 CMC, vol.25, no.2, pp.107-134, 2011

A Lie-Group Adaptive Method to Identify the Radiative Coefficients in Parabolic Partial Differential Equations

Chein-Shan Liu[1] **and Chih-Wen Chang**[2]

Abstract: We consider two inverse problems for estimating radiative coefficients $\alpha(x)$ and $\alpha(x, y)$, respectively, in $T_t(x, t) = T_{xx}(x, t) - \alpha(x)T(x, t)$, and $T_t(x, y, t) = T_{xx}(x, y, t) + T_{yy}(x, y, t) - \alpha(x, y)T(x, y, t)$, where α are assumed to be continuous functions of space variables. A Lie-group adaptive method is developed, which can be used to find α at the spatially discretized points, where we only utilize the initial condition and boundary conditions, such as those for a typical direct problem. This point is quite different from other methods, which need the overspecified final time data. Three-fold advantages can be gained by the present Lie-group adaptive method (LGAM): (i) no a priori information of radiative coefficients is required, (ii) no extra data are measured, and (iii) no complicated procedure is involved. The accuracy and efficiency of present method are confirmed by comparing the estimated results with some exact solutions for 1-D and 2-D cases.

Keywords: Inverse problem, Parameter identification, Lie-group adaptive method (LGAM), Spatial-dependence radiative coefficient, Iterative method

1 Introduction

The direct problem is to deduce an effect from a cause. In contrast, the inverse problem is to deduce a cause from an effect. An important example is the inverse problem of thermal physics, in which we seek to investigate the thermophysical properties of a heat conducting body by given measurements of temperature. Heat may conduct through the body in a manner which depends on the material properties of the body. If the material properties of the body were known exactly, then we could predict the temperature distribution from knowledge of the source. However, since we usually cannot directly measure these properties, we seek to infer them by observing the temperature in response to a collection of known sources.

[1] Department of Civil Engineering, National Taiwan University, Taipei 10617, Taiwan

[2] Grid Applied Technology Division, National Center for High-Performance Computing, Taichung 40763, Taiwan. Corresponding author, Tel.: +886-4-24620202x860. E-mail address: 0903040@nchc.narl.org.tw

In formulating such problems mathematically, we typically find that this problem amounts to that of determining one or more coefficients in a differential equation, or system of differential equations, given partial knowledge of certain special solutions of the equations. In the heat conduction problem, the conduction of heat in a body is governed by the equations of thermophysics, a system of partial differential equations in which the material properties of the body manifest themselves as coefficient functions in the equations. The measurements we can make amount to the knowledge of special solutions of these equations at special points of the body.

Inverse problems in differential equations have this general character. One has a certain definite kind of differential equation containing one or more unknown coefficient functions. From some limited knowledge about certain special solutions of these equations, we seek to determine the unknown coefficient functions. Problems of this sort arise in a variety of important application areas in engineering and sciences.

Inverse problems and their computations are presently becoming more and more important in many fields of engineering and sciences. They inevitably result in the mathematical models that are not well-posed in the sense of Hadamard, which means that one or more of the following well-posed properties are lost: for all admissible data the solution exists; for all admissible data the solution is unique; the solution depends continuously on the data. The problems that fail to meet these prerequisites are said to be ill-posed. The computations of ill-posed problems are usually more difficult than the well-posed problems because they are sensitive to the measurement errors of data.

Now, we consider an inverse problem of finding an unknown parameter $\alpha(x)$ in a one-dimensional heat conduction equation, of which for

Problem $\mathbf{P}_1$: one needs to find the temperature distribution $T(x, t)$, as well as the radiative coefficient function $\alpha(x)$ that simultaneously satisfy

$$\frac{\partial T(x,t)}{\partial t} = \frac{\partial^2 T(x,t)}{\partial x^2} - \alpha(x)T(x,t), \ 0 < x < \ell, t > 0, \tag{1}$$

$$T(0,t) = F_0(t), T(\ell,t) = F_\ell(t), \tag{2}$$

$$T(x,0) = f(x), \tag{3}$$

where $F_0(t)$, $F_\ell(t)$ and $f(x)$ are, respectively, the given/measured functions of left-boundary temperature, right-boundary temperature and initial temperature, and ℓ is a length of the heat conducting rod.

Mathematically speaking, Eqs. (1)-(3) form an underdetermined nonlinear system because both $\alpha(x)$ and $T(x, t)$ are unknown functions. Usually, $\alpha(x)$ can be estimated, provided that an extra measurement of data is available. For example, in order to estimate $\alpha(x)$, all the authors in the literature [e.g., Rundell (1987), Choulli(1996), Choulli (1997), Huang (2004), Chen (2006), Yang (2008), and Deng (2009, 2010)] required an overspecified final temperature measured at a time $t = t_f$:

$$T(x,t_f) = F_m(x). \tag{4}$$

Eq. (1) also appears in the optical tomography [Klibanov (1999)], of which $\alpha(x)$ is named the absorption coefficient, because it is a property of material which defines the amount of light absorbed by the material. In order to estimate $\alpha(x)$, Tadi, Klibanov and Cai (2002), and Tadi (2007) required two overspecified boundary data.

Yamamoto and Zou (2001) could recover both the initial temperature and radiative coefficient by an extra observation of the temperature inside a subregion, in addition to the above final data. Furthermore, Choulli and Yamamoto (2008) proved that the radiative coefficient, the initial temperature and a boundary coefficient can be simultaneously determined from the final overdetermination, provided that the radiative coefficient is a priori known in a small subdomain.

Most of the studies mainly relied on an iterative optimization formulation, and all these methods without exception required that some extra temperature or heat flux be measured. Most of them have a step to solve the heat conduction equation by assuming an absorption coefficient, and then the calculated temperature or heat flux at the selected measuring points are compared with the measured ones. The differences of these two results are thus used to determine the search step of solution for the next iterative procedure.

In the present paper, a novel method will be developed to estimate the unknown radiative coefficient $\alpha(x)$ of the above inverse problem, which merely requires the boundary conditions and initial condition given by Eqs. (2) and (3), as these used in the direct problem, without needing of an extra overspecified data. To the best knowledge of authors, in the open literature of estimation of unknown spatial-dependence parameter, there has no researchers to discuss this possibility.

We develop a novel Lie-group adaptive method (LGAM) for the inverse problem of parameter identification of $\alpha(x)$ governed by Eqs. (1)-(3), which is for a possible application in the heat conduction engineering by considering the measurement cost for impossibly mounting sensors in the material to measure the extra data. The LGAM has been successfully developed by Liu (2010, 2011), and Liu and Atluri (2010) and used for the inverse estimation problems for other type inverse problems.

We also deliberate the second inverse problem with

Problem P2: one needing to find the temperature distribution $T(x, y, t)$, as well as the radiative coefficient function $\alpha(x, y)$ that simultaneously satisfy

$$\frac{\partial T(x,y,t)}{\partial t} = \frac{\partial^2 T(x,y,t)}{\partial x^2} + \frac{\partial^2 T(x,y,t)}{\partial y^2} - \alpha(x,y)T(x,y,t),$$
$$0 < x < x_0,\, 0 < y < y_0,\, t > 0, \quad (5)$$

$$T(0,y,t) = F_0(y,t), T(x_0,y,t) = F_{x_0}(y,t),$$

$$T(x,0,t) = H_0(x,t), T(x,y_0,t) = H_{y_0}(x,t), \quad (6)$$

$$T(x,y,0) = f(x,y). \quad (7)$$

Besides, T and α, all other functions are given.

Liu (2006a, 2006b, 2006c) has extended the group-preserving scheme (GPS) developed previously by Liu (2001) for ODEs to solve the boundary value problems (BVPs). In the construction of the Lie group method for the calculations of BVPs, Liu (2006a) has introduced the idea of one-step GPS by utilizing the closure property of the Lie group, and hence, the new shooting method has been named the Lie-group shooting method (LGSM).

After that, Liu (2006d) has used this concept to establish a one-step estimation method to estimate the temperature-dependent heat conductivity, and then extended the Lie-group method to estimate the thermophysical properties of heat conductivity and heat capacity [e.g., Liu (2006e), Liu (2007), Liu, Liu and Hong (2007)]. The Lie-group method possesses a greater advantage than other numerical methods due to its group structure, and it is a powerful technique to solve the inverse problems of parameters identification. Liu (2008) has obtained very accurately estimated results by using the LGSM for the identification of heat conductivity. Some recent progress of the LGSM for inverse heat conduction problems, one can refer [e.g., Chang (2007), Liu (2009), Liu (2010a), Liu (2010b)]. For the problem governed by Eqs. (1)-(3), some estimated results about α are reported in this paper by using a Lie-group adaptive method (LGAM). In Sections 2-4, we first describe the Lie-group estimation theory for **Problem P**$_1$. Then, in Section 5 we directly write the required formulas for the Lie-group estimation theory for **Problem P**$_2$. Section 6 devotes to the numerical tests of 1-D and 2-D inverse problems, and the conclusions are given in Section 7.

2 The numerical method of line

For Eq. (1) we adopt the numerical method of line to discretize the differential term with respect to x by

$$\frac{\partial^2 T(x,t)}{\partial x^2}\Big|_{x=i\Delta x} = \frac{T_{i+1}(t) - 2T_i(t) + T_{i-1}(t)}{(\Delta x)^2}, \tag{8}$$

where $\Delta x = \ell/(n+1)$ with n the number of interior grid points, and $x_i = i\Delta x$ are the discretized coordinates of x, at which the temperature is discretized as $T_i(t) = T(x_i, t)$.

In doing so, we can obtain a system of ODEs for T_i with t as an independent variable:

$$\dot{T}_i(t) = \frac{T_{i+1}(t) - 2T_i(t) + T_{i-1}(t)}{(\Delta x)^2} - \alpha_i T_i(t),\ i = 1,\dots,n, \tag{9}$$

where $\alpha_i = \alpha(x_i)$ are the discretized quantities of $\alpha(x)$ at the spatial points x_i.

When $i = 1$, the term $T_0(t)$ appeared in Eq. (9) is determined by the first boundary condition in Eq. (2). Similarly, when $i = n$, the term $T_{n+1}(t)$ is determined by the second boundary condition in Eq. (2).

The known initial condition is given by

$$T_i(0) = f(x_i), \quad i = 1,\dots,n, \tag{10}$$

which is obtained from Eq. (3) by a discretization. In summary, we have totally n ODEs in Eq. (9) to solve the $2n$ unknowns $T_i(t)$ and α_i, $i = 1,\dots,n$.

Obviously, Eq. (9) alone is not enough to solve the unknowns $T_i(t)$ and α_i, $i = 1,\dots,n$, and we require to derive other equations to calculate α_i. After giving a necessary mathematical background of the LGAM in the next section, we will derive the linear equations in Section 4 to determine the unknown coefficients α_i through an iteration process.

3 Mathematical preliminaries

In order to explore our new method clearly, we first briefly sketch the group-preserving scheme (GPS) for ODEs and the one-step GPS in this section.

3.1 The GPS

Let us write Eq. (9) in a vector form:

$$\dot{\mathbf{T}} = \mathbf{f}(t, \mathbf{T}), \tag{11}$$

where

$$\mathbf{T} := \begin{bmatrix} T_1(t) \\ \vdots \\ T_n(t) \end{bmatrix}, \mathbf{f} := \begin{bmatrix} \frac{T_2-2T_1+T_0}{(\Delta x)^2} - \alpha_1 T_1 \\ \frac{T_3-2T_2+T_1}{(\Delta x)^2} - \alpha_2 T_2 \\ \vdots \\ \frac{T_n-2T_{n-1}+T_{n-2}}{(\Delta x)^2} - \alpha_{n-1} T_{n-1} \\ \frac{T_{n+1}-2T_n+T_{n-1}}{(\Delta x)^2} - \alpha_n T_n \end{bmatrix}. \tag{12}$$

$\mathbf{T}$ represents a vector form of the discretized temperatures at the interior grid points, and the components of $\mathbf{f}$ represent the right-hand side of Eq. (9). The dependence of $\mathbf{f}$ on t is due to the dependence of boundary condition (2) on t, i.e., $T_0 = F_0(t)$ and T_{n+1}=$F_\ell(t)$.
When both the vector $\mathbf{T}$ and its magnitude $\|\mathbf{T}\| := \sqrt{\mathbf{T}^t\mathbf{T}} = \sqrt{\mathbf{T}\cdot\mathbf{T}}$ are combined into a single augmented vector with dimension n+1:

$$\mathbf{X} = \begin{bmatrix} \mathbf{T} \\ \|\mathbf{T}\| \end{bmatrix}, \tag{13}$$

Liu (2001) has transformed Eq. (11) into an augmented differential equations system:

$$\dot{\mathbf{X}} = \mathbf{A}\mathbf{X}, \tag{14}$$

where

$$\mathbf{A} := \begin{bmatrix} \mathbf{0}_{n\times n} & \frac{\mathbf{f}(t,\mathbf{T})}{\|\mathbf{T}\|} \\ \frac{\mathbf{f}^t(t,\mathbf{T})}{\|\mathbf{T}\|} & 0 \end{bmatrix} \tag{15}$$

is an element of the Lie algebra $so(n,1)$ satisfying

$$\mathbf{A}^t\mathbf{g} + \mathbf{g}\mathbf{A} = 0, \tag{16}$$

and

$$\mathbf{g} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & -1 \end{bmatrix} \tag{17}$$

is a Minkowski metric. Here, $\mathbf{I}_n$ is the identity matrix of order n, and the superscript t denotes the transpose.

The augmented variable $\mathbf{X}$ can be viewed as a point in the Minkowski space M^{n+1}, satisfying the cone condition:

$$\mathbf{X}^T\mathbf{g}\mathbf{X} = \mathbf{T}\cdot\mathbf{T} - \|\mathbf{T}\|^2 = 0. \tag{18}$$

Then, Liu (2001) developed a group-preserving scheme (GPS) to guarantee that each $\mathbf{X}_k$ locates on the cone:

$$\mathbf{X}_{k+1} = \mathbf{G}(k)\mathbf{X}_k, \tag{19}$$

where $\mathbf{X}_k$ represents the numerical value of $\mathbf{X}$ at the discrete time t_k, and $\mathbf{G}(k)\in SO_o(n,1)$ satisfies

$$\mathbf{G}^T\mathbf{g}\mathbf{G} = \mathbf{g}, \tag{20}$$

$$\det\mathbf{G} \; = 1, \tag{21}$$

$$G_0^0 > 0, \tag{22}$$

where G_0^0 is the 00th component of $\mathbf{G}$.

3.2 One-step GPS

Throughout this paper we use the superscripted symbol $\mathbf{T}^0$ to denote the value of $\mathbf{T}$ at $t = 0$. In order to develop the Lie-group shooting method, we also give a parameter t_f of final time, and $\mathbf{T}^f$ denotes the value of $\mathbf{T}$ at $t = t_f$.

Applying scheme (19) to Eq. (14) with a specified initial condition $\mathbf{X}(0) = \mathbf{X}^0$, we can compute the solution $\mathbf{X}(t)$ by the GPS. Assuming that the time stepsize used in the GPS is $\Delta t = t_f/K$, and starting from an augmented initial condition $\mathbf{X}^0 = ((\mathbf{T}^0)^t, \left\|\mathbf{T}^0\right\|)^t \neq 0$, we can calculate $\mathbf{X}^f = ((\mathbf{T}^f)^t, \left\|\mathbf{T}^f\right\|)^t$ at a specified final time $t = t_f$. By applying Eq. (19) step-by-step, we can obtain

$$\mathbf{X}^f = \mathbf{G}_K(\Delta t)\cdots\mathbf{G}_1(\Delta t)\mathbf{X}^0, \tag{23}$$

However, let us recollect that each $\mathbf{G}_i$, $i = 1,\ldots,K$, is an element of the Lie group $SO_o(n,1)$, and by the closure property of the Lie group, $\mathbf{G}_K(\Delta t)\ldots\mathbf{G}_1(\Delta t)$ is also a Lie-group element denoted by $\mathbf{G}$. Thus, from Eq. (23) it follows that

$$\mathbf{X}^f = \mathbf{G}\mathbf{X}^0. \tag{24}$$

This is a one-step transformation from $\mathbf{X}^0$ to $\mathbf{X}^f$.

The remaining problem is how to calculate $\mathbf{G}$. While an exact solution of $\mathbf{G}$ is impossible, we can calculate an appropriate $\mathbf{G}$ through a numerical method by a generalized mid-point rule, which is obtained from an exponential mapping of **A** by taking the values of the argument variables of **A** at a generalized mid-point. The Lie group generated from $\mathbf{A}\in so(n,1)$ by an exponential mapping is

$$\mathbf{G} = \begin{bmatrix} \mathbf{I}_n + \frac{(a-1)}{\|\hat{\mathbf{f}}\|^2}\hat{\mathbf{f}}\hat{\mathbf{f}}^t & \frac{b\hat{\mathbf{f}}}{\|\hat{\mathbf{f}}\|} \\ \frac{b\hat{\mathbf{f}}^t}{\|\hat{\mathbf{f}}\|} & a \end{bmatrix}, \tag{25}$$

where

$$\hat{\mathbf{T}} = r\mathbf{T}^0 + (1-r)\mathbf{T}^f, \tag{26}$$

$$\hat{\mathbf{f}} = \mathbf{f}(\hat{t},\ \hat{\mathbf{T}}), \tag{27}$$

$$a = \cosh\left(\frac{t_f\left\|\hat{\mathbf{f}}\right\|}{\left\|\hat{\mathbf{T}}\right\|}\right), \quad b = \sinh\left(\frac{t_f\left\|\hat{\mathbf{f}}\right\|}{\left\|\hat{\mathbf{T}}\right\|}\right). \tag{28}$$

Here, we use the initial $\mathbf{T}^0$ and the final $\mathbf{T}^f$ through a suitable weighting factor r to evaluate **G**, where $r \in [0,\ 1]$ is a parameter, and $\hat{t} = (1-r)t_f$.To stress its dependence on r, we denote this **G** by $\mathbf{G}(r)$.

3.3 A universal one-step GPS

Let us define a new vector

$$\mathbf{F} := \frac{\hat{\mathbf{f}}}{\left\|\hat{\mathbf{T}}\right\|}, \tag{29}$$

such that Eqs. (25) and (28) can also be expressed as

$$\mathbf{G} = \begin{bmatrix} \mathbf{I}_n + \frac{(a-1)}{\|\mathbf{F}\|^2}\mathbf{F}\mathbf{F}^t & \frac{b\mathbf{F}}{\|\mathbf{F}\|} \\ \frac{b\mathbf{F}^t}{\|\mathbf{F}\|} & a \end{bmatrix}, \tag{30}$$

$$a = \cosh\left(t_f\|\mathbf{F}\|\right),\ b = \sinh\left(t_f\|\mathbf{F}\|\right). \tag{31}$$

From Eqs. (13), (24) and (30) it follows that

$$\mathbf{T}^f = \mathbf{T}^0 + \eta\mathbf{F}, \tag{32}$$

$$\left\|\mathbf{T}^f\right\| = a\left\|\mathbf{T}^0\right\| + b\frac{\mathbf{F}\cdot\mathbf{T}^0}{\|\mathbf{F}\|}, \tag{33}$$

where

$$\eta := \frac{(a-1)\mathbf{F}\cdot\mathbf{T}^0 + b\left\|\mathbf{T}^0\right\| \ \|\mathbf{F}\|}{\|\mathbf{F}\|^2}. \tag{34}$$

Eq. (32) is written as

$$\mathbf{F} = \frac{1}{\eta}(\mathbf{T}^f - \mathbf{T}^0) \ , \tag{35}$$

which being substituted into Eq. (33) and dividing both the sides by $\left\|\mathbf{T}^0\right\|$, we obtain

$$\frac{\left\|\mathbf{T}^f\right\|}{\|\mathbf{T}^0\|} = a + b\frac{(\mathbf{T}^f - \mathbf{T}^0)\cdot\mathbf{T}^0}{\|\mathbf{T}^f - \mathbf{T}^0\| \ \|\mathbf{T}^0\|} \ . \tag{36}$$

After inserting Eq. (35) for $\mathbf{F}$ into Eq. (31), a and b are now written as

$$a = \cosh\left(\frac{t_f\left\|\mathbf{T}^f - \mathbf{T}^0\right\|}{\eta}\right) \ , b = \sinh\left(\frac{t_f\left\|\mathbf{T}^f - \mathbf{T}^0\right\|}{\eta}\right) \ . \tag{37}$$

Let

$$\cos\theta := \frac{(\mathbf{T}^f - \mathbf{T}^0)\cdot\mathbf{T}^0}{\|\mathbf{T}^f - \mathbf{T}^0\| \ \|\mathbf{T}^0\|} \ , \tag{38}$$

$$S := t_f\left\|\mathbf{T}^f - \mathbf{T}^0\right\| \ , \tag{39}$$

and thus from Eqs. (36) and (37) it follows that

$$\frac{\left\|\mathbf{T}^f\right\|}{\|\mathbf{T}^0\|} = \cosh\left(\frac{S}{\eta}\right) + \cos\theta\sinh\left(\frac{S}{\eta}\right). \tag{40}$$

Upon defining

$$Z := \exp\left(\frac{S}{\eta}\right), \tag{41}$$

From Eq. (40) we can attain a quadratic equation for Z:

$$(1+\cos\theta)Z^2 - \frac{2\left\|\mathbf{T}^f\right\|}{\|\mathbf{T}^0\|}Z + 1 - \cos\theta = 0. \tag{42}$$

On the other hand, by inserting Eq. (35) for F into Eq. (34), we obtain

$$\left\|\mathbf{T}^f-\mathbf{T}^0\right\|^2=(a-1)(\mathbf{T}^f-\mathbf{T}^0)\cdot\mathbf{T}^0+b\left\|\mathbf{T}^0\right\|\ \left\|\mathbf{T}^f-\mathbf{T}^0\right\| . \tag{43}$$

Dividing both sides by $\left\|\mathbf{T}^0\right\|\ \left\|\mathbf{T}^f-\mathbf{T}^0\right\|$ and using Eqs. (37)-(39) and (41), we acquire another quadratic equation for Z:

$$(1+\cos\theta)Z^2-2\left(\cos\theta+\frac{\left\|\mathbf{T}^f-\mathbf{T}^0\right\|}{\|\mathbf{T}^0\|}\right)Z+\cos\theta-1=0 . \tag{44}$$

From Eqs. (42) and (44), the solution of Z is found to be

$$Z=\frac{(\cos\theta-1)\left\|\mathbf{T}^0\right\|}{\cos\theta\,\|\mathbf{T}^0\|+\|\mathbf{T}^f-\mathbf{T}^0\|-\|\mathbf{T}^f\|} . \tag{45}$$

From Eqs. (39) and (41) it follows that

$$\eta=\frac{t_f\left\|\mathbf{T}^f-\mathbf{T}^0\right\|}{\ln Z}. \tag{46}$$

Therefore, we come to an important result that between any two points $(\mathbf{T}^0,\ \left\|\mathbf{T}^0\right\|)$ and $(\mathbf{T}^f,\ \left\|\mathbf{T}^f\right\|)$ on the cone, there exists a Lie-group element $\mathbf{G}(t_f)\in SO_o(n,1)$ mapping $(\mathbf{T}^0,\ \left\|\mathbf{T}^0\right\|)$ onto $(\mathbf{T}^f,\ \left\|\mathbf{T}^f\right\|)$, which is given by

$$\begin{bmatrix}\mathbf{T}^f\\ \left\|\mathbf{T}^f\right\|\end{bmatrix}=\mathbf{G}(t_f)\begin{bmatrix}\mathbf{T}^0\\ \left\|\mathbf{T}^0\right\|\end{bmatrix} , \tag{47}$$

where $\mathbf{G}(t_f)$ is uniquely determined by $\mathbf{T}^0$ and $\mathbf{T}^f$ through the following equations:

$$\mathbf{G}(t_f)=\begin{bmatrix}\mathbf{I}_n+\frac{(a-1)}{\|\mathbf{F}\|^2}\mathbf{F}\mathbf{F}^t & \frac{b\mathbf{F}}{\|\mathbf{F}\|}\\ \frac{b\mathbf{F}^t}{\|\mathbf{F}\|} & a\end{bmatrix} , \tag{48}$$

$$a=\cosh\left(t_f\,\|\mathbf{F}\|\right) ,\ b=\sinh\left(t_f\,\|\mathbf{F}\|\right) , \tag{49}$$

$$\mathbf{T}=\frac{1}{\eta}(\mathbf{T}^f-\mathbf{T}^0)=\frac{\ln Z}{t_f}\frac{\mathbf{T}^f-\mathbf{T}^0}{\|\mathbf{T}^f-\mathbf{T}^0\|}. \tag{50}$$

In view of Eqs. (38) and (45), it can be seen that $\mathbf{G}(t_f)$ is fully determined by $\mathbf{T}^0$ and $\mathbf{T}^f$, and is independent on the vector field $\mathbf{f}$ in Eq. (11).

Notice that the above $\mathbf{G}(t_f)$ is different from the $\mathbf{G}(r)$ in Eq. (25). In order to stress its property as being a Lie-group mapping between the quantities spanned a whole time interval $[0, t_f]$, we write it to be $\mathbf{G}(t_f)$. Conversely, $\mathbf{G}(r)$ is a function of r. However, these two Lie-group elements $\mathbf{G}(r)$ and $\mathbf{G}(t_f)$ are both indispensable in our development of the Lie-group shooting method in the next section for the inverse problem of parameter identification.

The two Lie-group elements $\mathbf{G}(r)$ and $\mathbf{G}(t_f)$ are constructed by different manners. When the former is obtained by using the generalized mid-point rule, the latter is a universal mapping between $(\mathbf{T}^0,\ \|\mathbf{T}^0\|)$ and $(\mathbf{T}^f,\ \|\mathbf{T}^f\|)$ independent to the vector field $\mathbf{f}$, which means that such a mapping is applicable to all ODEs systems. It is interesting that by letting $\mathbf{G}(r) = \mathbf{G}(t_f)$, we can derive the required governing equation below. From this point of view, we may call our method the Lie-group shooting method (LGSM).

4 A Lie-group adaptive method for 1-D problem

4.1 A two-point Lie-group equation

Letting $\mathbf{G}(r) = \mathbf{G}(t_f)$ is essentially identical to letting the two $\mathbf{F}$'s in Eqs. (29) and (35) be equal, which leads to

$$\mathbf{T}^f = \mathbf{T}^0 + \frac{\eta}{\|\hat{\mathbf{T}}\|}\hat{\mathbf{f}}, \tag{51}$$

where

$$\|\hat{\mathbf{T}}\| = \|r\mathbf{T}^0 + (1-r)\mathbf{T}^f\| . \tag{52}$$

Up to here we have constructed a *Lie-group shooting equation* (51), which is a universal algebraic equation applicable to any vector field $\mathbf{f}$, and we may call it a natural field equation of global type. This equation involves four quantities of $\mathbf{T}^0$, $\mathbf{T}^f$, $\mathbf{f}$ and r, the last of which is a single parameter.

We can write $\hat{\mathbf{f}}$ explicitly,

$$\hat{\mathbf{f}} = \begin{bmatrix} \frac{\hat{T}_2 - 2\hat{T}_1 + \hat{T}_0}{(\Delta x)^2} - \alpha_1 \hat{T}_1 \\ \frac{\hat{T}_3 - 2\hat{T}_2 + \hat{T}_1}{(\Delta x)^2} - \alpha_2 \hat{T}_2 \\ \vdots \\ \frac{\hat{T}_n - 2\hat{T}_{n-1} + \hat{T}_{n-2}}{(\Delta x)^2} - \alpha_{n-1} \hat{T}_{n-1} \\ \frac{\hat{T}_{n+1} - 2\hat{T}_n + \hat{T}_{n-1}}{(\Delta x)^2} - \alpha_n \hat{T}_n \end{bmatrix}, \tag{53}$$

where $\hat{T}_i = rT_i^0 + (1-r)T_i^f = rf(x_i) + (1-r)T_i^f$, and $\hat{T}_0 = F_0(\hat{t})$ and $\hat{T}_{n+1} = F_\ell(\hat{t})$ with $\hat{t} = (1-r)t_f$.

From Eqs. (51) and (53), we can obtain a closed-form formula to calculate α_i:

$$\alpha_i = \frac{1}{\hat{T}_i}\left[\frac{\hat{T}_{i+1} - 2\hat{T}_i + \hat{T}_{i-1}}{(\Delta x)^2} - \frac{\|\hat{\mathbf{T}}\|}{\eta}(T_i^f - T_i^0)\right]. \tag{54}$$

Because we do not have a real target at t_f to be shot [that is, *we do not need the information from Eq. (4)*], we can take $r = 1$, and the above equation reduces to a simpler form:

$$\alpha_i = \frac{1}{f(x_i)}\left[\frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{(\Delta x)^2} - \frac{\|\mathbf{T}^0\|}{\eta}(T_i^f - T_i^0)\right], \tag{55}$$

where $T_i^0 = f(x_i)$ is the discretized initial condition. Eq. (55) can be used to find α_i, $i = 1,\ldots,n$.

4.2 An iterative procedure to estimate $\alpha(x)$

Now, the numerical procedures for estimating α_i are described as follows. We assume an initial value of α_i, for example, $\alpha_i = 1$. Substituting it into Eq. (9), we can apply the GPS to integrate it from $t = 0$ to $t = t_f$. Here t_f is a parameter chosen by the user. Then, we obtain T_i^f, and inserting it into Eq. (55), we can calculate a new α_i, which is then compared with the old α_i. If the difference of these two sets of α_i is smaller than a given criterion, then we stop the iteration and the final α_i is obtained. The numerical processes are summarized as follows:

(Step 1) Give an initial $\alpha_i = 1$.

(Step 2) For $j = 1, 2\ldots$, we repeat the following calculations. Calculate T_i^f by using the GPS to integrate Eq. (9) from $t = 0$ to $t = t_f$ [e.g., Liu (2001, 2005)]:

$$\mathbf{T}_{k+1} = \mathbf{T}_k + \eta_k \mathbf{f}_k, \tag{56}$$

where

$$a_k := \cosh\left(\frac{\Delta t\,\|\mathbf{f}_k\|}{\|\mathbf{T}_k\|}\right),\ b_k := \sinh\left(\frac{\Delta t\,\|\mathbf{f}_k\|}{\|\mathbf{T}_k\|}\right), \tag{57}$$

$$\eta_k := \frac{(a_k - 1)\mathbf{f}_k \cdot \mathbf{T}_k + b_k\|\mathbf{T}_k\|\ \|\mathbf{f}_k\|}{\|\mathbf{f}_k\|^2}, \tag{58}$$

and $\mathbf{f}$ is a vector form of the right-hand side of Eq. (9).

(Step 3) Insert the above calculated T_i^f denoted by $T_i^f(j)$ together with $T_i^0 = f(x_i)$ given by Eq. (10) into

$$\alpha_i^j = \frac{1}{f(x_i)}\left[\frac{f(x_{i+1})-2f(x_i)+f(x_{i-1})}{(\Delta x)^2} - \frac{\|\mathbf{T}^0\|}{\eta^j}\{T_i^f(j)-T_i^0\}\right], \tag{59}$$

where η^j is calculated from Eq. (46) by inserting $T_i^f(j)$ and $T_i^0 = f(x_i)$. If α_i^j converges according to a given convergence criterion:

$$C_j := \sqrt{\sum_{i=1}^{n}(\alpha_i^{j+1}-\alpha_i^j)^2} < \varepsilon, \tag{60}$$

then stop; otherwise, go to **(Step 2)**. Here, C_j is a measure of the convergence.

Basically, the LGSM for the present method is used in the time direction to derive Eq. (59) by supposing a fictitious target $T_i^f(j)$ at a time t_f. We can repeatedly use the time direction integrator GPS for Eq. (9) by inserting the calculated α_i to obtain the new final time data, which *are not obtained through the measurement*, and then we adjust α_i by Eq. (59). Because we have used an iteration process of a combination of the GPS and the LGSM to adjust α_i by the governing equations themselves, i.e., Eqs. (9) and (59), the present algorithm is quite simple, and is drastically different from other methods for the inverse problem of heat conduction equation, and in order to distinct it from the previous Lie-group shooting method with a really measured T_i^f as a target to adjust α_i, we may call the present method a *Lie-group adaptive method* (LGAM).

The philosophy of the solution methodology of the LGAM is that the *local in time equation* (9) and the *global in time equation* (59) must self-adapt to a situation that they are compatible.

5 A Lie-group adaptive method for 2-D problem

Eq. (5) is discretized by

$$\dot{T}_{i,j}(t) = \frac{T_{i+1,j}-2T_{i,j}+T_{i-1,j}}{(\Delta x)^2} + \frac{T_{i,j+1}-2T_{i,j}+T_{i,j-1}}{(\Delta y)^2} - \alpha_{i,j}T_{i,j},\ i,j=1,\ldots,n, \tag{61}$$

where $\Delta x = x_0/(n+1)$, $\Delta y = y_0/(n+1)$, $T_{i,j}(t) = T(x_i, y_j, t)$, $\alpha_{i,j} = \alpha(x_i, y_j)$ with $x_i = i\Delta x$ and $y_j = j\Delta y$.

From Section 4, it can be seen that the present LGAM is easily extended to the two-dimensional inverse problem. For saving space, we directly skip to the computational formulas. Similarly, the numerical processes are summarized as follows:

(Step 1) Give an initial $\alpha_{i,j} = 1$.

(Step 2) For $k = 1, 2\ldots$, we repeat the following calculations. Calculate $T^f_{i,j}$ by using the GPS to integrate Eq. (61) from $t = 0$ to $t = t_f$ by using the discretized boundary conditions and initial condition given in Eqs. (6) and (7).

(Step 3) Insert the above calculated $T^f_{i,j}$ denoted by $T^f_{i,j}(k)$ together with $T^0_{i,j} = f(x_i, y_j)$ discretized from Eq. (7) into

$$\alpha^k_{i,j} = \frac{1}{f_{i,j}}\left[\frac{f_{i+1,j}-2f_{i,j}+f_{i-1,j}}{(\Delta x)^2} + \frac{f_{i,j+1}-2f_{i,j}+f_{i,j-1}}{(\Delta y)^2} - \frac{\|\mathbf{T}^0\|}{\eta^k}\{T^f_{i,j}(k) - T^0_{i,j}\}\right],$$
$$i = 2,\ldots,n-1,\ j = 2,\ldots,n-1,$$

$$\alpha^k_{1,j} = \frac{1}{f_{1,j}}\left[\frac{f_{2,j}-2f_{1,j}+F_0(y_j,0)}{(\Delta x)^2} + \frac{f_{1,j+1}-2f_{1,j}+f_{1,j-1}}{(\Delta y)^2} - \frac{\|\mathbf{T}^0\|}{\eta^k}\{T^f_{1,j}(k) - T^0_{1,j}\}\right],$$
$$j = 2,\ldots,n-1,$$

$$\alpha^k_{1,1} =$$
$$\frac{1}{f_{1,1}}\left[\frac{f_{2,1}-2f_{1,1}+F_0(y_1,0)}{(\Delta x)^2} + \frac{f_{1,2}-2f_{1,1}+H_0(x_1,0)}{(\Delta y)^2} - \frac{\|\mathbf{T}^0\|}{\eta^k}\{T^f_{1,1}(k) - T^0_{1,1}\}\right],$$

$$\alpha^k_{1,n} =$$
$$\frac{1}{f_{1,n}}\left[\frac{f_{2,n}-2f_{1,n}+F_0(y_n,0)}{(\Delta x)^2} + \frac{H_{y0}(x_1,0)-2f_{1,n}+f_{1,n-1}}{(\Delta y)^2} - \frac{\|\mathbf{T}^0\|}{\eta^k}\{T^f_{1,n}(k) - T^0_{1,n}\}\right],$$

$$\alpha^k_{i,1} = \frac{1}{f_{i,1}}\left[\frac{f_{i+1,1}-2f_{i,1}+f_{i-1,1}}{(\Delta x)^2} + \frac{f_{i,2}-2f_{i,1}+H_0(x_i,0)}{(\Delta y)^2} - \frac{\|\mathbf{T}^0\|}{\eta^k}\{T^f_{i,1}(k) - T^0_{i,1}\}\right],$$
$$i = 2,\ldots,n-1,$$

$$\alpha^k_{n,1} =$$
$$\frac{1}{f_{n,1}}\left[\frac{F_{x0}(y_1,0)-2f_{n,1}+f_{n-1,1}}{(\Delta x)^2} + \frac{f_{n,2}-2f_{n,1}+H_0(x_n,0)}{(\Delta y)^2} - \frac{\|\mathbf{T}^0\|}{\eta^k}\{T^f_{n,1}(k) - T^0_{n,1}\}\right],$$

$$\alpha_{n,j}^k = \frac{1}{f_{n,j}} \left[\frac{F_{x0}(y_j,0)-2f_{n,j}+f_{n-1,j}}{(\Delta x)^2} + \frac{f_{n,j+1}-2f_{n,j}+f_{n,j-1}}{(\Delta y)^2} - \frac{\|\mathbf{T}^0\|}{\eta^k}\{T_{n,j}^f(k)-T_{n,j}^0\} \right], \quad j = 2,\dots,n-1,$$

$$\alpha_{n,n}^k =$$

$$\frac{1}{f_{n,n}} \left[\frac{F_{x0}(y_n,0)-2f_{n,n}+f_{n-1,n}}{(\Delta x)^2} + \frac{H_{y0}(x_n,0)-2f_{n,n}+f_{n,n-1}}{(\Delta y)^2} - \frac{\|\mathbf{T}^0\|}{\eta^k}\{T_{n,n}^f(k)-T_{n,n}^0\} \right],$$

$$\alpha_{i,n}^k = \frac{1}{f_{i,n}} \left[\frac{f_{i+1,n}-2f_{i,n}+f_{i-1,n}}{(\Delta x)^2} + \frac{H_{y0}(x_i,0)-2f_{i,n}+f_{i,n-1}}{(\Delta y)^2} - \frac{\|\mathbf{T}^0\|}{\eta^k}\{T_{i,n}^f(k)-T_{i,n}^0\} \right], \quad i = 2,\dots,n-1, \tag{62}$$

where $T_{i,j}^0 = f_{i,j} = f(x_i,y_j)$, and η^k is calculated from Eq. (46) by inserting $T_{i,j}^f(k)$ and $T_{i,j}^0$. If $\alpha_{i,j}^k$ converges according to a given convergence criterion:

$$C_k := \sqrt{\sum_{i=1}^n \sum_{j=1}^n (\alpha_{i,j}^{k+1} - \alpha_{i,j}^k)^2} < \varepsilon, \tag{63}$$

then stop; otherwise, go to **(Step 2)**.

6 Numerical examples

6.1 Example 1

Let us first employ the following example to demonstrate the process in Section 4. This example is given by

$$\alpha(x) = (x-3)^2,\ x \in (0,1). \tag{64}$$

Under the boundary conditions

$$T(0,t) = \exp(t+4.5), T(1,t) = \exp(t+2), \tag{65}$$

and the initial condition

$$T(x,0) = \exp\left(\frac{(x-3)^2}{2}\right), \tag{66}$$

and the exact solution of T is given by

$$T(x,t) = \exp\left(t + \frac{(x-3)^2}{2}\right). \tag{67}$$

We first apply the LGAM to this problem of the identification of $\alpha(x)$, where we used $\Delta x = 1/40$, and $t_f = 0.1$. The initial guess of $\alpha(x)$ is $\alpha_i = 1$. Under the stopping criterion with $\varepsilon = 10^{-3}$, the process is convergent within 16 iterations as shown in Fig. 1(a). Note that the convergence is very fast with an exponential decay. This behavior is rather promising to show that the LGAM is a powerful method. Moreover, the LGAM is insensitive to the initial guess of unknown parameter coefficients. In Fig. 1(b), we plot the tentative α_i for the first iteration, the third iteration, the fifth iteration and the seventh iteration, the last of which is already very close to the exact solution. The numerical solution of α_i is close to the exact one with the root-mean-square-error (RMSE) about 6.34×10^{-4}, and the maximum relative error about 7.91×10^{-4} as shown in Fig. 1(c).

6.2 Example 2

Let us then use the following example [Chen and Liu (2006)] to test the performance of LGAM:

$$\alpha(x) = 3 - 2\sin(2\pi x) + \frac{\cos^2(2\pi x)}{\pi^2},\ x \in (0,1). \tag{68}$$

Under the boundary conditions

$$T(0,t) = T(1,t) = \exp(-3t), \tag{69}$$

and the initial condition

$$T(x,0) = \exp\left(\frac{sin(2\pi x)}{2\pi^2}\right), \tag{70}$$

and the exact solution of T is given by

$$T(x,t) = \exp\left(\frac{sin(2\pi x)}{2\pi^2} - 3t\right). \tag{71}$$

We apply the LGAM to this problem of the identification of $\alpha(x)$, where we use $\Delta x = 1/50$, and $t_f = 0.1$. The initial guess of $\alpha(x)$ is $\alpha_i = 1$. Under the stopping criterion with $\varepsilon = 10^{-3}$, the process is convergent within 19 iterations. In Fig. 2(a), we compare the numerical solutions with the exact solutions. The numerical solution of α_i is close to the exact ones with the RMSE about 3.38×10^{-4}, and the maximum relative error about 2.64×10^{-3}as shown in Fig. 2(b).

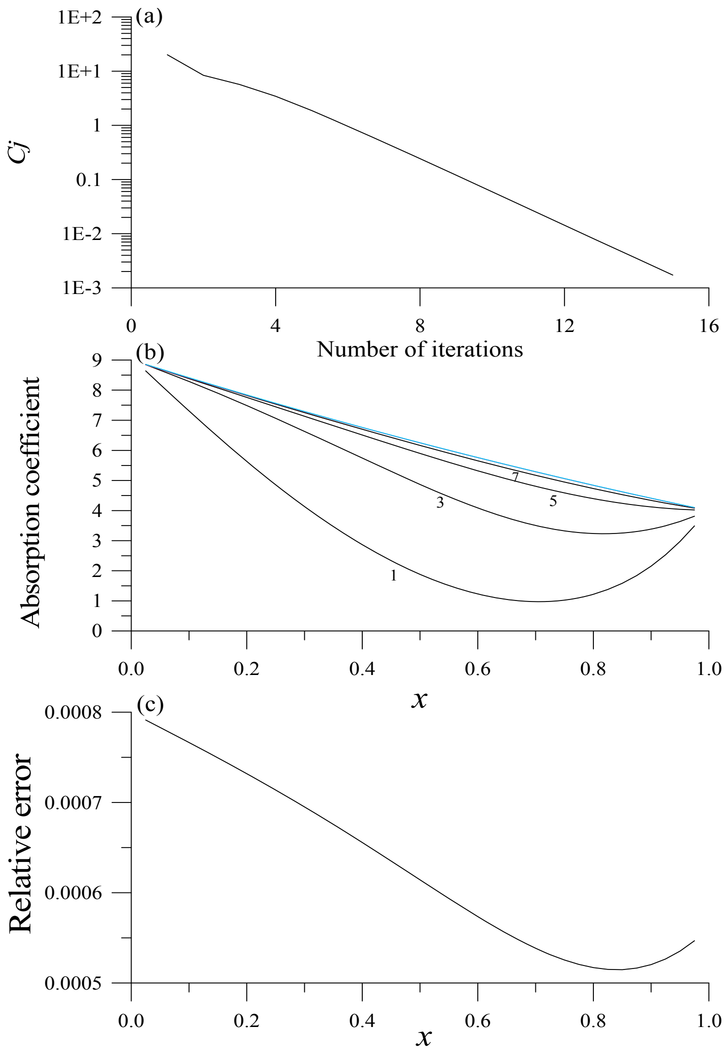


Figure 1: For Example 1: (a) the convergence speed, (b) comparing numerical and exact solutions, and (c) the relative error.

6.3 *Example 3*

Let us consider the following example [Yang, Yu and Deng (2008)]:

$$\alpha(x) = \frac{x^4 - 2x^3 + 13x^2 - 12x + 22}{(x - x^2)^2 + 20}, \; x \in (0,1). \tag{72}$$

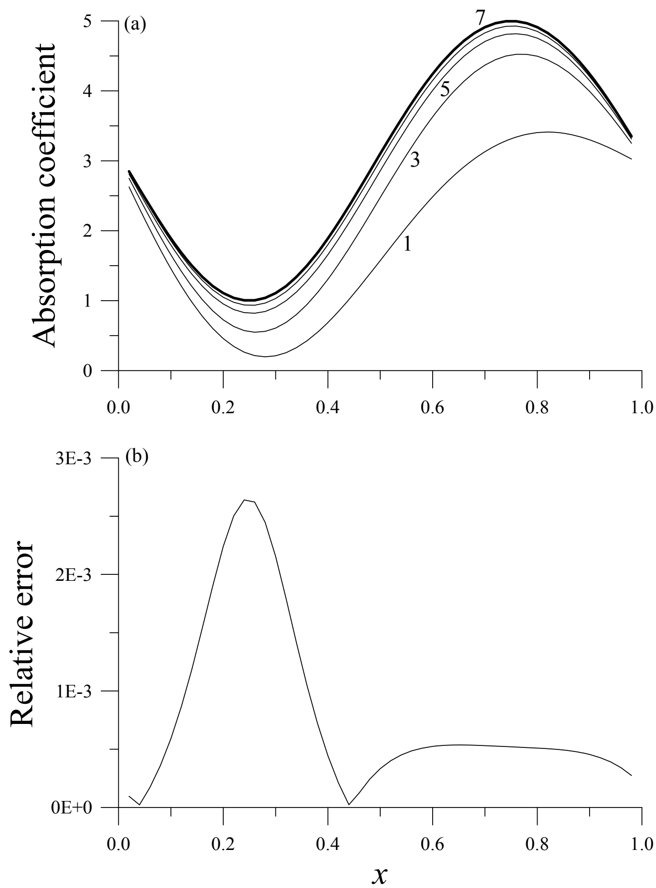


Figure 2: For Example 2 the first few iterative results of absorption coefficient are plotted in (a) by using the LGAM, and (b) displaying the relative error.

Under the boundary conditions

$$T(0,t) = T(1,t) = 20\exp(-t), \tag{73}$$

and the initial condition

$$T(x,0) = (x-x^2)^2 + 20, \tag{74}$$

and the exact solution of T is given by

$$T(x,t) = \exp(-t)[(x-x^2)^2 + 20]. \tag{75}$$

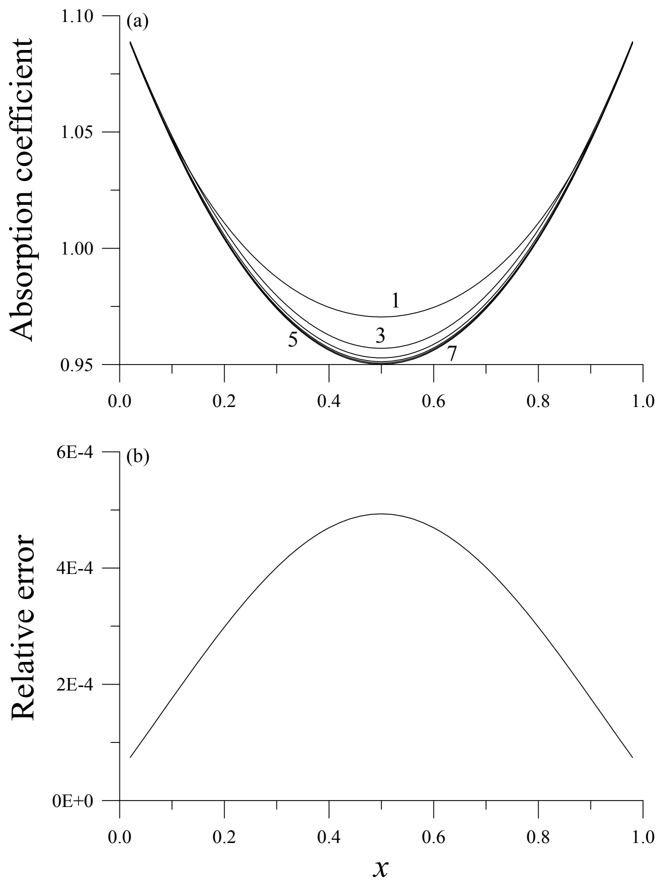


Figure 3: For Example 3 the first few iterative results of absorption coefficient are plotted in (a) by using the LGAM, and (b) displaying the relative error.

For the LGAM applied to this problem, we use $\Delta x = 1/50$ and $t_f = 0.1$. The initial guess of $\alpha(x)$ is $\alpha_i = 1$. Under the stopping criterion with $\varepsilon = 10^{-3}$, the process is convergent within 10 iterations. In Fig. 3(a), we compare the numerical solutions with the exact solutions. The numerical solution of α_i is close to the exact one with the RMES about 7.61×10^{-4}, and the maximum relative error about 4.93×10^{-4}as shown in Fig. 3(b). Besides, our results are better than that obtained by Yang, Yu and Deng (2008) as shown in Table 1, although we do not use the extra data measured at a final time. Furthermore, the numerical process as presented in Section 4.2 is much saving than those used by Yang, Yu and Deng (2008).

Table 1: For Example 3 comparing exact and the numerical solutions of present paper and that obtained by Yang, Yu and Deng (2008).

x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Exact $\alpha(x)$	1.0460	1.0040	0.9741	0.9561	0.9502	0.9561	0.9741	1.0040	1.0460
Present paper	1.0462	1.0043	0.9744	0.9566	0.9506	0.9566	0.9744	1.0043	1.0462
Yang, Yu and Deng (2008)	1.0462	1.0058	0.9756	0.9586	0.9542	0.9596	0.9761	1.0055	1.0457

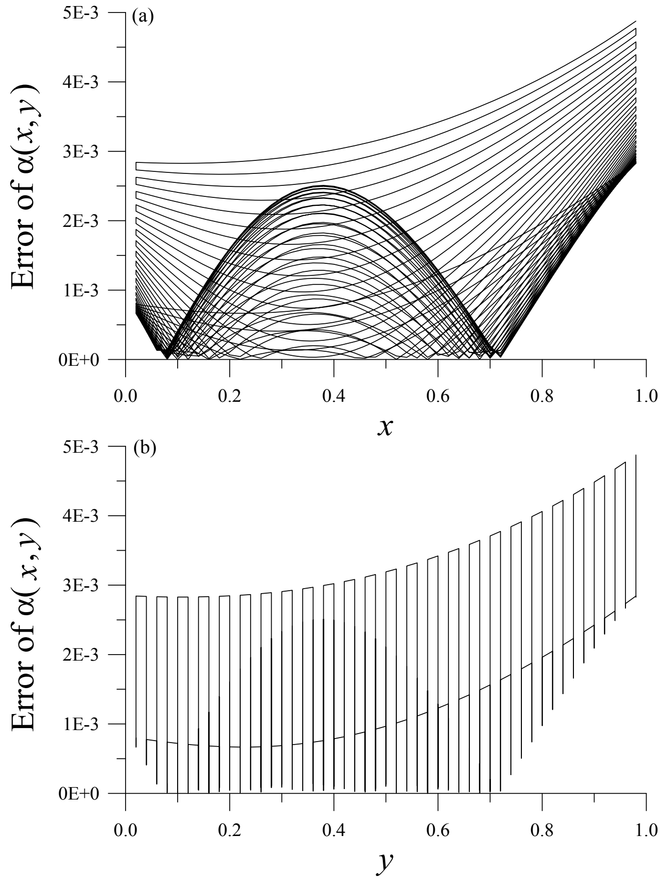


Figure 4: The numerical errors of LGAM solution for Example 4 plotted in (a) with respect to x by the errors projected along the y-axis, and in (b) with respect to y by the errors projected along the x-axis.

6.4 Example 4

Let us deliberate the following 2-D example:

$$\alpha(x,y) = 4(x^2 + y^2),\ (x,y) \in (0,1) \times (0,1). \tag{76}$$

The exact solution of T is

$$T(x,y,t) = \exp\left(x^2 + y^2 + 4t\right). \tag{77}$$

The required boundary conditions and initial condition are easily deduced from the above equation.

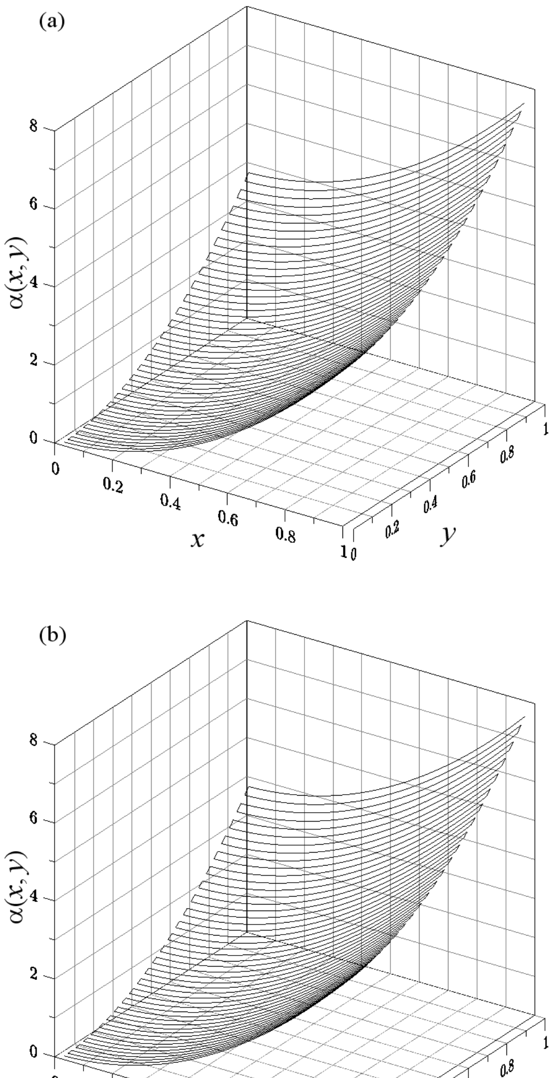
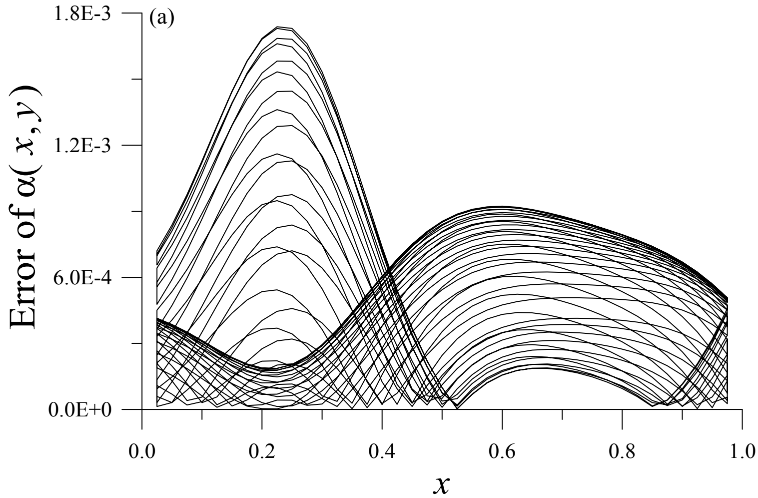


Figure 5: The exact and LGAM solutions for Example 4 of 2-D inverse problem are shown in (a) and (b).



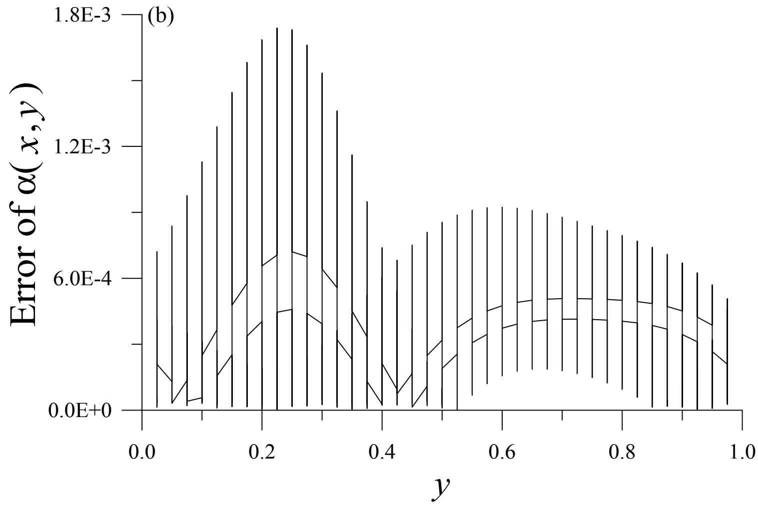


Figure 6: The numerical errors of LGAM solution for Example 5 plotted in (a) with respect to x by the errors projected along the y-axis, and in (b) with respect to y by the errors projected along the x-axis.

We apply the LGAM to this problem of the identification of $\alpha(x,y)$, where we use $\Delta x = \Delta y = 1/50$, and t_f = 0.01. The initial guess of $\alpha(x,y)$ is $\alpha_{i,j} = 1$. Under the stopping criterion with $\varepsilon = 10^{-2}$, the process is convergent within 61 iterations. The RMSE is about 1.87×10^{-3}. The numerical error is plotted with respect to x by projecting the errors along the y-axis as shown in Fig. 4(a), and the numerical error is plotted with respect to y by projecting the errors along the x-axis as shown

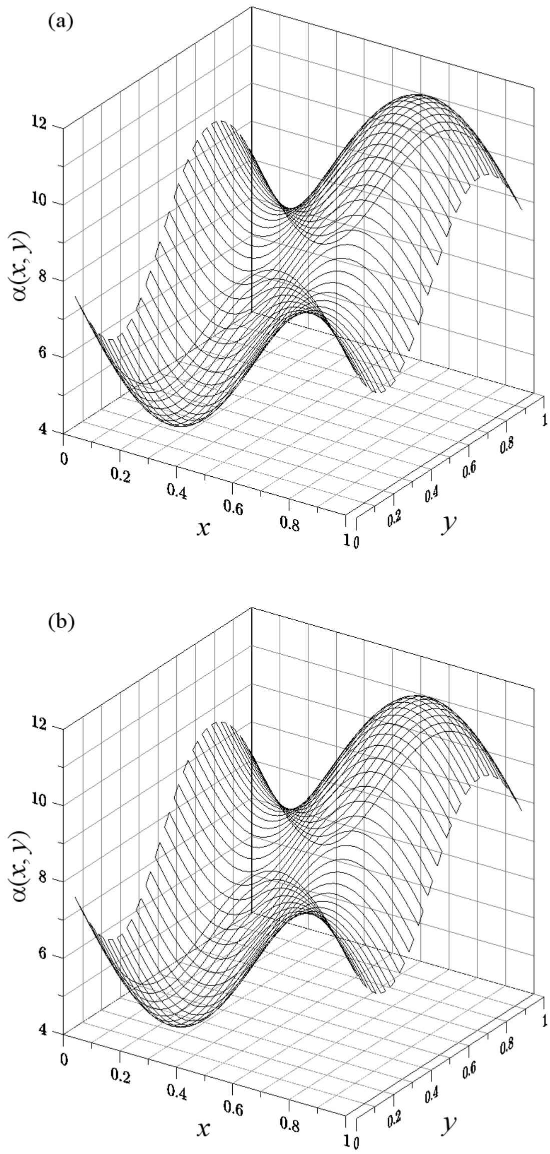


Figure 7: The exact and LGAM solutions for Example 5 of 2-D inverse problem are shown in (a) and (b).

in Fig. 4(b). The exact solution and numerical solution of α are plotted in Figs. 5(a) and 5(b) sequentially.

6.5 Example 5

Let us ponder another 2-D example as follows:

$$\alpha(x,y) = -2[\sin(2\pi x) + \sin(2\pi y)] + \frac{1}{\pi^2}[\cos^2(2\pi x) + \cos^2(2\pi y)] + 8, \quad (x,y) \in (0,1) \times (0,1). \quad (78)$$

The exact solution of T is

$$T(x,y,t) = \exp\left[\frac{\sin(2\pi x) + \sin(2\pi y)}{2\pi^2} - 8t\right]. \quad (79)$$

The required boundary conditions and initial condition are easily deduced from the above equation.

By applying the LGAM to this problem, we utilize $\Delta x = \Delta y = 1/40$ and $t_f = 0.01$. The initial guess of $\alpha(x,y)$ is $\alpha_{i,j} = 1$. Under the stopping criterion with $\varepsilon = 10^{-2}$, the process is convergent within 74 iterations. The RMSE is about 6.08×10^{-4}. The numerical error is plotted with respect to x by projecting the errors along the y-axis as shown in Fig. 6(a), and the numerical error is plotted with respect to y by projecting the errors along the x-axis as shown in Fig. 6(b). The exact solution and numerical solution of α are plotted in Figs. 7(a) and 7(b) sequentially.

7 Conclusions

A Lie-group adaptive method (LGAM) has been developed for the identification of the radiative coefficients in 1-D and 2-D parabolic inverse problems. The major advantages of the present method are that no a priori information about the functional form of radiative coefficients is necessary, and no extra measurement of data is required. In addition, the present method is easily extended to higher-dimensional and nonlinear problems, and also requires much less computational costs than other methods. The accuracy and efficiency of the present algorithm are confirmed by comparing the estimated results with exact solutions through five numerical examples. It is highly recommended this LGAM being used in the estimation of radiative coefficients, when there are no extra data available. The present methodology is quite simple and straightforward, which may provide an alternative option about the inverse problem of parameter identification of parabolic type PDEs. The *Lie-group shooting equation* supplemented an *inherent equation* to solve the unknown parameter, with a self-adapted manner. The philosophy of the LGAM is that the *local in time differential equation* (9) and the *global in time algebraic equation* (59) must self-adapt to a situation that they are compatible. This condition is also applicable to the 2-D inverse problem, where the *local in time differential equation* (61) and the *global in time algebraic equation* (62) must be compatible and in harmony by a self-adaptation.

Acknowledgement: The corresponding author would like to express his thanks to the National Science Council, ROC, for their financial supports under Grant Numbers, NSC 100-2221-E492-018. Besides, Taiwan's National Science Council project NSC 99-2221-E-002-074-MY3 granted to the first author is highly appreciated.

References

Chang, C.-W. and Liu, C.-S. (2009): A Fictitious Time Integration Method for Backward Advection-Dispersion Equation. *CMES: Computer Modeling in Engineering & Science*, Vol. 51, No. 3, pp. 261-276.

Chen, Q.; Liu, J. (2006): Solving an inverse parabolic problem by optimization from final measurement data. *J. Comp. Appl. Math.*, vol. 193, pp. 183–203.

Choulli, M.; Yamamoto, M. (1996): Generic well-posedness of an inverse parabolic problem-the Hölder space approach. *Inverse Problems*, vol. 12, pp. 195–205.

Choulli, M.; Yamamoto, M. (1997): An inverse parabolic problem with non-zero initial condition. *Inverse Problems*, vol. 13, pp. 19–27.

Choulli, M.; Yamamoto, M. (2008): Uniqueness and stability in determining the heat radiative coefficient, the initial temperature and a boundary coefficient in a parabolic equation. *Nonlinear Anal.*, vol. 69, pp. 3983–3998.

Deng, Z. C.; Yang, L.; Yu, J. N. (2009): Identifying the radiative coefficient of heat conduction equations from discrete measurement data. *Appl. Math. Letters*, vol. 22, pp. 495–500.

Deng, Z. C.; Yang, L.; Yu, J. N.; Luo, G. W. (2010): Identifying the radiative coefficient of an evolution type heat conduction equation by optimization method. *J. Math. Anal. Appl.*, vol. 362, pp. 210–223.

Huang, C. H.; Huang, C. Y. (2004): An inverse biotechnology problem in estimating the optical diffusion and absorption coefficients of tissue. *Int. J. Heat Mass Transfer*, vol. 47, pp. 447–457.

Klibanov, M. V.; Lucas, T. R. (1999): Numerical solutions of a parabolic inverse problem in optical tomography using experimental data. *SIAM J. Appl. Math.*, vol.

59, pp. 1763–1789.

Liu, C.-S. (2010): A Lie-group adaptive method for imaging a space-dependent rigidity coefficient in an inverse scattering problem of wave propagation. *CMC: Computers, Materials & Continua*, vol. 18, pp. 1–21.

Liu, C.-S. (2011): A self-adaptive LGSM to recover initial condition or heat source of one-dimensional heat conduction equation by using only minimal boundary thermal data. *Int. J. Heat Mass Transfer*, vol. 54, pp. 1305–1312.

Liu, C.-S.; Atluri, S. N. (2010): An iterative and adaptive Lie-group method for solving the Calderón inverse problem. *CMES: Computer Modeling in Engineering & Science*, vol. 64, pp. 299-326.

Liu, C.-S. (2006a): The Lie-group shooting method for nonlinear two-point boundary value problems exhibiting multiple solutions. *CMES: Computer Modeling in Engineering & Science*, vol. 13, pp. 149–163.

Liu, C.-S. (2006b): Efficient shooting methods for the second order ordinary differential equations. *CMES: Computer Modeling in Engineering & Science*, vol. 15, pp. 69–86.

Liu, C.-S. (2006c): The Lie-group shooting method for singularly perturbed two-point boundary value problems. *CMES: Computer Modeling in Engineering & Science*, vol. 15, pp. 179–196.

Liu, C.-S. (2001): Cone of non-linear dynamical system and group preserving schemes. *Int. J. of Non-Linear Mech.*, vol. 36, pp. 1047–1068.

Liu, C.-S. (2006d): One-step GPS for the estimation of temperature-dependent thermal conductivity. *Int. J. Heat Mass Transfer*, vol. 49, pp. 3084–3093.

Liu, C.-S. (2006e): An efficient simultaneous estimation of temperature-dependent thermophysical properties. *CMES: Computer Modeling in Engineering & Science*, vol. 14, pp. 77–90.

Liu, C.-S. (2007): Identification of temperature-dependent thermophysical properties in a partial differential equation subject to extra final measurement data. *Numer. Meth. Partial Diff. Eq.*, vol. 23, pp. 1083–1109.

Liu, C.-S.; Liu, L. W.; Hong, H. K. (2007): Highly accurate computation of spatial-dependent heat conductivity and heat capacity in inverse thermal problem. *CMES: Computer Modeling in Engineering & Science*, vol. 17, pp. 1–18.

Liu, C.-S. (2008): An LGSM to identify nonhomogeneous heat conductivity functions by an extra measurement of temperature. *Int. J. Heat Mass Transfer*, vol. 51, pp. 2603–2613.

Chang, J. R.; Liu, C.-S.; Chang, C.-W. (2007): A new shooting method for quasi-boundary regularization of backward heat conduction problems. *Int. J. Heat Mass*

Transfer, vol. 50, pp. 2325–2332.

Liu, C.-S. (2005): Nonstandard group-preserving schemes for very stiff ordinary differential equations. *CMES: Computer Modeling in Engineering & Science*, vol. 9, pp. 255-272.

Liu, C.-S. (2009): A two-stage LGSM to identify time-dependent heat source through an internal measurement of temperature. *Int. J. Heat Mass Transfer*, vol. 52, pp. 1635–1642.

Liu, C.-S. (2010): A highly accurate LGSM for severely ill-posed BHCP under a large noise on the final time data. *Int. J. Heat Mass Transfer*, vol. 53, pp. 4132–4140.

Liu, C.-S. (2010): The Lie-Group Shooting Method for Computing the Generalized Sturm-Liouville Problems. *CMES: Computer Modeling in Engineering & Science*, Vol. 56, No. 1, pp. 85-112.

Liu, C.-S. (2010): A two-stage Lie-group shooting method (TSLGSM) to identify time-dependent thermal diffusivity. *Int. J. Heat Mass Transfer*, vol. 53, pp. 4876–4884.

Liu, C.-S.; Atluri, S.N. (2010): An Iterative and Adaptive Lie-Group Method for Solving the Calderón Inverse Problem. *CMES: Computer Modeling in Engineering & Science*, Vol. 64, No. 3, pp. 299-326.

Rundell, W. (1987): The determination of a parabolic equation from initial and final data. *Proc. Amer. Math. Soc.*, vol. 99, pp. 637–642.

Tadi, M.; Klibanov, M. V.; Cai, W. (2002): An inversion method for parabolic equations based on quasireversibility. *Comp. Math. Appl.*, vol. 43, pp. 927–941.

Tadi, M. (2007): Iterative least-square method for 1-D inversion problems in optical tomography. *Int. J. Comp. Appl. Math.*, vol. 2, pp. 253–265.

Yamamoto, M.; Zou, J. (2001): Simultaneous reconstruction of the initial temperature and heat radiative coefficient. *Inverse Problems*, vol. 17, pp. 1181–1202.

Yang, L.; Yu, J. N.; Deng, Z. C. (2008): An inverse problem of identifying the coefficient of parabolic equation. *Appl. Math. Model.*, vol. 32, pp. 1984–1995.