

## A Meshless Numerical Method for Kirchhoff Plates under Arbitrary Loadings

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**Abstract:** This paper describes the combination of the method of fundamental solutions (MFS) and the dual reciprocity method (DRM) as a meshless numerical method to solve problems of Kirchhoff plates under arbitrary loadings. In the solution procedure, a arbitrary distributed loading is first approximated by either the multiquadrics (MQ) or the augmented polyharmonic splines (APS), which are constructed by splines and monomials. The particular solutions of multiquadrics, splines and monomials are all derived analytically and explicitly. Then, the complementary solutions are solved formally by the MFS. Furthermore, the boundary conditions of lateral displacement, slope, normal moment, and effective shear force are all given explicitly for the particular solutions of multiquadrics, splines and polynomials as well as the kernels of MFS. Finally, numerical experiments are carried out to validate these analytical formulas. In these numerical experiments, homogeneous problems are first considered to find the best location of the MFS sources by the way proposed by Tsai, Lin, Young and Atluri (2006). Then the corresponding nonhomogeneous problems are solved by the DRM based on both the MQ and APS. The numerical results demonstrate that the MQ is in general more accurate than the thin plate spline, or the first order APS, but less accurate than the high order APSs. Overall, this paper derives a meshless numerical method for solving problems of Kirchhoff plates under arbitrary loadings with all kinds of boundary conditions by both the MQ and APS.

**Keywords:** method of fundamental solutions, dual reciprocity method, multiquadrics, polyharmonic spline

### 1 Introduction

There is a traditional interest in the analysis of a Kirchhoff plate in equilibrium, which is governed by the biharmonic equation. The reasons are not only the re-

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duction of dimensionalities but also the mathematical attractions themselves. Analytical treatments have been intensively studied for over one hundred years. A comprehensive survey for these studies can be found in an excellent review paper of Meleshko (2003). Using the words of Jeffery (1920), all these problems “seem to be a branch of mathematical physics in which knowledge comes by the patient accumulation of special solutions rather than by the establishment of great general propositions.” Although analytical methods are so powerful, numerical methods are inevitable when arbitrary domains are considered.

In the recent years, the meshless methods have received a considerable attention to solve various partial differential equations. Roughly speaking, the meshless methods can be divided into two categories. The first one is domain-type methods in which both the differential equations and boundary conditions are approximated, such as the Kansa’s method (or multiquadrics (MQ) method) [Kansa (1990A, 1990B), Young, Chen and Wong (2005)] as well as the meshless local Petrov-Galerkin method (MLPG) [Wordelman, Aluru and Ravaioli (2000), Lin and Atluri (2000), Kim and Atluri (2000), Atluri (2004), Han and Atluri (2004)]. The second one is boundary-type methods where only boundary conditions are collocated, such as the boundary knot method [Chen and Tanaka (2002), Chen and Hon (2003)], the boundary particle method [Chen (2002)] and the method of fundamental solutions (MFS) [Kupradze and Aleksidze (1964), Mathon and Johnston (1977), Bogomolny (1985), Tsai, Young and Cheng (2002), Chen, Fan, Young, Murugesan and Tsai (2005), Hon and Wei (2005), Tsai, Lin, Young and Atluri (2006), Young and Ruan (2005), Young, Tsai, Lin and Chen (2006)]. Excellent reviews of the MFS are available in the recent literatures [Fairweather and Karageorghis (1998), Golberg and Chen (1998), Fairweather, Karageorghis, and Martin (2003), Cho, Golberg, Muleshkov, and Li (2004)].

The MFS has also been applied to the biharmonic equation [Fairweather and Karageorghis (1998)]. However, previous studies mainly concentrated on the essential boundary conditions and did not consider a meshless treatment for the particular solutions. For nonhomogeneous partial differential equations, Golberg (1995) first proposed the combination of the MFS and the dual reciprocity method (DRM) as a meshless numerical method to solve Poisson’s equation. Later, Golberg and Chen (1998) generalized the MFS-DRM for Helmholtz and diffusion problems. In their studies, the nonhomogeneous terms were first approximated by augmented polyharmonic splines (APS) [Duchon (1977)], which are constructed by the polyharmonic splines (PS) and monomials. Then, the corresponding particular solutions could be derived analytically. Muleshkov, Golberg, and Chen (1999) and Cheng (2000) gave reviews on these particular solutions for the splines. For monomials, Cheng, Lafe and Grilli (1994) as well as Golberg, Muleshkov, Chen and Cheng

(2003) studied their particular solutions. On the other hand, Golberg, Chen, Karur (1996) further improved the accuracy by replacing the thin plate spline (TPS), or the first order APS, by the MQ of Hardy (1971).

In this paper, we extend the MFS-DRM formulations to Kirchhoff plates in equilibrium, which are governed by the nonhomogeneous biharmonic equation. The particular solutions of the MQ, PS and monomials are all reviewed. Samaan and Rashed (2007) derived the particular solutions of the MQ for solving two-dimensional elastodynamic problems. For the PS and monomials, their particular solutions were studied respectively by Cheng (2000) and the author's recent study [Tsai (2008)]. It should be noted that the particular solutions of monomials are also applicable when the Chebyshev method [Golberg, Muleshkov, Chen and Cheng (2003); Reutskiy and Chen (2006); Tsai (2008)] is applied. Furthermore, the boundary conditions of lateral displacement, slope, normal moment, and effective shear force are all given explicitly and analytically in forms suitable for numerical implementations for the particular solutions as well as the kernels of MFS. Numerical results demonstrate that the MQ is more accurate than the TPS but less accurate than the high order APS.

A brief outline of the paper is as follows. We introduce the formulations of MFS-DRM for solving problems of Kirchhoff plates under arbitrary loading in Section 2. In Section 3, some numerical experiments are preformed and the issues of practically implementing the MFS-DRM are stated. Finally, the conclusions are summarized in Section 4.

## 2 MFS-DRM Formulation

### 2.1 Governing equations

Consider a Kirchhoff, or thin, plate in bending, with thickness  $h$  and midplane in the  $x_1$ - $x_2$  plane. According to the basic assumption of the Kirchhoff theory, the lateral deflection  $u$  is considered to be independent of  $x_3$ , and the transverse stressed are ignored. For homogeneous, isotropic, elastic plate, it is governed by the biharmonic equation [Timoshenko and Woinowsky-Krieger (1959)]

$$D\nabla^2\nabla^2u(\mathbf{x}) = q(\mathbf{x}) \text{ in } \Omega \quad (1)$$

where  $q(\mathbf{x})$  is the density of lateral force at  $\mathbf{x} = (x_1, x_2)$  and  $D = \frac{Eh^3}{12(1-\nu)}$  with  $E$  the Young's Modulus and  $\nu$  the Poisson's ratio of elasticity.

Also, some proper boundary conditions should be imposed:

$$\begin{aligned} B_1u(\mathbf{x}) &= \bar{u}_1(\mathbf{x}) \text{ on } \Gamma \\ B_2u(\mathbf{x}) &= \bar{u}_2(\mathbf{x}) \text{ on } \Gamma \end{aligned} \quad (2)$$

where  $\bar{u}_1(\mathbf{x})$  and  $\bar{u}_2(\mathbf{x})$  are the given boundary data and the boundary operators  $B_1$  and  $B_2$  are any two of the following operators:

$$\mathbf{K}_u(\bullet) = 1 \tag{3a}$$

$$\mathbf{K}_\theta(\bullet) = \frac{\partial(\bullet)}{\partial \mathbf{n}_x} \tag{3b}$$

$$\mathbf{K}_m(\bullet) = \nu \nabla_x^2(\bullet) + (1 - \nu) \frac{\partial^2(\bullet)}{\partial \mathbf{n}_x^2} \tag{3c}$$

$$\mathbf{K}_v(\bullet) = \frac{\partial \nabla_x^2(\bullet)}{\partial \mathbf{n}_x} + (1 - \nu) \frac{\partial}{\partial \mathbf{t}_x} \frac{\partial^2(\bullet)}{\partial \mathbf{n}_x \partial \mathbf{t}_x} \tag{3d}$$

where  $\frac{\partial}{\partial \mathbf{n}_x}$  and  $\frac{\partial}{\partial \mathbf{t}_x}$  are the normal and tangential derivatives, respectively, on the boundary point  $\mathbf{x}$ . In the above equations we denote  $\mathbf{K}_u(u(\mathbf{x}))$ ,  $\mathbf{K}_\theta(u(\mathbf{x}))$ ,  $\mathbf{K}_m(u(\mathbf{x}))$ , and  $\mathbf{K}_v(u(\mathbf{x}))$  the lateral displacement, the slope, the normal moment, and the effective shear force respectively. Practically, (3a)&(3b) are selected for clamped boundary condition, (3a)and(3c) for simply-supported boundary condition, and (3c)&(3d) for free boundary condition.

In the MFS-DRM formulation, we linearly decompose the solution into

$$u(\mathbf{x}) = u_h(\mathbf{x}) + u_p(\mathbf{x}) \tag{4}$$

where the particular solution,  $u_p(\mathbf{x})$ , satisfies

$$D \nabla^2 \nabla^2 u_p(\mathbf{x}) = q(\mathbf{x}) \text{ in } \Omega \tag{5}$$

and the homogenous solution,  $u_h(\mathbf{x})$ , satisfies

$$\nabla^2 \nabla^2 u_h(\mathbf{x}) = 0 \text{ in } \Omega \tag{6}$$

with the following boundary conditions:

$$\begin{aligned} B_1 u_h(\mathbf{x}) &= \bar{u}_1(\mathbf{x}) - B_1 u_p(\mathbf{x}) \text{ on } \Gamma \\ B_2 u_h(\mathbf{x}) &= \bar{u}_2(\mathbf{x}) - B_2 u_p(\mathbf{x}) \text{ on } \Gamma \end{aligned} \tag{7}$$

In the MFS-DRM formulation, the particular solution is first approximated by the DRM [Golberg (1995); Golberg and Chen (1998)]. Then, the homogeneous problem (6) & (7) become well posed and thus can be solved formally by the MFS [Fairweather and Karageorghis (1998)].

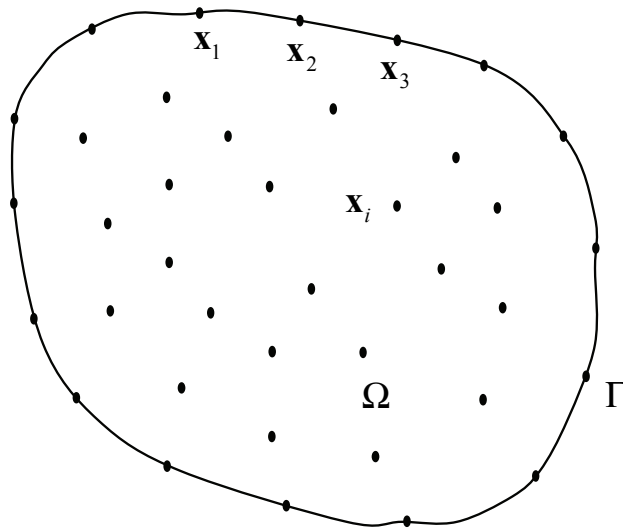


Figure 1: Geometry configuration of the DRM

### 2.2 Dual reciprocity method

Now, we are in a position to introduce the DRM. First of all, the force term  $q(\mathbf{x})$  should be approximated by

$$q(\mathbf{x}; \alpha^j, \beta^j) \cong \sum_{j=1}^M \alpha^j p^j(\mathbf{x}) + \sum_{j=1}^N \beta^j f(r_j) \tag{8}$$

where monomial basis consists of the family

$$\{p^1, p^2, \dots, p^M\} = \{1, x_1, x_2, x_1^2, x_2^2, x_1 x_2, x_1^3, \dots\} \tag{9}$$

and  $f(r_j)$  is equal to the MQ  $\sqrt{r_j^2 + c^2}$  or the  $n$ -th order PS  $r_j^{2n} \ln r_j$ . In the above statement,  $c$  is a shape parameter [Hardy (9171); Golberg, Chen, Karur (1996)], and  $r_j = \|\mathbf{x} - \mathbf{x}_j\|$  is the Euclidean distance between the coordinates  $\mathbf{x}$  and the prescribed points  $\mathbf{x}_j$  as depicted in Fig. 1.

Then, the  $M + N$  unknown coefficients,  $\alpha^j$  and  $\beta^j$ , can be determined by the collocation and constraint conditions as follows

$$q(\mathbf{x}_i; \alpha^j, \beta^j) \cong \sum_{j=1}^M \alpha^j p^j(\mathbf{x}_i) + \sum_{j=1}^N \beta^j f(r_{ij}) \text{ for } i = 1, 2, \dots, N \tag{10a}$$

$$\sum_{j=1}^N \beta^j p^i(\mathbf{x}_j) = 0 \text{ for } i = 1, 2, \dots, M \tag{10b}$$

where  $r_{ij} = \|\mathbf{x}_i - \mathbf{x}_j\|$ . Based on the theory of APS [Duchon (1977)], the PS in Eq. (8) should be augmented with  $n$ -th order monomials to ensure the solvability. On the other hand, Micchelli (1986) proved that the MQ is conditionally positive definite of order one, which indicates that only constant term is required for the MQ.

Then, the particular solution  $u^p(\mathbf{x})$  can be approximate as follows:

$$u^p(\mathbf{x}) \cong \sum_{j=1}^M \alpha^j P^j(\mathbf{x}) + \sum_{j=1}^N \beta^j F(r_j) \tag{11}$$

in which  $P^j(\mathbf{x})$ ,  $F(r_j)$  are governed by

$$D\nabla^2 \nabla^2 P^j(\mathbf{x}) = p^j(\mathbf{x}) \tag{12a}$$

$$D\nabla^2 \nabla^2 F(r_j) = f(r_j) \tag{12b}$$

Details of  $P^j(\mathbf{x})$  and  $F(r_j)$  will be given in the later subsection. Then, the boundary conditions of the particular solutions can be obtained by considering Eqs. (2) and (3) as follows:

$$B_1 u^p(\mathbf{x}) \cong \sum_{j=1}^M \alpha^j B_1 P^j(\mathbf{x}) + \sum_{j=1}^N \beta^j B_1 F(r_j) \tag{13a}$$

$$B_2 u^p(\mathbf{x}) \cong \sum_{j=1}^M \alpha^j B_2 P^j(\mathbf{x}) + \sum_{j=1}^N \beta^j B_2 F(r_j) \tag{13b}$$

More thorough discussion will be followed in the subsection of boundary conditions.

It should be noticed that the convergence of (8) and the solvability of the resulted linear equations from (10) have been mathematically investigated by Duchon (1977) and Micchelli (1986). However, few theoretical statements are addressed for the convergence of (11). Therefore, numerical validations are performed in this study.

### 2.3 Method of fundamental solutions

After the particular solution is solved, the boundary value problem of (6) & (7) becomes well posed. Thus, the homogeneous solution can be solved by the well-known MFS [Fairweather and Karageorghis (1998)]. In the spirits of MFS, the

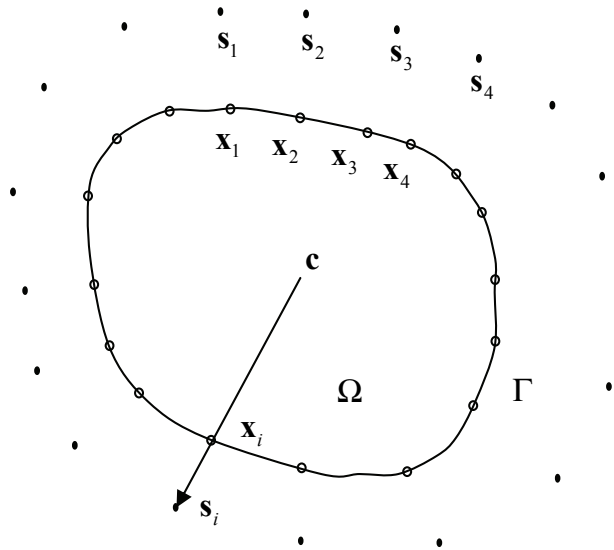


Figure 2: Geometry configuration of the MFS

complementary solution is represented approximately by

$$u_h(\mathbf{x}; \gamma_1^j, \gamma_2^j, \mathbf{s}_j) \cong \sum_{j=1}^L \gamma_1^j G_1(\mathbf{x}, \mathbf{s}_j) + \sum_{j=1}^L \gamma_2^j G_2(\mathbf{x}, \mathbf{s}_j) \tag{14}$$

where  $G_1(\mathbf{x}, \mathbf{s}_j) = r_j^2 \ln r_j$  and  $G_2(\mathbf{x}, \mathbf{s}_j) = \ln r_j$  are the kernels of MFS governed respectively by [Fairweather and Karageorghis (1998)]:

$$\nabla^2 \nabla^2 G_1 = 8\pi \delta(\mathbf{x} - \mathbf{s}) \tag{15a}$$

$$\nabla^2 G_2 = 2\pi \delta(\mathbf{x} - \mathbf{s}) \tag{15b}$$

with  $\delta(\mathbf{x} - \mathbf{s})$  the Dirac delta function and  $r_j = \|\mathbf{x} - \mathbf{s}_j\|$ .

The formula (14) is clear that it satisfies the governing equations analytically and has enough freedoms to fulfill any boundary conditions. To determine the unknowns,  $\gamma_1^j, \gamma_2^j, \mathbf{s}_j$ , boundary conditions in (7) should be imposed in suitable ways. Traditionally, the  $L$  source points  $\mathbf{s}_j$  can be treated either as unknown or *a priori* known. In which the first case results in a nonlinear optimization with  $4L$  unknowns,  $\gamma_1^j, \gamma_2^j, \mathbf{s}_j$ , [Fairweather and Karageorghis (1998)]. On the other hand, if the source points are considered as *a priori* known, the boundary conditions are

simply collocated at  $L$  boundary points  $\mathbf{x}_i$ . It results in a linear equations system as follows:

$$\bar{u}_1(\mathbf{x}_i) - B_1 u_p(\mathbf{x}_i) = \sum_{j=1}^L \gamma_1^j B_1 G_1(\mathbf{x}_i, \mathbf{s}_j) + \sum_{j=1}^L \gamma_2^j B_1 G_2(\mathbf{x}_i, \mathbf{s}_j) \text{ for } i = 1, 2, \dots, L \quad (16a)$$

$$\bar{u}_2(\mathbf{x}_i) - B_2 u_p(\mathbf{x}_i) = \sum_{j=1}^L \gamma_1^j B_2 G_1(\mathbf{x}_i, \mathbf{s}_j) + \sum_{j=1}^L \gamma_2^j B_2 G_2(\mathbf{x}_i, \mathbf{s}_j) \text{ for } i = 1, 2, \dots, L \quad (16b)$$

in which the  $B_1 G_1$ ,  $B_2 G_1$ ,  $B_1 G_2$  and  $B_2 G_2$  will be given in the subsection of boundary conditions. In Eq. (16), there are  $2L$  equations with  $2L$  unknowns,  $\gamma_1^j$ ,  $\gamma_2^j$ , and thus can be solved, in which the solvability was discussed in [Mathon and Johnston (1977), Bogomolny (1985)]. In this paper, we typically locate the boundary field points uniformly and place the source points stipulated out as depicted in Fig. 2 [Tsai, Lin, Young and Atluri (2006)]. In which we define the parameter of source location  $\lambda$  by

$$\mathbf{s}_i = \mathbf{c} + \lambda(\mathbf{x}_i - \mathbf{c}) \quad (17)$$

where  $\mathbf{c}$  is the geometric center.

Once the complementary and particular solutions are both obtained, we can get the desired solution by using (4).

### 2.4 Particular solutions

In this subsection, we address the particular solutions in (12). To be clearer, we are going to find particular solutions as follows:

$$D\nabla^2 \nabla^2 F(r) = f(r) \quad (18a)$$

$$D\nabla^2 \nabla^2 P(\mathbf{x}) = x_1^s x_2^t \quad (18b)$$

For  $f(r) = \sqrt{r^2 + c^2}$ , the solution of (18a) can be found in Samaan and Rashed (2007) as follows:

$$F(r) = \begin{cases} \frac{(-61c^4 + 48r^2c^2 + 4r^4)}{900} \sqrt{r^2 + c^2} + \frac{c^3(2c^2 - 5r^2)}{60} \ln \frac{c + \sqrt{r^2 + c^2}}{2c} + \frac{c^3(61c^2 - 25r^2)}{900} & \text{for } r > 0 \\ 0 & \text{for } r = 0 \end{cases} \quad (19)$$



And the particular solution of  $f(r) = r^{2n} \ln r$  has been derived by Cheng (2000) as follows:

$$F(r) = \frac{r^{2n+4}}{16(n+1)^2(n+2)^2D} \left( \ln r - \frac{1}{n+1} - \frac{1}{n+2} \right) \tag{20}$$

On the other hand, the particular solutions of (18b) can be found by inspecting the results of Laplacian given in Cheng, Lafe, and Grilli (1994) and they are

$$P(\mathbf{x}) = \sum_{k=0}^{\lfloor \frac{s}{2} \rfloor} \frac{(-1)^k (k+1) s! t! x_1^{s+2k+4} x_2^{t-2k}}{(s+2k+4)! (t-2k)! D} \tag{21}$$

This result was recently derived by Tsai (2008) by using the fractional calculus.

**2.5 Boundary conditions**

In practical implementations, the boundary conditions of lateral displacement, slope, normal moment, and effective shear force of the particular solutions of DRM as well as the kernels of MFS are required. In other words, we required the  $B_1 P^j$ ,  $B_2 P^j$ ,  $B_1 F$ ,  $B_2 F$ ,  $B_1 G_1$ ,  $B_2 G_1$ ,  $B_1 G_2$  and  $B_2 G_2$ .

First of all, we rewrite the boundary conditions in (3) in terms of the outward normal  $\mathbf{n}_x = (n_1, n_2)$  as follows [Tsai (2007)]:

$$\mathbf{K}_u(\bullet) = 1 \tag{22a}$$

$$\mathbf{K}_\theta(\bullet) = \frac{\partial(\bullet)}{\partial x_1} n_1 + \frac{\partial(\bullet)}{\partial x_2} n_2 \tag{22b}$$

$$\mathbf{K}_m(\bullet) = g_{11} \frac{\partial^2(\bullet)}{\partial x_1^2} + g_{12} \frac{\partial^2(\bullet)}{\partial x_1 \partial x_2} + g_{13} \frac{\partial^2(\bullet)}{\partial x_2^2} \tag{22c}$$

$$\mathbf{K}_v(\bullet) = g_{21} \frac{\partial^3(\bullet)}{\partial x_1^3} + g_{22} \frac{\partial^3(\bullet)}{\partial x_1^2 \partial x_2} + g_{23} \frac{\partial^3(\bullet)}{\partial x_1 \partial x_2^2} + g_{24} \frac{\partial^3(\bullet)}{\partial x_2^3} \tag{22d}$$

with

$$g_{11} = Dn_1^2 + \nu Dn_2^2 \tag{23a}$$

$$g_{12} = 2(1 - \nu) Dn_1 n_2 \tag{23b}$$

$$g_{13} = Dn_2^2 + \nu Dn_1^2 \tag{23c}$$

$$g_{21} = Dn_1(1 + n_2^2) - \nu Dn_1 n_2^2 \tag{23d}$$

$$g_{22} = \nu Dn_2(1 + n_1^2) + 2(1 - \nu) Dn_2^3 - Dn_1^2 n_2 \tag{23e}$$

$$g_{23} = \nu Dn_1(1 + n_2^2) + 2(1 - \nu)Dn_1^3 - Dn_2^2 n_1 \quad (23f)$$

$$g_{24} = Dn_2(1 + n_1^2) - \nu Dn_2 n_1^2 \quad (23g)$$

To complete the above equations, we require the partial derivatives  $\frac{\partial}{\partial x_i}$ ,  $\frac{\partial^2}{\partial x_i \partial x_j}$  &  $\frac{\partial^3}{\partial x_i \partial x_j \partial x_k}$  of  $P^j$ ,  $F$ ,  $G_1$  &  $G_2$  which are addressed in the Appendix.

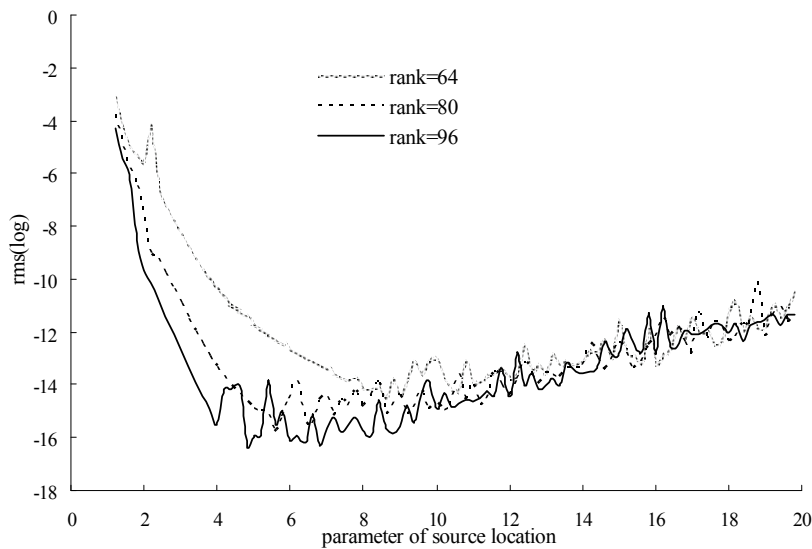


Figure 3: The errors of H Problem (case I).

### 3 Numerical Results

In order to validate the proposed MFS-DRM formulations, three numerical experiments with clamped, clamped & simply-supported, and clamped & free boundary conditions are first considered. Then, the method is applied to a problem of peanut-shaped domain. In order to understand the effect of the DRM, both homogeneous and nonhomogeneous problems are considered in all the four numerical experiments, and they are denoted by H Problem and NH Problem respectively. The exact solution of the H Problem is

$$u(\mathbf{x}) = \cos\left(\frac{\pi x_1}{2}\right) \sinh\left(\frac{\pi x_2}{2}\right) \quad (24)$$

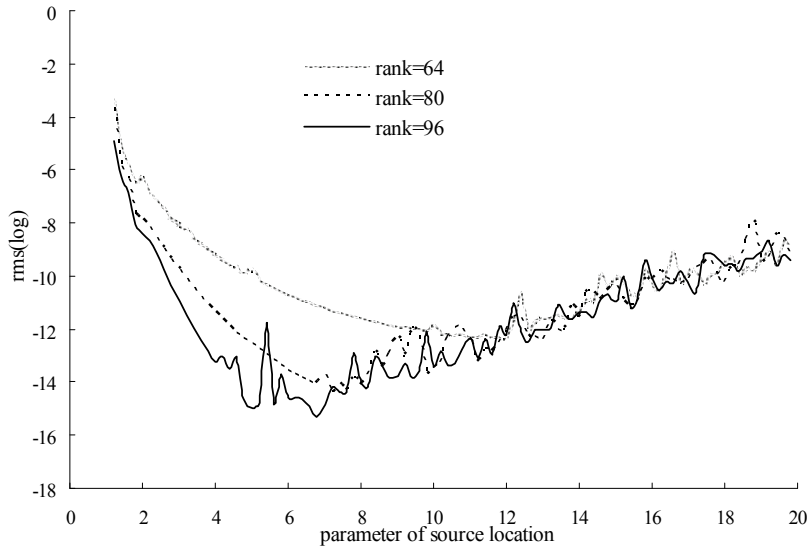


Figure 4: The errors of NH Problem (case I).

On the other hand, the NH Problem has exact solution

$$u(\mathbf{x}) = \sin\left(\frac{\pi x_1}{2}\right) \sin\left(\frac{\pi x_2}{2}\right) \tag{25}$$

which is the solution of

$$D\nabla^2\nabla^2 u = \frac{D\pi^4}{4} \sin\left(\frac{\pi x_1}{2}\right) \sin\left(\frac{\pi x_2}{2}\right) \tag{26}$$

In all the numerical experiments, boundary conditions are set up according to these exact solutions. Also,  $D = 1$  and  $\nu = 0.33$  are assumed.

Furthermore, the root-mean-square error (RMSE) is defined as

$$\sqrt{\frac{\sum_{j=1}^{\bar{N}} \sum_{i=1}^3 (u_{numerical}(\mathbf{x}_j) - u_{exact}(\mathbf{x}_j))^2}{3\bar{N}}} \tag{27}$$

where  $u_{numerical}(\mathbf{x}_j)$  is the numerical solutions obtained by the MFS-DRM equation (4) at  $\mathbf{x}_j$ ,  $u_{exact}(\mathbf{x}_j)$  is the corresponding exact solution, and  $\bar{N}$  is the number of total nodes considered.

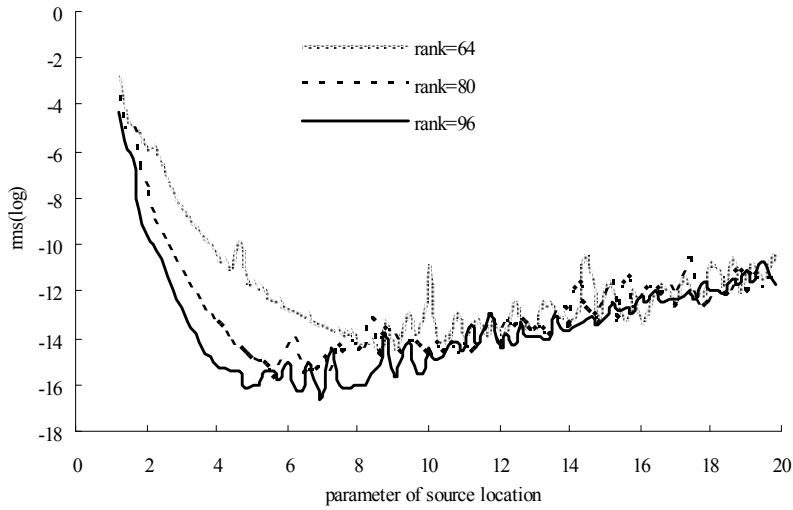


Figure 5: The errors of H Problem (case II).

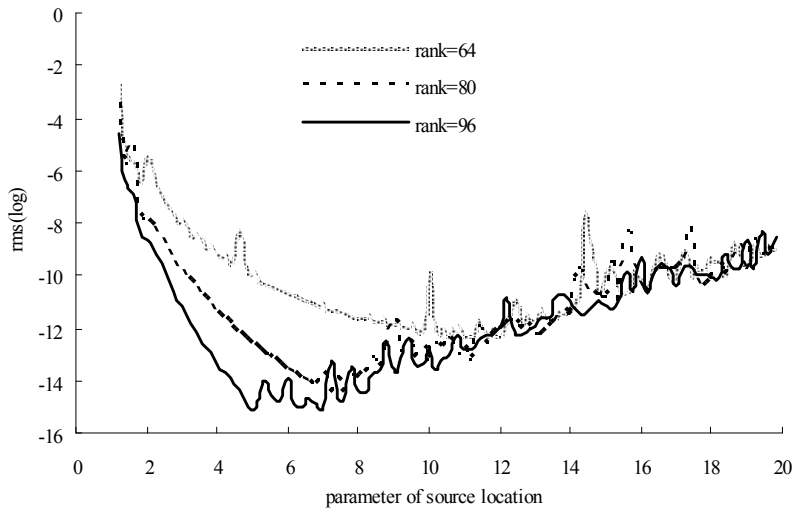


Figure 6: The errors of NH Problem (case II).

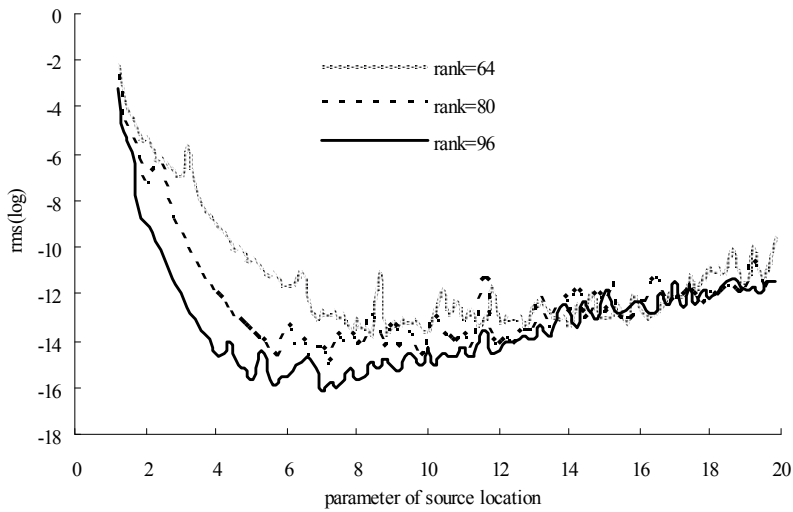


Figure 7: The errors of H Problem (case III).

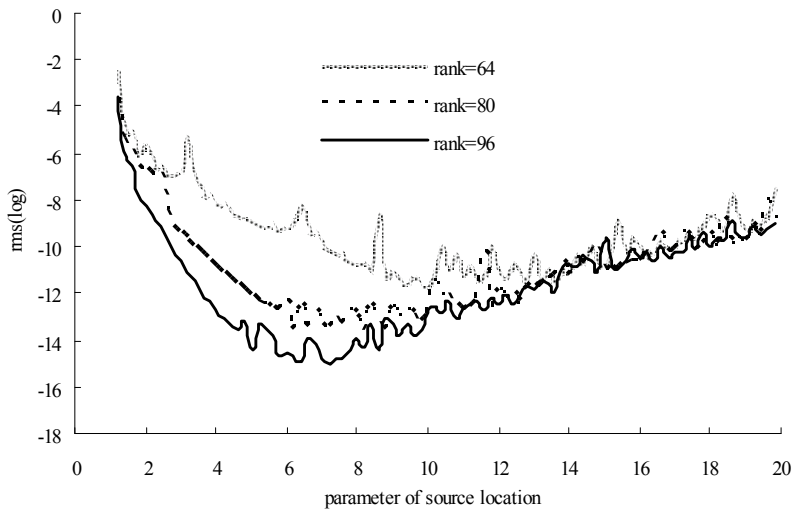


Figure 8: The errors of NH Problem (case III).

### 3.1 Case I: plate with clamped boundary condition

We consider a square plate of size  $1 \times 1$  suggested to clamped boundary conditions. Fig. 3 gives the RMSE versus the parameter of source location. It is found that farther sources give better accuracies before the capacity of the equation solver achieved,  $\lambda = 4 \sim 8$ . This phenomenon has been discovered by Tsai, Lin, Young and Atluri (2006). Furthermore, better accuracies are found for larger ranks, in which the rank is defined by  $2L$  in Eq. (16). In Fig. 3, the best RMSE is  $2.01 \times 10^{-16}$  for  $\lambda = 6.4$  and  $2L = 96$ . By using this optimal setting, we solve the NH problem several times by using the DRM based on the TPS, APS and MQ. Table 1 gives the resulted RMSEs, whose optimal value goes to  $4.81 \times 10^{-16}$  which is only slightly worse than the accuracy of the H problem. Moreover, it can be observed that the MQ is in general more accurate than the TPS, but less accurate than the high order APSs. And, the better accuracy is resulted for the higher order of augmented monomials within some allowable degree. Then we also solve the NH problem several times by using the best setting of the DRM and changing the parameters of the MFS ( $2L$  and  $\lambda$ ) as depicted in Fig. 4. It is of great interest that the same phenomenon also occurs. In other words, the conditioning of matrix resulted from (16) can be a fair guidance to locate the source for both the H Problem and NH Problem.

Table 1: RMSEs of different DRM for Case I.

	N=81	N=121	N=169
TPS	1.12E-07	1.80E-07	1.03E-07
APS (n=2)	1.12E-08	3.40E-08	1.03E-08
APS (n=4)	4.00E-10	2.53E-10	4.90E-11
APS (n=6)	5.72E-12	5.54E-12	7.47E-13
APS (n=8)	5.53E-14	1.02E-14	2.82E-15
APS (n=10)	2.41E-06	1.53E-15	4.81E-16
APS (n=12)	3.60E-06	9.79E-08	7.32E-16
MQ	1.82E-08	1.47E-08	1.25E-09

### 3.2 Case II: plate with clamped and simply-supported boundary conditions

Then, we consider the same two problems by imposing simply-supported boundary condition on one edge. Fig. 5 and Fig. 6 describe the errors for the H Problem and NH Problem, respectively. The best RMSEs are  $10^{-14} \sim 10^{-17}$  and  $10^{-13} \sim 10^{-16}$  when  $\lambda = 4 \sim 8$  for the H Problem and NH Problem, respectively. Similarly, better accuracies are found for larger rank. Similarly, the accuracy of the NH problem

Table 2: RMSEs of different DRM for Case II.

	N=81	N=121	N=169
TPS	1.10E-07	4.87E-08	2.54E-08
APS (n=2)	3.15E-09	4.22E-09	1.64E-09
APS (n=4)	2.85E-10	7.57E-11	2.24E-11
APS (n=6)	7.08E-12	1.17E-12	2.17E-13
APS (n=8)	3.98E-14	1.34E-14	3.89E-15
APS (n=10)	3.45E-06	2.63E-15	3.05E-15
APS (n=12)	2.06E-06	3.22E-07	3.48E-15
MQ	7.08E-09	2.36E-09	3.43E-10

Table 3: RMSEs of different DRM for Case III.

	N=81	N=121	N=169
TPS	4.26E-08	1.80E-08	8.39E-09
APS (n=2)	1.48E-09	2.83E-09	7.75E-10
APS (n=4)	9.45E-11	2.35E-11	4.77E-12
APS (n=6)	2.66E-12	5.04E-13	6.64E-14
APS (n=8)	1.28E-14	3.33E-15	1.09E-15
APS (n=10)	7.45E-07	8.47E-16	7.61E-16
APS (n=12)	1.26E-06	8.42E-08	7.56E-16
MQ	3.35E-09	1.34E-09	1.20E-10

Table 4: RMSEs of different DRM for Case IV.

	N=21	N=29	N=39
TPS	3.94E-07	3.00E-07	6.42E-08
APS (n=2)	2.73E-09	7.52E-10	9.44E-10
APS (n=4)	9.65E-11	1.64E-11	4.68E-11
APS (n=6)	DIV	3.92E-06	1.00E-12
MQ	5.29E-08	9.35E-09	7.81E-10

is only slightly worse than that of the H problem by using higher order APS. This excellent performance of the DRM is discovered for the first time to the best knowledge of the author. Furthermore, Table 2 address the RMSEs obtained by different DRMs, which also shows that the MQ is more accurate than the TPS, but less accurate than the high order APSs. Overall, these studies demonstrate that the MFS-

DRM is able to solve the plate loading problems with simply-supported boundary conditions accurately and stably.



Figure 9: The geometry of the peanut-shape plate.

### 3.3 Case III: plate with clamped and free boundary conditions

Then, we replace the simply-supported boundary condition of the previous case by free boundary condition. Fig. 7 gives the RMSEs of the H Problems. After obtaining the best location of source, we fix the MFS sources and solve the NH problem by various DRM. Table 4 describe the RMSEs of the NH problem obtained by the DRM based on the TPS, APS and MQ, which is generally similar to the previous two examples. Then, by using the optimal setting of the DRM ( $N = 169$  and  $n = 12$ ) we solve the NH problem by different numbers and locations of the MFS sources as addressed in Fig. 8. These results indicate that the proposed MFS-DRM is capable of solving plate loading problems with free boundary conditions.

### 3.4 Case IV: A peanut-shaped plate

In order to demonstrate the flexibility of the proposed numerical method to treat irregular domains, a peanut-shaped plate, depicted in Fig. 9, subjected to clamped boundary condition is chosen as the last problem. The exact solution is the same as the previous cases. Table 4 gives the RMSEs for different DRM based on the optimal settings of the MFS. Also, they behave very similar to the previous examples and excellent accuracy can be observed.

## 4 Conclusions

In this paper, the method of fundamental solutions (MFS) and the dual reciprocity method (DRM) are combined as a meshless numerical method to solve Kirchhoff plates under arbitrary loadings. In the DRM, the arbitrary distributed loadings are



approximated by either the multiquadrics or the augmented polyharmonic splines. Then, the nonhomogeneous solutions can be represented by a series of the analytical particular solutions of these basis functions. The complementary solutions are then solved by the MFS. Furthermore, the boundary conditions of lateral displacement, slope, normal moment, and effective shear force are all given explicitly for the particular solutions of multiquadrics, splines and monomials of the DRM as well as the kernels of the MFS.

To validate the proposed numerical method, three numerical experiments of clamped, clamped & simply-supported, and clamped & free boundary conditions are carried out. Both homogeneous and nonhomogeneous problems are solved by the MFS and MFS-DRM respectively. From the numerical results, it is found that the strategy of locating sourced proposed in Tsai, Lin, Young, and Atluri (2006) can be applied to all kinds of boundary conditions for both homogeneous and nonhomogeneous problems. Furthermore, we found that the multiquadrics is in general more accurate than the thin plate spline but less accurate than the high order augmented polyharmonic splines when a nonhomogeneous problem is solved. However, a shape parameter has to be determined for the MQ and much effort is required to implement the particular solutions of the augmented monomials. Then, the method is applied to a problem of peanut-shaped domain to demonstrate the flexibility of the proposed numerical method to treat irregular domains. From these results, it is convinced that the MFS-DRM is a suitable meshless numerical method to solve Kirchhoff plates under arbitrary loadings without integrations and singularities.

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## Appendix

When applying the MFS-DRM, the partial derivatives of  $F(r)$ ,  $P(\mathbf{x})$ ,  $G_1(r)$  and  $G_2(r)$  are required to find the boundary conditions of lateral displacement, slope, normal moment, and effective shear force. Here, we only consider partial derivatives of radial function since the partial derivatives of the polynomial  $P(\mathbf{x})$  is straightforward. First of all, we introduce a connection between the partial derivatives and the radial operation  $\left(\frac{d}{rdr}\right)^i$  on a radial function through  $G_1(r)$ .

$$\frac{\partial G_1}{\partial x_i} = x_i \left(\frac{d}{rdr}\right) G_1 \quad (\text{A1})$$

$$\frac{\partial^2 G_1}{\partial x_i \partial x_j} = x_i x_j \left(\frac{d}{rdr}\right)^2 G_1 + \delta_{ij} \left(\frac{d}{rdr}\right) G_1 \quad (\text{A2})$$

$$\frac{\partial^3 G_1}{\partial x_i \partial x_j \partial x_k} = x_i x_j x_k \left(\frac{d}{rdr}\right)^3 G_1 + (\delta_{ij} x_k + \delta_{jk} x_i + \delta_{ki} x_j) \left(\frac{d}{rdr}\right)^2 G_1 \tag{A3}$$

where  $r = \sqrt{x_1^2 + x_2^2}$ . Eqs. (A1)~(A3) indicate that to find the partial derivatives of a radial function, only the formulas involving its radial operation  $\left(\frac{d}{rdr}\right)^i$  are required, which are listed in the following.

$$\left(\frac{d}{rdr}\right) G_1 = 1 + 2 \ln r \tag{A4}$$

$$\left(\frac{d}{rdr}\right)^2 G_1 = \frac{2}{r^2} \tag{A5}$$

$$\left(\frac{d}{rdr}\right)^3 G_1 = \frac{-4}{r^4} \tag{A6}$$

and

$$\left(\frac{d}{rdr}\right) G_2 = \frac{1}{r^2} \tag{A7}$$

$$\left(\frac{d}{rdr}\right)^2 G_2 = \frac{-2}{r^4} \tag{A8}$$

$$\left(\frac{d}{rdr}\right)^3 G_2 = \frac{8}{r^6} \tag{A9}$$

For the particular solutions of the  $n$ -th order PS, the formulas are

$$\left(\frac{d}{rdr}\right) F = \frac{r^{2n+2} [2(n+1)(n+2) \ln r - (3n+5)]}{16D(n+1)^3(n+2)^2} \tag{A10}$$

$$\left(\frac{d}{rdr}\right)^2 F = \frac{r^{2n} [2(n+1)(n+2) \ln r - (2n+3)]}{8D(n+1)^2(n+2)^2} \tag{A11}$$

$$\left(\frac{d}{rdr}\right)^3 F = \frac{r^{2n-2} [2n(n+1)(n+2) \ln r - (n^2-2)]}{4D(n+1)^2(n+2)^2} \tag{A12}$$

And if  $F$  is the particular solutions of the MQ, we need the following the formulas

$$\left(\frac{d}{rdr}\right) F = \begin{cases} \frac{(-6c^4 + 28c^2 r^2 + 4r^4) \sqrt{r^2 + c^2} + c^3 (6c^2 - 25r^2)}{180Dr^2} - \frac{c^3}{6D} \ln \frac{c + \sqrt{r^2 + c^2}}{2c} & \text{for } r > 0 \\ 0 & \text{for } r = 0 \end{cases} \tag{A13}$$

$$\left(\frac{d}{rdr}\right)^2 F = \begin{cases} \frac{(2c^4+4c^2r^2+2r^4)\sqrt{r^2+c^2}-c^3(2c^2+5r^2)}{30Dr^4} & \text{for } r > 0 \\ \frac{c}{8} & \text{for } r = 0 \end{cases} \quad (\text{A14})$$

$$\left(\frac{d}{rdr}\right)^3 F = \begin{cases} \frac{(-4c^6-7c^4r^2-2c^2r^4+r^6)+c^3(4c^2+5r^2)\sqrt{r^2+c^2}}{15Dr^6\sqrt{r^2+c^2}} & \text{for } r > 0 \\ \frac{1}{24c} & \text{for } r = 0 \end{cases} \quad (\text{A15})$$

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