u_r^{(q-k)}}{r\partial\theta}-\frac{u_\theta^{(q-k)}}{r}\right)+\sigma_{zz}^{(k)}\frac{\partial u_z^{(q-k)}}{\partial z}\right\}$$

$$S_{z\theta}^{(q-1)}=\sum_{k=1}^{q-1}\left\{\sigma_{zr}^{(k)}\frac{\partial u_\theta^{(q-k)}}{\partial r}+\sigma_{z\theta}^{(k)}\left(\frac{\partial u_\theta^{(q-k)}}{r\partial\theta}+\frac{u_r^{(q-k)}}{r}\right)+\sigma_{zz}^{(k)}\frac{\partial u_\theta^{(q-k)}}{\partial z}\right\},$$

$$\varepsilon_{rr}^{(q)}=\frac{\partial u_r^q}{\partial r}+\frac{1}{2}\sum_{k=1}^{q-1}\left\{\left(\frac{\partial u_r^{(k)}}{\partial r}\frac{\partial u_r^{(q-k)}}{\partial r}\right)+\left(\frac{\partial u_\theta^{(k)}}{\partial r}\frac{\partial u_\theta^{(q-k)}}{\partial r}\right)+\left(\frac{\partial u_z^{(k)}}{\partial r}\frac{\partial u_z^{(q-k)}}{\partial r}\right)\right\},$$

$$\varepsilon_{r\theta}^{(q)}=\frac{1}{2}\left(\frac{\partial u_\theta^{(q)}}{\partial r}+\frac{\partial u_r^{(q)}}{r\partial\theta}-\frac{u_\theta^{(q)}}{r}\right)+$$

$$\frac{1}{2}\sum_{k=1}^{q-1}\left\{\frac{\partial u_r^{(k)}}{\partial r}\left(\frac{\partial u_r^{(q-k)}}{\partial\theta}-u_\theta^{(q-k)}\right)+\frac{\partial u_\theta^{(k)}}{\partial r}\left(\frac{\partial u_\theta^{(q-k)}}{r\partial\theta}+\frac{u_r^{(q-k)}}{r}\right)+\frac{\partial u_z^{(k)}}{\partial r}\frac{\partial u_z^{(q-k)}}{\partial\theta}\right\},$$

$$\varepsilon_{rz}^{(q)}=\frac{1}{2}\left(\frac{\partial u_r^{(q)}}{\partial z}+\frac{\partial u_z^{(q)}}{\partial r}\right)+\frac{1}{2}\sum_{k=1}^{q-1}\left\{\frac{\partial u_r^{(k)}}{\partial r}\frac{\partial u_r^{(q-k)}}{\partial z}+\frac{\partial u_\theta^{(k)}}{\partial r}\frac{\partial u_\theta^{(q-k)}}{\partial z}+\frac{\partial u_z^{(k)}}{\partial r}\frac{\partial u_z^{(q-k)}}{\partial z}\right\},$$

$$\varepsilon_{\theta\theta}^{(q)}=\frac{\partial u_\theta^{(q)}}{r\partial\theta}+\frac{u_r^{(q)}}{r}+\frac{1}{2}\sum_{k=1}^{q-1}\left\{\frac{1}{r^2}\left(\frac{\partial u_r^{(k)}}{\partial\theta}-u_\theta^{(k)}\right)\left(\frac{\partial u_r^{(q-k)}}{\partial\theta}-u_\theta^{(q-k)}\right)+\right.$$

$$\left.\frac{1}{r^2}\left(\frac{\partial u_\theta^{(k)}}{\partial\theta}+u_r^{(k)}\right)\left(\frac{\partial u_\theta^{(q-k)}}{\partial\theta}+u_r^{(q-k)}\right)+\frac{1}{r^2}\left(\frac{\partial u_z^{(k)}}{\partial\theta}\right)\left(\frac{\partial u_z^{(q-k)}}{\partial\theta}\right)\right\}\quad,$$

$$\varepsilon_{\theta z}^{(q)}=\frac{1}{2}\left(\frac{\partial u_z^{(q)}}{r\partial\theta}+\frac{\partial u_\theta^{(q)}}{\partial z}\right)+\frac{1}{2}\sum_{k=1}^{q-1}\left\{\frac{1}{r}\frac{\partial u_r^{(k)}}{\partial z}\left(\frac{\partial u_r^{(q-k)}}{\partial\theta}-u_\theta^{(q-k)}\right)+\right.$$

$$\frac{1}{r}\frac{\partial u_{\theta}^{(k)}}{\partial z}\left(\frac{\partial u_{\theta}^{(q-k)}}{\partial \theta}+u_r^{(q-k)}\right)+\frac{1}{r}\frac{\partial u_z^{(k)}}{\partial \theta}\frac{\partial u_z^{(q-k)}}{\partial z}\right\},$$

$$\varepsilon_{zz}^{(q)}=\frac{\partial u_z^{(q)}}{\partial z}+\frac{1}{2}\sum_{k=1}^{q-1}\left\{\left(\frac{\partial u_r^{(k)}}{\partial z}\frac{\partial u_r^{(q-k)}}{\partial z}\right)+\left(\frac{\partial u_{\theta}^{(k)}}{\partial z}\frac{\partial u_{\theta}^{(q-k)}}{\partial z}\right)+\left(\frac{\partial u_z^{(k)}}{\partial z}\frac{\partial u_z^{(q-k)}}{\partial z}\right)\right\} \tag{18}$$

Due to linearity, the constitutive relations (4) are satisfied by each approximation separately:

$$\begin{aligned}&\sigma_{rr}^{(q)}=A_{11}^{*}\varepsilon_{rr}^{(q)}+A_{12}^{*}\varepsilon_{\theta\theta}^{(q)}+A_{13}^{*}\varepsilon_{zz}^{(q)};\ \sigma_{\theta\theta}^{(q)}=A_{12}^{*}\varepsilon_{rr}^{(q)}+A_{11}^{*}\varepsilon_{\theta\theta}^{(q)}+A_{13}^{*}\varepsilon_{zz}^{(q)};\\&\sigma_{zz}^{(q)}=A_{13}^{*}\varepsilon_{rr}^{(q)}+A_{13}^{*}\varepsilon_{\theta\theta}^{(q)}+A_{33}^{*}\varepsilon_{zz}^{(q)};\ \sigma_{r\theta}^{(q)}=\left(A_{11}^{*}-A_{22}^{*}\right)\varepsilon_{r\theta}^{(q)};\\&\sigma_{rz}^{(q)}=2G^{*}\varepsilon_{rz}^{(q)};\ \sigma_{\theta z}^{(q)}=2G^{*}\varepsilon_{\theta z}^{(q)},\end{aligned} \tag{19}$$

Now we write the boundary conditions given on the lateral surface of the cylinder for the first approximation by the physical components of the stress tensor:

$$\sigma_{(ir)}^{(1)}+f_1\frac{\partial \sigma_{(ir)}^{(0)}}{\partial r}+\phi_1\frac{\partial \sigma_{(ir)}^{(0)}}{\partial z}+\gamma_{\theta}\sigma_{(i)\theta}^{(0)}+\gamma_z\sigma_{(i)z}^{(0)}=0, \tag{20}$$

where $(i)=r,\theta,z$. In Eq. (20) replacing (i) with r,θ and z we obtain the explicit form of the corresponding contact conditions in the considered approximation. Moreover, in Eq. (20) the following notation is used:

$$\gamma_z=-f'(t_3)\cos(\theta),\quad f_1=f(t_3)\cos(\theta),\quad \phi_1=-Rf'(t_3)\cos(\theta),$$

$$\gamma_{\theta}=\frac{f(t_3)}{R}\sin(\theta),\quad f'(t_3)=\frac{df(t_3)}{dt_3},\quad f''(t_3)=\frac{d^2f(t_3)}{dt_3^2}. \tag{21}$$

Consider the satisfaction of the end conditions (8). To simplify the discussion we rewrite these conditions in the Cartesian system of coordinates, $Ox_1x_2x_3$:

$$\sigma_{3n}\left(\delta_n^j+\frac{\partial u_j}{\partial x_n}\right)\bigg|_{S_0}n_{0j}=p;\ \ \sigma_{3n}\left(\delta_n^j+\frac{\partial u_j}{\partial x_n}\right)\bigg|_{S_\ell}n_{\ell j}=-p \tag{22}$$

where δ_n^j is the Kronecker symbol. The other notation used in (22) is conventional.

According to equations (12) and (13), we can write the following expressions from the conditions (22):

$$\sigma_{3n}\left(\delta_n^j+\frac{\partial u_j}{\partial x_n}\right)\bigg|_{S_0} n_{0j}=\sigma_{3n}(x_1,x_2,-\varepsilon\pi x_1,t)\left(\delta_n^1+\frac{\partial u_1(x_1,x_2,-\varepsilon\pi x_1,t)}{\partial x_n}\right)n_{01}$$
$$+\sigma_{3n}(x_1,x_2,-\varepsilon\pi x_1,t)\left(\delta_n^3+\frac{\partial u_3(x_1,x_2,-\varepsilon\pi x_1,t)}{\partial x_n}\right)n_{03}=-p$$

$$\sigma_{3n}\left(\delta_n^j+\frac{\partial u_j}{\partial x_n}\right)\bigg|_{S_\ell} n_{\ell j}=\sigma_{3n}(x_1,x_2,\ell+\varepsilon\pi x_1,t)\left(\delta_n^1+\frac{\partial u_1(x_1,x_2,\ell+\varepsilon\pi x_1,t)}{\partial x_n}\right)n_{\ell 1}$$
$$+\sigma_{3n}(x_1,x_2,\ell+\varepsilon\pi x_1,t)\left(\delta_n^3+\frac{\partial u_3(x_1,x_2,\ell+\varepsilon\pi x_1,t)}{\partial x_n}\right)n_{\ell 3}=p \tag{23}$$

Using the expansions:

$$\sigma_{in}(x_1,x_2,-\varepsilon\pi x_1,t)=\sum_{q=0}^{\infty}\varepsilon^q\sigma_{in}^{(q)}(x_1,x_2,-\varepsilon\pi x_1,t)=\sigma_{in}^{(0)}(x_1,x_2,0,t)+$$

$$\varepsilon\left(\sigma_{in}^{(1)}(x_1,x_2,0,t)+(-\pi x_1)\frac{\partial\sigma_{in}^{(0)}(x_1,x_2,0,t)}{\partial x_3}\right)+O\left((\varepsilon\pi)^2\right);$$

$$\frac{\partial u_m(x_1,x_2,-\varepsilon\pi x_1,t)}{\partial x_j}=\sum_{q=0}^{\infty}\varepsilon^q\frac{\partial u_m^{(q)}(x_1,x_2,-\varepsilon\pi x_1,t)}{\partial x_j}=\frac{\partial u_m^{(0)}(x_1,x_2,0,t)}{\partial x_j}+$$

$$\varepsilon\left(\frac{\partial u_m^{(1)}(x_1,x_2,0,t)}{\partial x_j}+(-\pi x_1)\frac{\partial^2 u_m^{(0)}(x_1,x_2,0,t)}{\partial x_3\partial x_j}\right)+O\left((\varepsilon\pi)^2\right);$$

$$\sigma_{in}(x_1,x_2,\ell+\varepsilon\pi x_1,t)=\sum_{q=0}^{\infty}\varepsilon^q\sigma_{in}^{(q)}(x_1,x_2,\ell+\varepsilon\pi x_1,t)=\sigma_{in}^{(0)}(x_1,x_2,\ell,t)+$$

$$\varepsilon\left(\sigma_{in}^{(1)}(x_1,x_2,\ell,t)+(\pi x_1)\frac{\partial\sigma_{in}^{(0)}(x_1,x_2,\ell,t)}{\partial x_3}\right)+O\left((\varepsilon\pi)^2\right);$$

$$\frac{\partial u_m(x_1,x_2,\ell+\varepsilon\pi x_1,t)}{\partial x_j}=\sum_{q=0}^{\infty}\varepsilon^q\frac{\partial u_m^{(q)}(x_1,x_2,\ell+\varepsilon\pi x_1,t)}{\partial x_j}=\frac{\partial u_m^{(0)}(x_1,x_2,\ell,t)}{\partial x_j}+$$

$$\varepsilon\left(\frac{\partial u_m^{(1)}(x_1,x_2,\ell,t)}{\partial x_j}+(\pi x_1)\frac{\partial^2 u_m^{(0)}(x_1,x_2,\ell,t)}{\partial x_3\partial x_j}\right)+O\left((\varepsilon\pi)^2\right), \tag{24}$$

we obtain the following expression for the end conditions in (22):

$$\left\{-\sigma_{3k}^{(0)}\left(\delta_k^3+\frac{\partial u_3^{(0)}}{\partial x_k}\right)+\varepsilon\left[-\pi\sigma_{3k}^{(0)}\left(\delta_k^1+\frac{\partial u_1^{(0)}}{\partial x_k}\right)-\sigma_{3k}^{(0)}\left(\frac{\partial u_3^{(1)}}{\partial x_k}-\pi x_1\frac{\partial^2 u_3^{(0)}}{\partial x_3\partial x_k}\right)-\right.\right.$$

$$\left.\left.\left(\sigma_{3k}^{(1)}-\pi x_1\frac{\partial\sigma_{3k}^{(0)}}{\partial x_k}\right)\left(\delta_3^k+\frac{\partial u_3^{(0)}}{\partial x_k}\right)\right]+O\left(\varepsilon^2\right)\right\}_{(x_1,x_2,0)}=p$$

$$\left\{\sigma_{3k}^{(0)}\left(\delta_k^3+\frac{\partial u_3^{(0)}}{\partial x_k}\right)+\varepsilon\left[\pi\sigma_{3k}^{(0)}\left(\delta_k^1+\frac{\partial u_1^{(0)}}{\partial x_k}\right)+\sigma_{3k}^{(0)}\left(\frac{\partial u_3^{(1)}}{\partial x_k}+\pi x_1\frac{\partial^2 u_3^{(0)}}{\partial x_3\partial x_k}\right)+\right.\right.$$

$$\left.\left.\left(\sigma_{3k}^{(1)}+\pi x_1\frac{\partial\sigma_{3k}^{(0)}}{\partial x_k}\right)\left(\delta_3^k+\frac{\partial u_3^{(0)}}{\partial x_k}\right)\right]+O\left(\varepsilon^2\right)\right\}_{(x_1,x_2,\ell)}=-p. \tag{25}$$

In a similar manner, we can write the following expansions for the physical components $u_{(i)}$ of the displacement vector at the ends of the cylinder:

$$u_{(i)}^{(0)}(x_1,x_2,0,t)+\varepsilon\left(u_{(i)}^{(1)}(x_1,x_2,0,t)+(-\pi x_1)\frac{\partial u_{(i)}^{(0)}(x_1,x_2,0,t)}{\partial x_3}\right)+O\left((\varepsilon\pi)^2\right)=0,$$

$$u_{(i)}^{(0)}(x_1,x_2,\ell,t)+\varepsilon\left(u_{(i)}^{(1)}(x_1,x_2,\ell,t)+(\pi x_1)\frac{\partial u_{(i)}^{(0)}(x_1,x_2,\ell,t)}{\partial x_3}\right)+O\left((\varepsilon\pi)^2\right)=0. \tag{26}$$

$(i)=r,\theta,z$

We assume that the coefficient of ε^q in the expansion (26) for $(i)=r$; θ is equal to zero. Consequently, according to this assumption, we obtain the end conditions for the first and subsequent approximations for the displacements u_r and u_θ.

Taking the estimation $\left(\delta_k^3 + \partial u_3^{(0)}/\partial x_k\right) \approx \delta_k^3$, $\left(\delta_k^1 + \partial u_1^{(0)}/\partial x_k\right) \approx \delta_k^1$ and the expansions (23)-(26) into account, we obtain the following end conditions for the stresses for the zeroth and first approximations from the condition (22).

For the zeroth approximation:

$$\sigma_{33}^{(0)}(x_1,x_2,0) = \sigma_{33}^{(0)}(x_1,x_2,\ell) - p. \tag{27}$$

For the first approximation:

$$\pi\sigma_{31}^{(0)}(x_1,x_2,0,t) + \sigma_{3k}^{(0)}(x_1,x_2,0,t)\left(\frac{\partial u_3^{(1)}(x_1,x_2,0,t)}{\partial x_k} - \pi x_1 \frac{\partial^2 u_3^{(0)}(x_1,x_2,0,t)}{\partial x_3 \partial x_k}\right) +$$
$$\left(\sigma_{33}^{(1)}(x_1,x_2,0,t) - \pi x_1 \frac{\partial \sigma_{33}^{(0)}(x_1,x_2,0,t)}{\partial x_3}\right) = 0,$$

$$\pi\sigma_{31}^{(0)}(x_1,x_2,\ell,t) + \sigma_{3k}^{(0)}(x_1,x_2,\ell,t)\left(\frac{\partial u_3^{(1)}(x_1,x_2,\ell,t)}{\partial x_k} + \pi x_1 \frac{\partial^2 u_3^{(0)}(x_1,x_2,\ell,t)}{\partial x_3 \partial x_k}\right) +$$
$$\left(\sigma_{33}^{(1)}(x_1,x_2,\ell,t) + \pi x_1 \frac{\partial \sigma_{33}^{(0)}(x_1,x_2,\ell,t)}{\partial x_3}\right) = 0. \tag{28}$$

Thus, rewriting the condition (27) in the cylindrical system of coordinates $Or\theta z$ we obtain the condition (17):

According to (15), (16) and (17), the values related to the zeroth approximation are determined as follows:

$$\sigma_{zz}^{(0)} = -p, \quad \sigma_{(ij)}^{(0)} = 0, \;\; for \;\; (ij) \neq zz, \tag{29}$$

It follows from (29) that in the zeroth approximation the components of the displacement vector can be presented as follows:

$$u_r^{(0)} = a(t)r + a_0, \quad u_\theta^{(0)} = b_0, \quad u_z^{(0)} = c(t)z + c_0, \tag{30}$$

where a_0, b_0 and c_0 are constants, $a(t)$ and $c(t)$ are functions, t is a time. The functions $a(t)$ and $c(t)$ can be easily determined from equations (19) and (29).

Now we consider determination of the values related to the first approximation. Taking the expression (29) into account the following field equations are obtained from Eq. (18) for this approximation:

$$\frac{\partial \sigma_{rr}^{(1)}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}^{(1)}}{\partial \theta} + \frac{\partial \sigma_{rz}^{(1)}}{\partial z} + \frac{1}{r}\left(\sigma_{rr}^{(1)} - \sigma_{\theta\theta}^{(1)}\right) + \sigma_{zz}^{(0)} \frac{\partial^2 u_z^{(1)}}{\partial z^2} = 0,$$

$$\frac{\partial \sigma_{r\theta}^{(1)}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta\theta}^{(1)}}{\partial \theta}+\frac{\partial \sigma_{\theta z}^{(1)}}{\partial z}+\frac{2}{r}\sigma_{r\theta}^{(1)}+\sigma_{zz}^{(0)}\frac{\partial^2 u_\theta^{(1)}}{\partial z^2}=0,$$

$$\frac{\partial \sigma_{rz}^{(1)}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta z}^{(1)}}{\partial \theta}+\frac{\partial \sigma_{zz}^{(1)}}{\partial z}+\frac{1}{r}\sigma_{rz}^{(1)}+\sigma_{zz}^{(0)}\frac{\partial^2 u_z^{(1)}}{\partial z^2}=0.$$

$$\varepsilon_{rr}^{(1)}=\frac{\partial u_r^{(1)}}{\partial r},\quad \varepsilon_{\theta\theta}^{(1)}=\frac{\partial u_\theta^{(1)}}{r\partial \theta}+\frac{u_r^{(1)}}{r},\quad \varepsilon_{zz}^{(1)}=\frac{\partial u_z^{(1)}}{\partial z},$$

$$\varepsilon_{r\theta}^{(1)}=\frac{1}{2}\left(\frac{\partial u_r^{(1)}}{r\partial \theta}+\frac{\partial u_\theta^{(1)}}{\partial r}-\frac{u_\theta^{(1)}}{r}\right),$$

$$\varepsilon_{\theta z}^{(1)}=\frac{1}{2}\left(\frac{\partial u_\theta^{(1)}}{\partial z}+\frac{\partial u_z^{(1)}}{r\partial \theta}\right),\quad \varepsilon_{zr}^{(1)}=\frac{1}{2}\left(\frac{\partial u_z^{(1)}}{\partial r}+\frac{\partial u_r^{(1)}}{\partial z}\right). \tag{31}$$

The following conditions on the lateral surface of the cylinder are obtained from (20) and (29):

$$\sigma_{rr}^{(1)}(R,\theta,t_3,t)=0,\quad \sigma_{r\theta}^{(1)}(R,\theta,t_3,t)=0,\quad \sigma_{rz}^{(1)}(R,\theta,t_3,t)=2\pi\sigma_{zz}^{(0)}\cos(\alpha z)\cos\theta, \tag{32}$$

According to Eqs. (26) and (28), the end conditions for the first approximation can be written as follows:

$$\sigma_{zz}^{(1)}(r,\theta,0,t)+\sigma_{zz}^{(0)}\frac{\partial u_z^{(1)}(r,\theta,0,t)}{\partial z}=0,\quad u_r^{(1)}(r,\theta,0,t)=0,\quad u_\theta^{(1)}(r,\theta,0,t)=0$$

$$\sigma_{zz}^{(1)}(r,\theta,\ell,t)+\sigma_{zz}^{(0)}\frac{\partial u_z^{(1)}(r,\theta,\ell,t)}{\partial z}=0,\quad u_r^{(1)}(r,\theta,\ell,t)=0,\quad u_\theta^{(1)}(r,\theta,\ell,t)=0. \tag{33}$$

Thus, the equations (31), (19) and (5) and the conditions (32) and (33) complete the formulation of the problem for determination of the values of the first approximation. For the solution to this problem we apply the Laplace transform:

$$\bar{\psi}=\int_0^\infty \psi(t)\,e^{-st}dt \tag{34}$$

with parameter $s>0$, to all equations and relations related to the first approximation. After this application to equation (31), the boundary conditions (32) (in which $\sigma_{zz}^{(0)}$ must be replaced with $\sigma_{zz}^{(0)}/s$) and (33) are valid for the Laplace transforms of the corresponding sought-for quantities, whilst the constitutive relations (19) are transformed to the following ones:

$$\begin{aligned}
&\bar{\sigma}_{rr}^{(1)}=\bar{A}_{11}^{*}\bar{\varepsilon}_{rr}^{(1)}+\bar{A}_{12}^{*}\bar{\varepsilon}_{\theta\theta}^{(1)}+\bar{A}_{13}^{*}\bar{\varepsilon}_{zz}^{(1)};\ \bar{\sigma}_{\theta\theta}^{(1)}=\bar{A}_{12}^{*}\bar{\varepsilon}_{rr}^{(1)}+\bar{A}_{11}^{*}\bar{\varepsilon}_{\theta\theta}^{(1)}+\bar{A}_{13}^{*}\bar{\varepsilon}_{zz}^{(1)};\\
&\bar{\sigma}_{zz}^{(1)}=\bar{A}_{13}^{*}\bar{\varepsilon}_{rr}^{(1)}+\bar{A}_{13}^{*}\bar{\varepsilon}_{\theta\theta}^{(1)}+\bar{A}_{33}^{*}\bar{\varepsilon}_{zz}^{(1)};\ \bar{\sigma}_{r\theta}^{(1)}=\left(\bar{A}_{11}^{*}-\bar{A}_{22}^{*}\right)\bar{\varepsilon}_{r\theta}^{(1)};\\
&\bar{\sigma}_{rz}^{(1)}=2\bar{G}^{*}\bar{\varepsilon}_{rz}^{(1)};\ \bar{\sigma}_{\theta z}^{(1)}=2\bar{G}^{*}\bar{\varepsilon}_{\theta z}^{(1)},
\end{aligned}\tag{35}$$

where

$$\begin{Bmatrix}\bar{A}_{ij}^{*}\\ \bar{G}^{*}\end{Bmatrix}\bar{\varphi}(s)=\begin{Bmatrix}A_{ij0}\\ G_{0}\end{Bmatrix}\bar{\varphi}(s)+\begin{Bmatrix}\bar{A}_{ij1}(s)\\ \bar{G}_{1}(s)\end{Bmatrix}\bar{\varphi}(s)\tag{36}$$

As has been noted above, Eqs. (31) - (36) coincide with the corresponding equations of the TDLTS. Therefore, to solve the obtained equation systems, according to the monograph by Guz (1999), in the cylindrical system of coordinates we can use the following representations:

$$\bar{u}_{r}^{(1)}=\frac{1}{r}\frac{\partial}{\partial\theta}\psi-\frac{\partial^{2}}{\partial r\partial z}\chi,\quad \bar{u}_{\theta}^{(1)}=-\frac{\partial}{\partial r}\psi-\frac{1}{r}\frac{\partial^{2}}{\partial\theta\,\partial z}\chi,$$

$$\bar{u}_{z}^{(1)}=\left(\bar{A}_{13}^{*}+\bar{G}^{*}\right)^{-1}\left(\bar{A}_{11}^{*}\Delta_{1}+\left(\bar{G}^{*}+\sigma_{zz}^{(0)}\right)\frac{\partial^{2}}{\partial z^{2}}\right)\chi,\quad \Delta_{1}=\frac{\partial^{2}}{\partial r^{2}}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^{2}}\frac{\partial^{2}}{\partial\theta^{2}}.\tag{37}$$

The functions ψ and χ are determined from the equations:

$$\left(\Delta_{1}+\xi_{1}^{2}\frac{\partial^{2}}{\partial z^{2}}\right)\psi=0,\quad \left(\Delta_{1}+\xi_{2}^{2}\frac{\partial^{2}}{\partial z^{2}}\right)\left(\Delta_{1}+\xi_{3}^{2}\frac{\partial^{2}}{\partial z^{2}}\right)\chi=0,\tag{38}$$

where

$$\xi_{1}^{2}=\frac{2\bar{G}^{*}}{\bar{A}_{11}^{*}-\bar{A}_{12}^{*}},\quad \xi_{2,3}^{2}=c\pm\left(c^{2}-\frac{\left(\bar{A}_{33}^{*}+\sigma_{zz}^{(0)}\right)\left(\bar{G}^{*}+\sigma_{zz}^{(0)}\right)}{\bar{A}_{11}^{*}\bar{G}^{*}}\right)^{\frac{1}{2}},$$

$$2\bar{A}_{11}^{*}\bar{G}^{*}c=\bar{A}_{11}^{*}\left(\bar{A}_{33}^{*}+\sigma_{zz}^{(0)}\right)+\bar{G}^{*}\left(\bar{G}^{*}+\sigma_{zz}^{(0)}\right)-\left(\bar{A}_{13}^{*}+\bar{G}^{*}\right)^{2}.\tag{39}$$

Taking the expressions of the right sides of the conditions (32) and (33) we find the solution to the equation (38) as follows:

$$\psi = B_1 I_1(\xi_1 \alpha r)\sin(\alpha z)\sin\theta, \quad \chi = [B_2 I_1(\xi_2 \alpha r) + B_3 I_1(\xi_3 \alpha r)]\cos(\alpha z)\cos\theta, \tag{40}$$

where $I_1(x)$ is the first order Bessel function of a purely imaginary argument and B_1, B_2 and B_3 are unknown constants.

Substituting these solutions into relations (37) and (35) we obtain the following expressions for Laplace transform of the south values:

$$\bar{u}_r^{(1)} = \left[B_1 \frac{1}{r} I_1(\xi_1 \alpha r) + B_2 \xi_2 \alpha^2 I_1'(\xi_2 \alpha r) + B_3 \xi_3 \alpha^2 I_1'(\xi_3 \alpha r)\right] \sin(\alpha z)\cos\theta,$$

$$\bar{u}_\theta^{(1)} = \left[-B_1 \xi_1 \alpha I_1'(\xi_1 \alpha r) - \frac{\alpha}{r} B_2 I_1(\xi_2 \alpha r) - \frac{\alpha}{r} B_3 I_1(\xi_3 \alpha r)\right] \sin(\alpha z)\sin\theta,$$

$$\bar{u}_z^{(1)} = \left[B_2 D_2 \alpha^2 I_1(\xi_2 \alpha r) + B_3 D_3 \alpha^2 I_1(\xi_3 \alpha r)\right] \cos(\alpha z)\cos\theta,$$

$$D_2 = \frac{\bar{A}_{11}^* \xi_2^2 - \bar{G}^* - \sigma_{zz}^{(0)}}{\bar{A}_{13}^* + \bar{G}^*}, \quad D_3 = \frac{\bar{A}_{11}^* \xi_3^2 - \bar{G}^* - \sigma_{zz}^{(0)}}{\bar{A}_{13}^* + \bar{G}^*}$$

$$\bar{\sigma}_{rr}^{(1)} = \left\{B_1 \left[-\left(\bar{A}_{11}^* - \bar{A}_{12}^*\right) \frac{1}{r^2} I_1(\xi_1 \alpha r) + \left(\bar{A}_{11}^* - \bar{A}_{12}^*\right) \frac{\xi_1 \alpha}{r} I_1'(\xi_1 \alpha r)\right] + \right.$$

$$B_2 \left[\bar{A}_{11}^* \alpha^3 \xi_2^2 I_1''(\xi_2 \alpha r) + \bar{A}_{12}^* \left(\frac{-\alpha}{r^2} I_1(\xi_2 \alpha r) + \frac{\xi_2 \alpha^2}{r} I_1'(\xi_2 \alpha r)\right) - \bar{A}_{13}^* D_2 \alpha^3 I_1(\xi_2 \alpha r)\right] +$$

$$\left. B_3 \left[\bar{A}_{11}^* \alpha^3 \xi_3^2 I_1''(\xi_3 \alpha r) + \bar{A}_{12}^* \left(\frac{-\alpha}{r^2} I_1(\xi_3 \alpha r) + \frac{\xi_3 \alpha^2}{r} I_1'(\xi_3 \alpha r)\right) - \bar{A}_{13}^* D_3 \alpha^3 I_1(\xi_3 \alpha r)\right]\right\}$$

$$\sin(\alpha z)\cos\theta;$$

$$\bar{\sigma}_{r\theta}^{(1)} = \left\{B_1 \left[\frac{1}{2}\left(\bar{A}_{11}^* - \bar{A}_{12}^*\right)\left(-\xi_1^2 \alpha^2 I_1''(\xi_1 \alpha r) - \frac{1}{r^2} I_1(\xi_1 \alpha r) + \frac{\xi_1 \alpha}{r} I_1'(\xi_1 \alpha r)\right)\right] + \right.$$

$$B_2\left[\left(\bar{A}^*_{11}-\bar{A}^*_{12}\right)\left(\frac{\alpha}{r^2}I_1(\xi_2\alpha r)-\frac{\xi_2\alpha^2}{r}I'_1(\xi_2\alpha r)\right)\right]+$$

$$B_3\left[\left(\bar{A}^*_{11}-\bar{A}^*_{12}\right)\left(\frac{\alpha}{r^2}I_1(\xi_3\alpha r)-\frac{\xi_3\alpha^2}{r}I'_1(\xi_3\alpha r)\right)\right]\Bigg\}\sin(\alpha z)\sin\theta;$$

$$\bar{\sigma}^{(1)}_{rz}=\Big\{B_1\bar{G}^*\frac{\alpha}{r}I_1(\xi_1\alpha r)+B_2\bar{G}^*\alpha^3\xi_2 I'_1(\xi_2\alpha r)\,(1+D_2)+$$

$$B_3\bar{G}^*\alpha^3\xi_3 I'_1(\xi_3\alpha r)\,(1+D_3)\Big\}\cos(\alpha z)\cos\theta;$$

$$\bar{\sigma}^{(1)}_{zz}=$$

$$\Bigg\{B_2\left[\bar{A}^*_{13}\left(\xi_2^2\alpha^3 I''_1(\xi_2\alpha r)-\frac{\alpha}{r^2}I_1(\xi_2\alpha r)+\frac{\alpha^2\xi_2}{r}I'_1(\xi_2\alpha r)\right)-\bar{A}^*_{33}D_2\alpha^3 I_1(\xi_2\alpha r)\right]+$$

$$B_3\left[\bar{A}^*_{13}\left(\xi_3^2\alpha^3 I''_1(\xi_3\alpha r)-\frac{\alpha}{r^2}I_1(\xi_3\alpha r)+\frac{\alpha^2\xi_3}{r}I'_1(\xi_3\alpha r)\right)-\bar{A}^*_{33}D_3\alpha^3 I_1(\xi_3\alpha r)\right]\Bigg\}$$

$$\sin(\alpha z)\cos\theta. \tag{41}$$

Here the notation $I'(x)=dI(x)/dx$, $I''(x)=d^2I(x)/dx^2$ is used. The solution (41) to the problem under consideration satisfies automatically the end condition (33). Replacing the unknowns B_1, B_2 and B_3 with $\alpha^2 B_1\ (=C_1)$, $\alpha^3 B_2(=C_2)$ and $\alpha^3 B_3(=C_3)$, respectively, we obtain the following algebraic equation from the boundary condition (32) for determination of these unknowns:

$$\bar{\sigma}^{(1)}_{rr}(R,\theta,t_3,t)=0\ \Rightarrow C_1a_{11}(\alpha R)+C_2a_{12}(\alpha R)+C_3a_{13}(\alpha R)=0,$$

$$\bar{\sigma}^{(1)}_{r\theta}(R,\theta,t_3,t)=0\ \Rightarrow C_1a_{21}(\alpha R)+C_2a_{22}(\alpha R)+C_3a_{23}(\alpha R)=0,$$

$$\bar{\sigma}^{(1)}_{rz}(R,\theta,t_3,t)=2\pi\sigma^{(0)}_{zz}\frac{1}{s}\ \ \Rightarrow C_1a_{21}(\alpha R)+C_2a_{22}(\alpha R)+C_3a_{23}(\alpha R)=2\pi\sigma^{(0)}_{zz}\frac{1}{s}, \tag{42}$$

Note that expressions for the coefficients $a_{ij}(i;j=1,2,3)$ can be easily obtained from equation (41). Thus, with the foregoing we determine completely the Laplace transforms of the values related to the first approximation. The Laplace transform of the values of the second and subsequent approximations in (14) can also be determined as the values of the first approximation by taking the obvious changes into account. However, as shown in the works by Akbarov (1998), Akbarov and Yahnioglu (2001) and others, for stability loss problems, consideration of only the zeroth and first approximation is sufficient, because accounting for the second and subsequent approximations does not change the values of the critical parameters.

The original of the south values is determined by employing the method by Schapery (1966), according to which, for instance, the original of the displacement $u_r^{(1)}(r,\theta,t_3,t)$ is determined through the expression:

$$u_r^{(1)}(r,\theta,t_3,t) \approx \left(s\bar{u}_r^{(1)}(r,\theta,t_3,s)\right)\Big|_{s=1/(2t)} \tag{43}$$

Now we consider the selection of the stability loss criterion. In the present investigation, the case will be understood under stability loss, where:

$$\max_{\substack{t_3\in(0,\ell)\\ r\in(0,R)\\ \theta\in(0,2\pi)}} \left|u_r^{(1)}(r,\theta,t_3,t)\right| \to \infty \text{ as } t\to t_{cr.} \text{ (or as } p\to p_{cr.} \text{ for the pure elastic case).} \tag{44}$$

Thus, the values of the critical time or the values of the critical force are determined from the initial imperfection criterion (44).

4 Approximate Equations of the Stability Loss of the Cylinder-Beam obtained from 3D equations (31) - (36) by average-integrating procedure

4.1 Bernoulli Beam Theory

For the case under consideration the approximate stability loss equations for the cylinder-beam can be derived from equations (31)-(36) by using the Bernoulli hypothesis, according to which, the displacements of the cylinder are presented as follows:

$$u_r^{(1)} = u_r^{(1)}(\theta,z,t) = w(z)\cos\theta, u_\theta^{(1)} = u_\theta^{(1)}(\theta,z,t) = -w(z)\sin\theta,$$

$$u_z^{(1)} = u_z^{(1)}(r,\theta,z,t) = -r\frac{dw(z)}{dz}\cos\theta. \tag{45}$$

In writing the expression (45), it is assumed that the elongation of the middle line of the cylinder is very small with respect to the term $-rdw(z)/dz\cos\theta$ and is ignored. Thus, according to (31) and (45), we obtain: $\varepsilon_{zz}^{(1)} = -r\cos\theta\frac{d^2w(z)}{dz^2}$, $\varepsilon_{(ij)}^{(1)} = 0$ for $(ij) \neq zz$, $\sigma_{rr}^{(1)} = \sigma_{\theta\theta}^{(1)} = \sigma_{r\theta}^{(1)} = 0$,

$$\sigma_{zz}^{(1)} = E_3^*\varepsilon_{zz}^{(1)} = -r\cos\theta E_3^*\frac{d^2w(z)}{dz^2}, \tag{46}$$

where E_3^* is an operator and $E_3^*\varphi(t) = E_{30}\varphi(t) + \int\limits_0^t E_{31}(t-\tau)\varphi(\tau)d\tau$, where E_{30} is the instantaneous value of the modulus of elasticity in the direction of the Oz axis of the cylinder material and $E_{31}(t)$ is a function which determines the relaxation of this material.

Assume that:

$$\sigma_{rz}^{(1)} \neq 0, \quad \sigma_{\theta z}^{(1)} \neq 0. \tag{47}$$

Taking the relations (46) and (47) into account we obtain the following equation from (31):

$$\frac{\partial\sigma_{rz}^{(1)}}{\partial z} + \sigma_{zz}^{(0)}\frac{d^2w}{dz^2}\cos\theta = 0, \quad \frac{\partial\sigma_{\theta z}^{(1)}}{\partial z} - \sigma_{zz}^{(0)}\frac{d^2w}{dz^2}\sin\theta = 0,$$

$$\frac{\partial\sigma_{rz}^{(1)}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta z}^{(1)}}{\partial\theta} + \frac{1}{r}\sigma_{rz}^{(1)} + \frac{\partial\sigma_{zz}^{(1)}}{\partial z} + \sigma_{zz}^{(0)}\frac{d^3w}{dz^3}(-r\cos\theta) = 0. \tag{48}$$

Using the expressions of the equations in (48) we can write the presentations:

$$\sigma_{rz}^{(1)} = s_{rz}^{(1)}\cos\theta, \quad \sigma_{\theta z}^{(1)} = s_{\theta z}^{(1)}\sin\theta, \quad \sigma_{zz}^{(1)} = s_{zz}^{(1)}\cos\theta, \tag{49}$$

according to which, we obtain the following equations from (48) and (49):

$$\frac{ds_{rz}^{(1)}}{dz} + \sigma_{zz}^{(0)}\frac{d^2w}{dz^2} = 0, \quad \frac{ds_{\theta z}^{(1)}}{dz} - \sigma_{zz}^{(0)}\frac{d^2w}{dz^2} = 0. \tag{50}$$

Multiplying the last equation in (48) with $r^2\cos\theta$ and integrating with respect to r in the interval $[0,R]$ and with respect to θin the interval $[0,2\pi]$ we can write the following transformations:

$$\int\limits_0^R\int\limits_0^{2\pi}\frac{1}{r}\frac{\partial\sigma_{\theta z}^{(1)}}{\partial\theta}r^2\cos\theta drd\theta = \pi\int\limits_0^R s_{rz}^{(1)}rdr; \quad \int\limits_0^R\int\limits_0^{2\pi}\frac{1}{r}\sigma_{rz}^{(1)}r^2\cos\theta drd\theta = \pi\int\limits_0^R s_{rz}^{(1)}rdr;$$

$$\int_0^R\int_0^{2\pi}\frac{\partial\sigma_{rz}^{(1)}}{\partial r}r^2\cos\theta drd\theta=\pi\int_0^R\frac{ds_{rz}^{(1)}}{dr}r^2dr=\pi\left[\int_0^R\frac{d}{dr}\left(s_{rz}^{(1)}r^2\right)dr-2\int_0^R s_{rz}^{(1)}rdr\right]=$$

$$\pi\, s_{rz}^{(1)}\Big|_{r=R}R^2-2\pi\int_0^R s_{rz}^{(1)}rdr=\pi\sigma_{zz}^{(0)}R^2\cos(\alpha z)-2\pi\int_0^R s_{rz}^{(1)}rdr,$$

$$\int_0^R\int_0^{2\pi}\frac{\partial\sigma_{zz}^{(1)}}{\partial z}r^2\cos\theta drd\theta=\pi\int_0^R E_3^*\frac{d^3w}{dz^3}(-r^3)dr=-\frac{\pi R^4}{4}E_3^*\frac{d^3w}{dz^3}=-JE_3^*\frac{d^3w}{dz^3},\tag{51}$$

$$J=\frac{\pi R^4}{4}.$$

Here, the last condition in (32), i.e. the condition:

$$\sigma_{rz}^{(1)}(R,\theta,t_3,t)=2\pi\sigma_{zz}^{(0)}\cos(\alpha z)\cos\theta,\Rightarrow s\,_{rz}^{(1)}\Big|_{r=R}=2\pi\sigma_{zz}^{(0)}\cos(\alpha z)$$

is used. Thus, taking the calculations given in (51), the equation (50) and the estimation $\left|\sigma_{zz}^{(0)}\right|<<|E_3^*1|$ into account we obtain the following equation for the displacement $w(z)$ from the last equation in (48):

$$-E_3^*J\frac{d^4w}{dz^4}+\pi R^2\sigma_{zz}^{(0)}\frac{d^2w}{dz^2}=R^2\alpha\sigma_{zz}^{(0)}\sin(\alpha z)\tag{52}$$

Applying the presentations (45) and (46) to the end conditions (33) and doing the integration over the area of the end cross sections we obtain the following ones for the approximate Bernoulli approach:

$$w|_{z=0;\ell}=0,\quad \frac{d^2w}{dz^2}\Big|_{z=0;\ell}=0.\tag{53}$$

Thus, using the notation $P=-\pi R^2\sigma_{zz}^{(0)}$ we obtain the classical stability loss equation of the Bernoulli beam from which the Euler critical force within the scope of the initial imperfection criterion is obtained.

For the solution to the boundary value problems (52) and (53) we apply the Laplace transform (34) to these equations. Using the convolution theorem for transformation of the term $E_3^* d^4 w / dz^4$ we obtain the same equation and end conditions written for $\bar{w}$. For instance, after this transformation we obtain the following equation instead of equation (52):

$$-\bar{E}_3^* J \frac{d^4 \bar{w}}{dz^4} + \pi R^2 \sigma_{zz}^{(0)} \frac{d^2 \bar{w}}{dz^2} = \frac{1}{s} R^2 \alpha \sigma_{zz}^{(0)} \sin(\alpha z) \tag{54}$$

According to the end condition (53), the solution to the equation (54) is taken as follows:

$$\bar{w} = \bar{A}(s) \sin \alpha z \tag{55}$$

Substituting (55) into (54) and doing some mathematical calculations we obtain the following expression for the unknown $\bar{A}(s)$:

$$\bar{A}(s) = -\frac{P\ell}{s\pi^2} \left[P - \frac{\pi^2 \bar{E}_3^* J}{\ell^2} \right]^{-1}, \quad P = -\pi R^2 \sigma_{zz}^{(0)}. \tag{56}$$

Employing the method by Schapery (1966), the original of $\bar{A}(s)$ is determined as follows:

$$A(t) = \left(s\bar{A}(s) \right) \Big|_{s=(1/(2t))} \tag{57}$$

For the cases where the cylinder material is pure elastic we obtain from (56) the following expression:

$$A = -\frac{P\ell}{\pi^2} \left[P - \frac{\pi^2 E_3 J}{\ell^2} \right]^{-1}. \tag{58}$$

According to (44), the stability loss criterion for the Bernoulli beam can be expressed as follows:

$A(t) \to \infty$ as $t \to t_{cr}$(for the case where the cylinder material is viscoelastic),

$A \to \infty$ as $P \to P_{cr}$(for the case where the cylinder material is pure elastic). (59)

The following expression follows from (58) and (59) for the critical force which coincides with the Euler critical force:

$$P_{Eu.cr} = \frac{\pi^2 E_3 J}{\ell^2} \tag{60}$$

4.2 Third Order Refined Beam Theory

For the third order refined beam theory we use the same modification of the theory by Kromm (1955) for the cylinder-beam, according to which, the displacements $u_r^{(1)}$ and $u_\theta^{(1)}$ are presented as in (45), but the displacement $u_z^{(1)}$ is presented as follows:

$$u_z^{(1)} = u_z^{(1)}(r,\theta,z,t) = -r\frac{dw(z)}{dz}\cos\theta + \frac{1}{4}\left(R^2 r\cos\theta - \frac{1}{3}r^3\cos^3\theta\right)\phi(z). \qquad (61)$$

Substituting the expressions for $u_r^{(1)}$, $u_\theta^{(1)}$ and $u_z^{(1)}$ into equations (31) we obtain the expression for strains and stresses given below:

$$\varepsilon_{rz}^{(1)} = \frac{1}{8}R^2\cos\theta\left(1-\frac{r^2}{R^2}\cos^2\theta\right)\phi(z), \quad \varepsilon_{\theta z}^{(1)} = -\frac{1}{8}R^2\sin\theta\left(1-\frac{r^2}{R^2}\cos^2\theta\right)\phi(z),$$

$$\varepsilon_{zz}^{(1)} = -r\cos\theta\frac{d^2w(z)}{dz^2} + \frac{1}{4}\left(R^2 r\cos\theta - \frac{r^3\cos\theta}{3}\right)\frac{d\phi}{dz}, \quad \varepsilon_{rr}^{(1)} = \varepsilon_{\theta\theta}^{(1)} = \varepsilon_{r\theta}^{(1)} = 0,$$

$$\sigma_{rr}^{(1)} = \sigma_{\theta\theta}^{(1)} = \sigma_{r\theta}^{(1)} = 0, \quad \sigma_{rz}^{(1)} = 2G^*\varepsilon_{rz}^{(1)} = G^*\frac{1}{4}R^2\cos\theta\left(1-\frac{r^2}{R^2}\cos^2\theta\right)\phi(z),$$

$$\sigma_{\theta z}^{(1)} = 2G^*\varepsilon_{\theta z}^{(1)} = -G^*\frac{1}{4}R^2\sin\theta\left(1-\frac{r^2}{R^2}\cos^2\theta\right)\phi(z),$$

$$\sigma_{zz}^{(1)} = E_3^*\varepsilon_{zz}^{(1)} = E_3^*\left(-r\cos\theta\frac{d^2w(z)}{dz^2} + \frac{1}{4}\left(R^2 r\cos\theta - \frac{r^3\cos\theta}{3}\right)\frac{d\phi}{dz}\right). \qquad (62)$$

It follows from the expressions given in (62) that the equations (48) and (49) also hold for the considered case. First, we apply the Laplace transformation to the equations (48), (49) and (62) and doing the integration procedure in the previous case we obtain the following equation for the function $\bar{w}(z,s)$:

$$-E_3^*J\left[1+\frac{32}{27}\frac{\sigma_{zz}^{(0)}}{G^*}\right]\frac{d^4\bar{w}(z,s)}{dz^4} + \pi R^2\sigma_{zz}^{(0)}\frac{d^2\bar{w}(z,s)}{dz^2} = \frac{\alpha R^2}{s}\sigma_{zz}^{(0)}\sin\alpha z, \qquad (63)$$

Note that the estimation $\left(1+\sigma_{zz}^{(0)}/E_3^*\right) \approx 1$ has also been taken into account in obtaining the equation (63). Thus, using the presentation (53) we obtain from (63) the following expression for $\bar{A}(s)$:

$$\bar{A}(s) = -\frac{P\ell}{s\pi^2}\left[P\left(1+\frac{8}{27}\frac{E_3^*}{G^*}\left(\frac{\pi R}{\ell}\right)^2\right)-\frac{\pi^2 E_3^* J}{\ell^2}\right]^{-1}, \quad P = -\pi R^2 \sigma_{zz}^{(0)}. \tag{64}$$

By employing the method (57) we determine the function$A(t)$. For determination of the values of the critical parameters we can also use the expression (59). The following expression can be written for the unknown A from equation (64) for the case where the material of the cylinder is pure elastic:

$$A = -\frac{P\ell}{\pi^2}\left[P\left(1+\frac{8}{27}\frac{E_3}{G}\left(\frac{\pi R}{\ell}\right)^2\right)-\frac{\pi^2 E_3 J}{\ell^2}\right]^{-1}. \tag{65}$$

From (65) the following expression for the critical force for the case where the cylinder's material is a pure elastic one can be written:

$$P_{cr.} = \frac{\pi^2 E_3 J}{\ell^2}\left(1+\frac{8}{27}\frac{E_3}{G}\left(\frac{\pi R}{\ell}\right)^2\right)^{-1}. \tag{66}$$

5 Numerical results and discussions

We assume that the cylinder is made from viscoelastic unidirectional fibrous composite material and its fibers lie along the Oz axis. In the discussions below, the values related to the matrix and the fibers will be denoted by upper indices (1) and (2), respectively. The material of the fibers is supposed to be pure elastic with Young's modulus $E^{(2)}$, Poisson coefficient $\nu^{(2)}$ and shear modulus $\mu^{(2)}$, but the material of the matrix is supposed to be linearly viscoelastic with operators:

$$E^{*(1)}\varphi = E_0^{(1)}\left[\varphi(t) - \omega_0 \Pi_\beta^*(-\omega_0-\omega_\infty)\varphi\right],$$

$$\nu^{*(1)}\varphi = \nu_0^{(1)}\left[\varphi(t) + \frac{1-2\nu_0^{(1)}}{2\nu_0^{(1)}}\omega_0 \Pi_\beta^*(-\omega_0-\omega_\infty)\varphi\right],$$

$$\mu^{*(1)}\varphi = \mu_0^{(1)}\left[\varphi(t) - \frac{3}{2(1+\nu_0^{(1)})}\omega_0 \Pi_\beta^*(-\frac{3}{2(1+\nu_0^{(1)})}\omega_0-\omega_\infty)\varphi\right], \tag{67}$$

where $E_0^{(1)}$, $\nu_0^{(1)}$ are the instantaneous values of Young's modulus and the Poisson coefficient, respectively, $\mu_0^{(1)}$ is the instantaneous value of shear modulus, β, ω_0

and ω_∞ are the rheological parameters of the matrix material, Π^*_β is the fractional exponential operator of Rabotnov (1977), and this operator is determined as:

$$\Pi^*_\beta(x)\,\phi = \int_0^t \Pi_\beta(x, t-\tau)d\tau, \tag{68}$$

where

$$\Pi_\beta(x,t) = t^\beta \sum_{n=0}^{\infty} \frac{x^n t^{n(1+\beta)}}{\Gamma((1+n)(1+\beta))}, \quad -1 < \beta < 0, \tag{69}$$

where $\Gamma(x)$ is the Gamma function.

We introduce the dimensionless rheological parameter $\omega\,(=\omega_\infty/\omega_0)$ and the dimensionless time $t'\left(=\omega_0^{1/(1+\beta)}t\right)$ and assume that $\nu^{(2)} = \nu_0^{(1)} = 0.3$, $\eta^{(2)} = 0.5$, where $\eta^{(2)}$ is the fiber concentration in the composite under consideration.

It is known that, within the scope of the continuum approach this composite can be taken as a homogeneous transversal isotropic one, the isotropy axis of which coincides with the Oz axis. According to Christensen (1979) and many others, by replacing the mechanical constants of the components of a composite with Laplace transform of the corresponding operators in the expressions of the effective mechanical properties, we determine the Laplace transform of the effective operators. Therefore, in the Laplace transform of the constitutive relations (35), instead of $\bar{A}^*_{ij}$ and $\bar{G}^*$ we write these expressions. For the considered composite material, the expressions for $\bar{A}^*_{ij}$ and $\bar{G}^*$ are determined as follows:

$$\bar{A}^*_{33} = \bar{E}^*_3 + 4\left(\bar{\nu}^*_{31}\right)^2 \bar{K}^*_{21}, \quad \bar{A}^*_{13} = 2\bar{\nu}^*_{31}\bar{K}^*_{12}, \quad \bar{A}^*_{11} = \bar{\mu}^*_{12} + \bar{K}^*_{12}, \quad \bar{A}^*_{12} = -\bar{\mu}^*_{12} + \bar{K}^*_{12},$$

$$\bar{G}^* = \bar{\mu}^{(1)} \frac{\mu^{(2)}(1+\eta^{(2)}) + \bar{\mu}^{(1)}(1-\eta^{(2)})}{\mu^{(2)}(1-\eta^{(2)}) + \bar{\mu}^{(1)}(1+\eta^{(2)})}, \tag{70}$$

where

$$\bar{K}^*_{12} = \bar{K}^{(1)} + \frac{1}{3}\bar{\mu}^{(1)} + \eta^{(2)}\left[\left(\frac{1}{3}\left(\mu^{(2)} - \bar{\mu}^{(1)}\right) + K^{(2)} - \bar{K}^{(1)}\right)^{-1} + \frac{\left(1-\eta^{(2)}\right)}{\bar{K}^{(1)} + \frac{4}{3}\bar{\mu}^{(1)}}\right]^{-1},$$

$$\bar{E}^*_3 = \eta^{(2)}E^{(2)} + (1-\eta^{(2)})\bar{E}^{(1)} + \frac{4\eta^{(2)}(1-\eta^{(2)})(\nu^{(2)} - \bar{\nu}^{(1)})^2\bar{\mu}^{(1)}}{(1-\eta^{(2)})\bar{\mu}^{(1)}(K^{(2)} + \mu^{(2)}/3)^{-1} + \eta^{(2)}\bar{\mu}^{(1)}(\bar{K}^{(1)} + \bar{\mu}^{(1)}/3)^{-1}},$$

$$\bar{\nu}^*_{31} = \eta^{(2)}\nu^{(2)} + (1-\eta^{(2)})\bar{\nu}^{(1)} +$$

$$\frac{4\eta^{(2)}(1-\eta^{(2)})(\nu^{(2)}-\bar{\nu}^{(1)})\left[\bar{\mu}^{(1)}(\bar{K}^{(1)}+\bar{\mu}^{(1)}/3)^{-1}-\bar{\mu}^{(1)}(K^{(2)}+\mu^{(2)}/3)^{-1}\right]}{(1-\eta^{(2)})\bar{\mu}^{(1)}(K^{(2)}+\mu^{(2)}/3)^{-1}+\eta^{(2)}\bar{\mu}^{(1)}(\bar{K}^{(1)}+\bar{\mu}^{(1)}/3)^{-1}},$$

$$\bar{\mu}_{12}^{*}=\bar{\mu}^{(1)}\left\{1+\eta^{(2)}\left[\frac{\bar{\mu}^{(1)}}{\mu^{(2)}-\bar{\mu}^{(1)}}+\frac{\bar{K}^{(1)}+7\bar{\mu}^{(1)}/3}{\bar{K}^{(1)}+8\bar{\mu}^{(1)}/3}\right]^{-1}\right\}. \tag{71}$$

Here the following notation is used:

$$\bar{K}^{(1)}=\frac{\bar{E}^{(1)}}{3(1-2\bar{\nu}^{(1)})},\quad K^{(2)}=\frac{E^{(2)}}{3(1-2\nu^{(2)})},\quad \mu^{(2)}=\frac{E^{(2)}}{2(1+\nu^{(2)})}$$

$$\bar{E}^{*(1)}=E_0^{(1)}\left[1-\bar{\Pi}_{\beta}(-\omega)\right],\quad \bar{\nu}^{*(1)}=\nu_0^{(1)}\left[1+\frac{1-2\nu_0^{(1)}}{2\nu_0^{(1)}}\bar{\Pi}(-\omega)\right],$$

$$\bar{\mu}^{*(1)}=\mu_0^{(1)}\left[1-\frac{3}{2(1+\nu_0^{(1)})}\Pi_{\beta}(-\frac{3}{2(1+\nu_0^{(1)})}-\omega)\right],\quad \bar{\Pi}_{\beta}(-x)=\frac{1}{s^{1+\beta}+x}. \tag{72}$$

Thus, within the framework of the foregoing preparation we consider the numerical results and first examine the pure elastic stability loss under $t'=0$ and $t'=\infty$, because in the viscoelastic stability loss of the cylinder the intensity of the external compressed force p must satisfy the inequality:

$$p_{cr.\infty}<p<p_{cr.0}, \tag{73}$$

where $p_{cr.0}(p_{cr.\infty})$ is the critical force obtained at $t'=0(t'=\infty)$.

We compare the results obtained within the scope of the TDLTS with the corresponding ones obtained within the scope of the Bernoulli beam theory and within the scope of the third order refined beam theory. Through $p'_{3D.c.0}\left(=p_{3D.cr.0}/E_0^{(1)}\right)$ and $p'_{3D.c.\infty}\left(=p_{3D.cr.\infty}/E_0^{(1)}\right)$ we denote the critical values of the intensity of the dimensionless forces obtained by employing TDLTS. Here, and below, the subindices 0 and ∞ indicate that the critical values of the forces are calculated at $t'=0$ and at $t'=\infty$ respectively. According to the equations (60) and (66), the corresponding critical values of the intensity of the compressed forces obtained within the scope of the Bernoulli beam theory (denoted by $p'_{E.c.0}$ and $p'_{E.c.\infty}$) and within the scope of the third order refined beam theory (denoted by $p'_{R.c.0}$ and $p'_{R.c.\infty}$) can be calculated by the use of the following expressions:

$$p'_{E.c.0} = \frac{E_{30}}{4E_0^{(1)}}\left(\frac{\pi R}{\ell}\right)^2, \quad p'_{R.c.0} = \frac{E_{30}}{4E_0^{(1)}}\left(\frac{\pi R}{\ell}\right)^2\left(1+\frac{8}{27}\frac{E_{30}}{G_0}\right)^{-1},$$

$$p'_{E.c.\infty} = \frac{E_{3\infty}}{4E_0^{(1)}}\left(\frac{\pi R}{\ell}\right)^2, \quad p'_{R.c.\infty} = \frac{E_{3\infty}}{4E_0^{(1)}}\left(\frac{\pi R}{\ell}\right)^2\left(1+\frac{8}{27}\frac{E_{3\infty}}{G_\infty}\right)^{-1}, \tag{74}$$

where $E_{30} = \bar{E}_3^*|_{s=\infty}$, $E_{3\infty} = \bar{E}_3^*|_{s=0}$, $G_0 = \bar{G}^*\big|_{s=\infty}$ and $G_\infty = \bar{G}^*\big|_{s=0}$.

Also, introduce the parameter $\rho\,(=\pi R/\ell)$ and consider the cases where $\rho = 0.1$, 0.2 and 0.3.

Table 1 shows the values of $p'_{3D.c.0}$, $p'_{R.c.0}$ and $p'_{E.c.0}$. But Tables 2, 3 and 4 show the values of $p'_{3D.c.\infty}$, $p'_{R.c.\infty}$ and $p'_{E.c.\infty}$ calculated under $\omega = 0.5$, 1.0 and 2.0 respectively. These values are obtained for various values of $E^{(2)}/E_0^{(1)}$ and ofρ. It follows from these results that the values of $p'_{R.c.0}(p'_{R.c.\infty})$ are very near to the corresponding values of $p'_{3D.c.0}(p'_{3D.c.\infty})$ and the difference between them is insignificant. However the values of $p'_{E.c.0}(p'_{E.c.\infty})$ are greater than the corresponding values of $p'_{3D.c.0}(p'_{3D.c.\infty})$ and the difference between them becomes more significant with $E^{(2)}/E_0^{(1)}$ and ρ.

Consequently, it can be concluded from the foregoing results that for the pure elastic stability loss problems of the considered cylinder, the third order refined beam theory gives results which are sufficiently close to the corresponding ones obtained within the scope of the TDLTS.

Now we consider the case where the cylinder material is viscoelastic and analyze the critical time $t'_{cr.}$ To compare the values of $t'_{cr.}$given by the TDLTS with those given by approximate beam theories, we must select the values of $p'\left(=p/E_0^{(1)}\right)$ which fall in the intervals $[p'_{E.c.\infty}, p'_{E.c.0}]$, $[p'_{R.c.\infty}, p'_{R.c.0}]$ and $[p'_{3D.c.\infty}, p'_{3D.c.0}]$ simultaneously. However, as follows from the data given in Tables 1- 4, in many cases such common values of $p'\left(=p/E_0^{(1)}\right)$ exist only for the intervals $[p'_{R.c.\infty}, p'_{R.c.0}]$ and $[p'_{3D.c.\infty}, p'_{3D.c.0}]$. For this reason, we will compare only the critical time obtained by employing TDLTS (denoted by $t'_{3D.cr.}$) with that obtained by employing the third order refined beam theory (denoted by $t'_{Rf.cr.}$).

Thus, consider the data given in Tables 5 – 7 which show the values of the $t'_{3D.cr.}$ and $t'_{Rf.cr.}$ obtained for the cases where $\rho = 0.1$, 0.2 and 0.3 respectively for various values of p' and β under $\omega = 0.5$, $E^{(2)}/E_0^{(1)} = 5$. Moreover, for a clearer illustration of the results related to the critical time in Fig. 2 the graphs of the

Table 1: The values of $p'_{3D.c.0.}$, $p'_{R.c.0.}$ and $p'_{E.c.0.}$obtained for various values of $E^{(2)}/E_0^{(1)}$ and ρ.

$\frac{E^{(2)}}{E_0^{(1)}}$	$\rho = \pi R/\ell$								
	0.1			0.2			0.3		
	$p'_{3D.c.0}$	$p'_{R.c.0}$	$p'_{E.c.0}$	$p'_{3D.c.0}$	$p'_{R.c.0}$	$p'_{E.c.0}$	$p'_{3D.c.0}$	$p'_{R.c.0}$	$p'_{E.c.0}$
1	0.00247	0.00248	0.00250	0.00963	0.00970	0.01000	0.0207	0.02104	0.02249
5	0.00740	0.00741	0.00750	0.0284	0.02867	0.03000	0.0603	0.06114	0.06750
10	0.0134	0.01351	0.01375	0.0510	0.05135	0.05500	0.1055	0.10668	0.12375
20	0.0254	0.02547	0.02615	0.0931	0.09358	0.10500	0.1841	0.18536	0.23625
30	0.03709	0.03714	0.03875	0.1317	0.13211	0.15500	0.2501	0.25095	0.34875

Table 2: The values of $p'_{3D.c.\infty.}$, $p'_{R.c.\infty.}$ and $p'_{E.c.\infty.}$obtained for various values of $E^{(2)}/E_0^{(1)}$ and ρunder $\omega = 0.5$.

$\frac{E^{(2)}}{E_0^{(1)}}$	$\rho = \pi R/\ell$								
	0.1			0.2			0.3		
	$p'_{3D.c.\infty}$	$p'_{R.c.\infty}$	$p'_{E.c.\infty}$	$p'_{3D.c.\infty}$	$p'_{R.c.\infty}$	$p'_{E.c.\infty}$	$p'_{3D.c.\infty}$	$p'_{R.c.\infty}$	$p'_{E.c.\infty}$
1	0.00163	0.00163	0.00166	0.00611	0.00614	0.00667	0.01248	0.01258	0.01501
5	0.00621	0.00622	0.00666	0.02073	0.02073	0.02667	0.03661	0.03649	0.06001
10	0.01138	0.01137	0.01291	0.03368	0.03353	0.05167	0.05307	0.05244	0.11626
20	0.02018	0.02013	0.02541	0.05016	0.04959	0.10167	0.06963	0.06802	0.22876
30	0.02740	0.02728	0.03791	0.06026	0.05925	0.15167	0.07812	0.07567	0.34126

Table 3: The values of $p'_{3D.c.\infty.}$, $p'_{R.c.\infty.}$ and $p'_{E.c.\infty.}$ obtained for various values of $E^{(2)}/E_0^{(1)}$ and ρ under $\omega = 1.0$.

$\frac{E^{(2)}}{E_0^{(1)}}$	$\rho = \pi R/\ell$								
	0.1			0.2			0.3		
	$p'_{3D.c.\infty}$	$p'_{R.c.\infty}$	$p'_{E.c.\infty}$	$p'_{3D.c.\infty}$	$p'_{R.c.\infty}$	$p'_{E.c.\infty}$	$p'_{3D.c.\infty}$	$p'_{R.c.\infty}$	$p'_{E.c.\infty}$
1	0.00184	0.00185	0.00187	0.00709	0.00713	0.00750	0.01494	0.01511	0.01689
5	0.00664	0.00665	0.00687	0.02411	0.02420	0.02750	0.04711	0.04734	0.06189
10	0.01235	0.01236	0.01312	0.04207	0.04212	0.05250	0.07606	0.07598	0.11814
20	0.02295	0.02294	0.02562	0.07007	0.06984	0.10250	0.11348	0.11239	0.23064
30	0.03256	0.03252	0.03812	0.09091	0.09031	0.15250	0.13671	0.13458	0.34314

Table 4: The values of $p'_{3D.c.\infty.}$, $p'_{R.c.\infty.}$ and $p'_{E.c.\infty.}$ obtained for various values of $E^{(2)}/E_0^{(1)}$ and ρ under $\omega = 2.0$.

$\frac{E^{(2)}}{E_0^{(1)}}$	$\rho = \pi R/\ell$								
	0.1			0.2			0.3		
	$p'_{3D.c.\infty}$	$p'_{R.c.\infty}$	$p'_{E.c.\infty}$	$p'_{3D.c.\infty}$	$p'_{R.c.\infty}$	$p'_{E.c.\infty}$	$p'_{3D.c.\infty}$	$p'_{R.c.\infty}$	$p'_{E.c.\infty}$
1	0.00206	0.00206	0.00208	0.00796	0.00801	0.00833	0.01698	0.01720	0.01876
5	0.00692	0.00693	0.00708	0.02560	0.02614	0.02833	0.05315	0.05363	0.06376
10	0.01285	0.01287	0.01333	0.04645	0.04666	0.05333	0.09021	0.09064	0.12001
20	0.02421	0.02429	0.02583	0.08161	0.08168	0.10333	0.14589	0.14563	0.23251
30	0.03495	0.03495	0.03833	0.11077	0.11059	0.15333	0.18577	0.18445	0.34501

dependencies among $t'_{3D.cr.}$, $t'_{Rf.cr.}$ and p' are given for various values of βin the case where $\rho = 0.1$, $\omega = 0,5$. It follows from the results given in Fig. 2 and from the other numerical results (which are not given here) that the graphs of the analyzed dependencies constructed for various values of β have a common intersection point. Before (after) this intersection point the increase in the absolute values of β causes an increase (decrease) in the values of the critical time. Fig. 3 illustrates the parts of the graphs which appear after the mentioned intersection point.

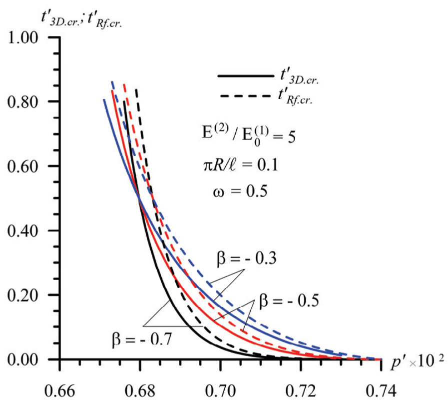

Figure 3: The parts of the graphs given in Fig. 2 which arise after the intersection point

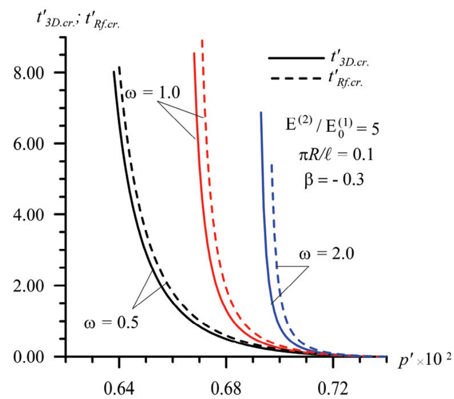

Figure 4: The graphs of the dependencies among $t'_{3D.cr}$, $t'_{Rf.cr.}$ and dimensionless intensity of the ompressed force p' constructed for various values of rheological parameter ω.

Note that the parameter β is one of the rheological parameters of the matrix material and characterizes the mechanical behavior of this viscoelastic material around the initial state of the deformation, i.e. in the near vicinity of $t' = 0$. But the other dimensionless rheological parameter ω characterizes the mechanical properties of the matrix material around $t' = \infty$. Thus, the values of $E_\infty^{(1)}$ ($\nu_\infty^{(1)}$) increase (decrease) with ω. According to the well-known mechanical consideration, it can be predicted that the values of $p'_{cr.\infty}$ must increase with ω. This prediction is proved by the data given in Tables 2 – 4. Moreover, the influence of the parameter ω on the graphs of the dependencies among $t'_{3D.cr.}$, $t'_{Rf.cr.}$ and p' is illustrated by the graphs given in Fig. 4.

Analyses of the results given in Tables 5 – 7 show that in the case under consideration, i.e. in the case where $E^{(2)}/E_0^{(1)} = 5$, $\omega = 0.5$, $\rho = 0.1$, 0.2 and 0.3 the values of $t'_{Rf.cr.}$ are greater than the corresponding values of $t'_{3D.cr.}$. The difference between

Table 5: The values of $t'_{3D.cr.}$(upper number) and $t'_{Rf.cr.}$(lower number) obtained for various values of p' and β under $\rho = 0.1$, $\omega = 0.5$ and $E^{(2)}/E_0^{(1)} = 5$.

β	$p' \times 10^2$								
	0.637	0.650	0.660	0.670	0.680	0.690	0.700	0.710	0.720
	$t'_{3D.cr.}/t'_{Rf.cr.}$								
-0.3	10.59 / 11.24	3.341 / 3.508	1.726 / 1.810	0.967 / 1.015	0.561 / 0.590	0.328 / 0.346	0.187 / 0.199	0.100 / 0.108	0.046 / 0.051
-0.5	35.92 / 39.06	7.143 / 7.648	2.835 / 3.030	1.259 / 1.347	0.588 / 0.631	0.277 / 0.299	0.126 / 0.138	0.052 / 0.058	0.018 / 0.020
-0.7	620.9 / 713.9	42.05 / 47.13	9.015 / 10.07	2.331 / 2.609	0.655 / 0.738	0.187 / 0.212	0.050 / 0.058	0.011 / 0.014	0.0019 / 0.0025

$t'_{Rf.cr.}$ and $t'_{3D.cr.}$ depends not only on the rheological parameters ω and β, but also on the values of the intensity of the external compressive forces p' and, as can be predicted, there exist the following relations:

$$t'_{3D.cr} \to 0 \text{ as } p' \to p'_{3D.c.0}; \quad t'_{3D.cr} \to \infty \text{ as } p' \to p'_{3D.c.\infty},$$

$$t'_{Rf.cr} \to 0 \text{ as } p' \to p'_{R.c.0}; \, t'_{Rf.cr} \to \infty \text{ as } p' \to p'_{R.c.\infty}. \tag{75}$$

Table 6: The values of $t'_{3D.cr.}$(upper number) and $t'_{Rf.cr.}$(lower number) obtained for various values of p' and β under $\rho = 0.2$, $\omega = 0.5$ and $E^{(2)}/E_0^{(1)} = 5$.

β	$p' \times 10^2$								
	2.250	2.300	2.350	2.400	2.450	2.500	2.550	2.600	2.650
	$t'_{3D.cr.}/t'_{Rf.cr.}$								
-0.3	5.704 / 5.998	3.465 / 3.660	2.227 / 2.366	1.475 / 1.578	0.990 / 1.068	0.664 / 0.724	0.439 / 0.486	0.282 / 0.318	0.173 / 0.200
-0.5	15.10 / 16.20	7.517 / 8.117	4.049 / 4.408	2.275 / 2.501	1.301 / 1.447	0.743 / 0.840	0.417 / 0.480	0.225 / 0.266	0.113 / 0.139
-0.7	146.4 / 165.7	45.79 / 52.04	16.33 / 18.81	6.247 / 7.315	2.462 / 2.941	0.969 / 1.187	0.370 / 0.468	0.132 / 0.174	0.042 / 0.059

The results given in Tables 5 – 7, and many other results (which are not given here), show that the insignificant difference between $p'_{3D.c.0}$ and $p'_{R.c.0}$, as well as the insignificant difference between $p'_{3D.c.\infty}$ and $p'_{R.c.\infty}$, causes the significant difference between the values of $t'_{3D.cr.}$ and $t'_{Rf.cr.}$. In this case, if $p'_{3D.c.0} > p'_{R.c.0}$(or if $p'_{3D.c.0} < p'_{R.c.0}$) then the values of $t'_{3D.cr.}$obtained for p'which are close to the $p'_{3D.c.0}$ or to the $p'_{R.c.0}$, are greater (less) than the corresponding values of $t'_{Rf.cr.}$. Moreover, if $p'_{3D.c.\infty} > p'_{R.c.\infty}$(or if $p'_{3D.c.\infty} < p'_{R.c.\infty}$) then the values of $t'_{3D.cr.}$obtained for the p'which is close to the $p'_{3D.c.\infty}$ or to the $p'_{R.c.\infty}$, are greater (less) than the corresponding values of $t'_{Rf.cr.}$. The results illustrated in Tables 5 – 7 correspond to the

Table 7: The values of $t'_{3D.cr.}$(upper number) and $t'_{Rf.cr.}$ (lower number)obtained for various values of p' and β under $\rho = 0.3$, $\omega = 0.5$ and $E^{(2)}/E_0^{(1)} = 5$.

β	$p' \times 10^2$								
	4.100	4.200	4.300	4.400	4.500	4.600	4.700	4.800	4.900
	$t'_{3D.cr.}/t'_{Rf.cr.}$								
-0.3	8.151 / 8.208	5.634 / 5.730	4.074 / 4.178	3.035 / 3.137	2.308 / 2.403	1.779 / 1.866	1.383 / 1.462	1.080 / 1.151	0.843 / 0.908
-0.5	24.89 / 25.13	14.84 / 15.20	9.428 / 9.769	6.246 / 6.541	4.257 / 4.504	2.957 / 3.161	2.078 / 2.247	1.469 / 1.608	1.040 / 1.153
-0.7	336.9 / 342.4	142.3 / 148.0	66.80 / 70.87	33.62 / 36.31	17.74 / 19.50	9.670 / 10.81	5.373 / 6.119	3.015 / 3.504	1.695 / 2.013

case where $p'_{3D.c.0} < p'_{R.c.0}$ and $p'_{3D.c.\infty} < p'_{R.c.\infty}$. Therefore in the all cases considered in these tables the values of $t'_{3D.cr.}$ are less than the corresponding values of $t'_{Rf.cr.}$.

Consider the numerical results related to the other aforementioned cases, namely the case where $p'_{3D.c.0} < p'_{R.c.0}$ and $p'_{3D.c.\infty} > p'_{R.c.\infty}$. The results given in Tables 1 and 2 show that the values of$p'_{3D.c.0}$, $p'_{R.c.0}$, $p'_{3D.c.\infty}$ and $p'_{R.c.\infty}$ obtained, for example for the case where$\rho = 0.1$, $E^{(2)}/E_0^{(1)} = 10$ and $\omega = 0.5$ satisfy these inequalities. The corresponding values of $t'_{3D.cr.}$ and $t'_{Rf.cr.}$obtained for the case mentioned are given in Table 8. Consequently, according to the foregoing discussion, the following relations must occur:

$$t'_{3D.cr.} > t'_{Rf.cr.} \text{ for } p' \text{ which is close to the values of } p'_{3D.c.0} \text{ or } p'_{R.c.0};$$

$$t'_{3D.cr.} < t'_{Rf.cr.} \text{ for } p' \text{ which is close to the values of } p'_{3D.c.\infty} \text{ or } p'_{R.c.\infty}. \quad (76)$$

The analysis of the results given in Table 8 proves the reliability of the relation (76).

Table 8: The values of $t'_{3D.cr.}$(upper number) and $t'_{Rf.cr.}$(lower number) obtained for various values of p' and β under $\rho = 0.1$, $\omega = 0.5$ and $E^{(2)}/E_0^{(1)} = 10$.

β	$p' \times 10^2$								
	1.15	1.16	1.17	1.18	1.19	1.20	1.21	1.22	1.23
	$t'_{3D.cr.}/t'_{Rf.cr.}$								
-0.3	73.51 / 68.95	27.39 / 26.73	14.47 / 14.35	8.865 / 8.869	5.881 / 5.920	4.094 / 4.141	2.937 / 2.984	2.148 / 2.191	1.588 / 1.627
-0.5	541.1 / 494.7	135.8 / 131.34	55.61 / 54.96	28.00 / 28.02	15.76 / 15.91	9.493 / 9.648	5.964 / 6.098	3.849 / 3.958	2.522 / 2.608

Compare the values of $t'_{3D.cr.}$ with the corresponding ones calculated by employing the critical deformation method by Gerard and Gilbert (1958). According to this method, it is assumed that the critical deformation of the viscoelastic cylinder is equal to the critical deformation of the corresponding elastic cylinder. Consequently, using this assumption the critical deformation for the pure elastic cylinder is determined within the scope of the TDLTS. Note that in the considered case the critical deformation mentioned corresponds to $p'_{3D.c.0}$. According to this determination, using the relation $p'_{3D.c.0} = p/E_3^*$ the critical time is determined for the selected values of p. The values of the dimensionless critical time (denoted by $t'_{cdm.cr.}$) determined by employing the critical deformation method are given in Table 9. These values are calculated for the case where $\rho = 0.1$, $E^{(2)}/E_0^{(1)} = 5$ and $\omega = 0.5$. At the same time, in this table the corresponding values of $t'_{3D.cr.}$ are also illustrated. Comparison of the values of $t'_{cdm.cr.}$with the corresponding values of $t'_{3D.cr.}$ shows that the critical deformation method is not acceptable for determination of the critical time for the stability loss of the cylinder made from viscoelastic composite material.

Table 9: The values of $t'_{cdm.cr.}$(upper number) and $t'_{3D.cr.}$(lower number) obtained for various values of p' and β under $\rho = 0.1$, $\omega = 0.5$ and $E^{(2)}/E_0^{(1)} = 5$.

β	$p' \times 10^2$						
	0.660	0.670	0.680	0.690	0.700	0.710	0.720
	$t'_{cdm.cr.}/t'_{3D.cr.}$						
-0.3	$\frac{52.77}{1.726}$	$\frac{52.77}{0.967}$	$\frac{1.179}{0.561}$	$\frac{0.532}{0.328}$	$\frac{0.262}{0.187}$	$\frac{0.127}{0.100}$	$\frac{0.055}{0.046}$
-0.5	$\frac{340.2}{2.835}$	$\frac{7.577}{1.259}$	$\frac{1.662}{0.588}$	$\frac{0.545}{0.277}$	$\frac{0.202}{0.126}$	$\frac{0.074}{0.052}$	$\frac{0.023}{0.018}$
-0.7	$\frac{26326}{9.015}$	$\frac{46.40}{2.331}$	$\frac{3.701}{0.655}$	$\frac{0.578}{0.187}$	$\frac{0.110}{0.050}$	$\frac{0.020}{0.011}$	$\frac{0.0029}{0.0019}$

6 Conclusions

In the present paper, within the scope of the three-dimensional geometrically non-linear field equations of the theory for viscoelastic transversal isotropic bodies, an approach has been developed

for 3D stability loss analyses of the solid cylinder made from viscoelastic composite material. It is supposed that the cylinder has an initial infinitesimal imperfection and a time for which this infinitesimal imperfection starts to increase and grows indefinitely, is taken as a critical time for viscoelastic problems. To study the corresponding non-linear boundary value problem, the boundary-form disturbance method and small parameter method are employed simultaneously. As a result of these applications, the solution to the mentioned non-linear problem is reduced to the solution to the corresponding series linear problem. By direct verification it is proven that the equations and relations related to the first and subsequent approximations coincide with the equations and relations of the well known TDLTS.

Based on the previous work of the first author it is noted that for investigation of the stability loss problem it is enough to use only the zeroth and first approximations. By the average-integrating procedure the corresponding approximate stability loss equations within the scope of the Bernoulli and third order refined beam theories is derived from the equations and relations of the mentioned first approximations. The final numerical results are obtained by employing Laplace transform and variation separation methods for the case where the cylinder is made from the viscoelastic composite consisting of the viscoelastic matrix and unidirectional elastic fibers lying along the cylinder. At first, the numerical results related to the pure elastic stability loss of the cylinder at $t = 0$ and $t = \infty$ are considered. It is established that the critical forces obtained within the scope of the third order refined beam theory are very close to the corresponding results obtained within the scope of the 3D approach. However, the critical time obtained within the scope of the 3D approach can be significantly greater or less than the corresponding values of the critical time obtained within the scope of the third order refined beam theory. The difference between the critical times obtained by employing 3D and third order refined beam theories becomes more non-negligible if the values of the external compressed force are close to the critical compressed force obtained at $t = \infty$. Consequently, to investigate the stability loss of the cylinders made from viscoelastic composites, it is necessary in many cases to use the 3D approach developed in the present paper.

The approach developed in the present paper can be easily extended for investigation of the stability loss of cylinders made from more complicated-type time-dependent materials.

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