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A Differential Quadrature Method for Multi-Dimensional Inverse Heat Conduction Problem of Heat Source

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Abstract: In this paper, we employ the differential quadrature method (DQM) to tackle the inverse heat conduction problem (IHCP) of heat source. These advantages of this numerical approach are that no a priori presumption is made on the functional form of the estimates, and that evaluated heat source can be obtained directly in the calculation process. Seven examples show the effectiveness and accuracy of our algorism in providing excellent estimates of unknown heat source from the given data. We find that the proposed scheme is applicable to the IHCP of heat source. Even though the noise is added to the exact temperature, the DQM is still robust against disturbance.

Keywords: Differential quadrature method (DQM), Heat source, Inverse heat conduction problem, Heat conduction equation, Ill-posed problem

1 Introduction

The partial differentiation equation usually describes a physical phenomenon in the engineering and science. For example, the Navier-Stokes equation, the heat conduction equation, the vibration equation, the wave equation and so forth. Inverse heat conduction problems (IHCPs) are important in engineering and science because they played a vital role in the various industrial applications such as in a heat exchanger, casting processes, semiconductor heating measurements, melting processes and so on. It is well known that the inverse problems are usually unstable and hard to solve. Therefore, some researches on inverse problems can consult from Beck (1970) proposed a new finite difference to tackle the nonlinear IHCP,

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and Hesel and Hills (1989) employed an adjoin formulation to solve the multidimensional steady-state IHCPs, and Alifanov, Artyukhin and Rumyantsev (1995) used the iterative regularization method to resolve the inverse heat transfer problems.

The differential quadrature method (DQM) was proposed by [Bellman and Casti (1971) ; Bellman, Kashef and Casti (1972)], and it is applicable to solve those nonlinear partial differential equations. Over the past few decades, many researchers used the DQM to apply many science and engineering disciplines, such as solid mechanics [Wang (1995) ; Karami and Malekzadeh (2002)], fluid mechanics [Shu and Richards (1992) ; Shu, Chew and Richards (1995)], vibration mechanics [Choi, Wu and Chou (2000) ; Malekzadeha and Vosoughic (2009)]. Many approaches have been proposed for tackling the estimation of thermal problems, for example, Yeung and Lam (1996) utilized the second-order finite difference method to resolve the inverse determination of thermal conductivity. After that, Telejko and Malinowski (2004) employed the finite element method to the thermal conductively identification, but they did not add the noise in the problem. Thereafter, Farcas and Lesnic (2006) used the BEM for determination of heat source and obtained reasonable results; however, their numerical results were sensitive to the noise. Later, Chang and Chang (2006) addressed the finite volume method to calculate the inverse determination of thermal conductivity; nevertheless, their results were not good and did not consider the noisy effect. After that, Char, Chang and Tai (2007) proposed a hybrid numerical method to predict the unknown apace and time dependent of heat source, and obtained the acceptable results. Then, Char, Chang, Tai (2008) utilized the DQM to resolve the inverse determination of thermal conductivity in one-dimensional slab and attained good results.

In the past several years, many scholars investigated a lot of numerical methods to IHCPs with the heat source. Recently, Hematiyan and Karami (2008) have employed the meshless BEM to resolve the heat source domain integrals without the domain discretization. Later, Liu (2008) proposed a modified genetic algorithm to solve the unknown heat source, but his approach needed to spend much time on selecting the best individual genes. Thereafter, Yang and Fu (2010) used the simplified the Tikhonov regularization method to evaluate the one-dimensional heat source; however, their numerical results were not goog and did not consider the noisy effect. Recently, Lin (2011) have adopted the sequential algorithm to solve the one-dimensional multiple heat sources; nevertheless, the results are inaccurate. After that, Yang, Dehghan, Yu and Luo (2011) proposed a numerical method on the basis of the Landweber iteration was designed to deal with the operator equation and some typical numerical experiments; however, their results were not good and merely deliberate the one-dimensional problems.

For the high-dimensional problem, Coles and Murio (2001) addressed a numerical marching scheme to solve the simultaneous recovery of the diffusively coefficient, spatial source term, temperature, and heat flux distributions in the two-dimensional IHCP and obtained acceptable results with the noise. However, the numerical method was complex, and the exact heat source term was wrong.

The paper is summarized as follows. In section 2, we presented the multi-dimensional nonlinear and nonhomogeneous heat conduction problems (HCPs). Then, we explain the DQM theory in Section 3, and use the DQM to discretize the governing equation. Section 4 shows seven examples to estimate the unknown heat source item. Finally, we draw some important conclusions in Section 5.

2 Formulation of the heat conduction problems

First, we consider the one-dimensional nonlinear and nonhomogeneous heat conduction problem (HCP) is respectively given by the following equations:

$$\frac{\partial u}{\partial t} = \nabla u + f + F(u) \text{ in } \Omega,\ (x,t) \in \Omega := [0,\ell] \times [0,T], \tag{1}$$

$$u(0,t) = \phi(t), \quad u(\ell,t) = \varphi(t), \quad t \in [0,T], \tag{2}$$

$$u(x,0) = \Phi(x), \quad x \in [0,\ell], \tag{3}$$

where u is the temperature of slab, ℓ is the length of slab, t is the time, f is the heat source, and $F(u)$ is a nonlinear function of u.

Besides, we further ponder the difficult two-dimensional and three-dimensional nonhomogenous HCP are respectively given by the following equations:

$$\frac{\partial u}{\partial t} = \nabla(\alpha \nabla u) + fg, \tag{4}$$

$$u = u_B \text{ on } \Gamma_{\mathrm{B}}, \tag{5}$$

$$u = u_i \text{ on } \Gamma_{\mathrm{i}}, \tag{6}$$

where u is a scalar temperature field of heat distribution, g is a function of t, and f is the heat source. We take a bounded domain D in R^j, j = 2, 3 and a spacetime domain $\Omega = D \times (0,t)$ in R^{j+1} for a time $t > 0$, and write two surfaces $\Gamma_B = \partial D \times [0,t]$ and $\Gamma_i = \partial D \times \{t\}$ of the boundary $\partial\Omega.\Delta$ represents the j-dimensional Laplacian operator, j = 2, 3. While Eqs. (4)-(6) constitute a j-dimensional HCP for the given boundary data u_B: $\Gamma_B \mapsto R$ and the initial data u_i: $\Gamma_{\mathrm{i}} \mapsto R$.

3 Differential quadrature method

Pondering a one-dimensional function $f(x)$ on the area $a \leq x \leq b$. To approximate the derivate of a smooth function at a discrete point x_i in the domain, the DQM employs the weighted linear sum of all function values at all discrete points in the x direction. Then, the$m$th-order derivatives $f(x)$ with respect to x_i at point i can be formulated as

$$\frac{d^m f(x_i)}{dx^m} = \sum_{i=1}^{N} C_{i,j}^m f(x_j), \quad i = 1, \ldots, N, \tag{7}$$

where $f(x_j)$ are the function values at the jth sampling point x_j, N is the number of discrete points, and $C_{i,j}^m$are the unknown weighting coefficients of the mth order derivative at discrete point x_i, in which $m \leq N-1$.

Shu and Richards (1995) provided a convenient and recurrent formula for determining the following these derivative weighting coefficients:

$$C_{i,j}^1 = \frac{M(x_i)}{(x_i - x_j) \cdot M(x_j)}, \text{ for } i \neq j, \text{ and } i, j = 1, \ldots., N, \tag{8}$$

$$C_{i,j}^m = m \cdot \left[C_{i,j}^{m-1} \cdot C_{i,j}^1 - \frac{C_{i,j}^{m-1}}{(x_i - x_j)} \right], \text{ for } 2 \leq m \leq N-1, i \neq j, \text{ and } i, j = 1, \ldots., N, \tag{9}$$

$$C_{i,j}^m = - \sum_{\substack{j=1 \\ i \neq j}}^{N} C_{i,j}^m, \text{ for } 1 \leq m \leq N-1 \text{ and } i = 1, \ldots.N, \tag{10}$$

where

$$M(x_i) = \prod_{j=1, i \neq j}^{N} (x_i - x_j). \tag{11}$$

Note that in accordance with the principle of the DQM, the locations of the sampling grid point x_i can be arbitrarily determined.

4 Numerical examples

We employ the DQM to the determination of IHCP with heat source through numerical instances. Then, we are also interested in the stability of our algorism when the input measured data are polluted by the random noise for different problems. We can evaluate the stability by increasing the different levels of random disturbance in the exact temperature:

$$\hat{\mathbf{u}} = \mathbf{u} \cdot [1 + \omega R(i)], \tag{12}$$

where $\mathbf{u}$ is the exact temperature. We utilize the function rand_number given in Matlab to generate the noisy data $R(i)$, which are random numbers in [-1, 1], and ω means the level of absolute noise. Then, the noisy data $\hat{\mathbf{u}}$ are used in the calculations.

4.1 Example 1

We consider the one-dimensional nonhomogenous HCP is as follows:

$$u_t = u_{xx} + f, \; 0 < x < \ell, 0 < t < 1, \tag{13}$$

with the boundary conditions

$$u(0,t) = u(\ell,t) = 0, \tag{14}$$

and the initial condition

$$u(x,0) = \sin(\pi x). \tag{15}$$

The exact temperature and the heat source are given by

$$u(x,t) = (2 - e^{-\pi^2 t}) \sin(\pi x), \tag{16}$$

$$f(x) = 2\pi^2 \sin(\pi x). \tag{17}$$

A straightforward derivation in accordance with the concept of DQM results in

$$f(x_i) = \frac{(u_{i,j} - u_{i,j-1})}{\Delta t} - \sum_{k=1}^{N} C_{j,k}^{[2]} u(x_k, t_j). \tag{18}$$

Under the following parameters: ℓ = 1, N = 21, Δx = 0.05, t = 1, and Δt = 0.05. Fig. 1 shows the numerical results and numerical errors with noises of ω = 0, 0.01, 0.03, 0.05, and the maximum error is about 0.4. The present results are also better than that calculated by Adrian and Lesnic (2005), of which the maximum error is about 1.5 (see Fig. 5 of the above cited paper), under a noise of ω= 0.05. To the authors' best knowledge, there has been no open report that the numerical methods can calculate this inverse problem well as the DQM.

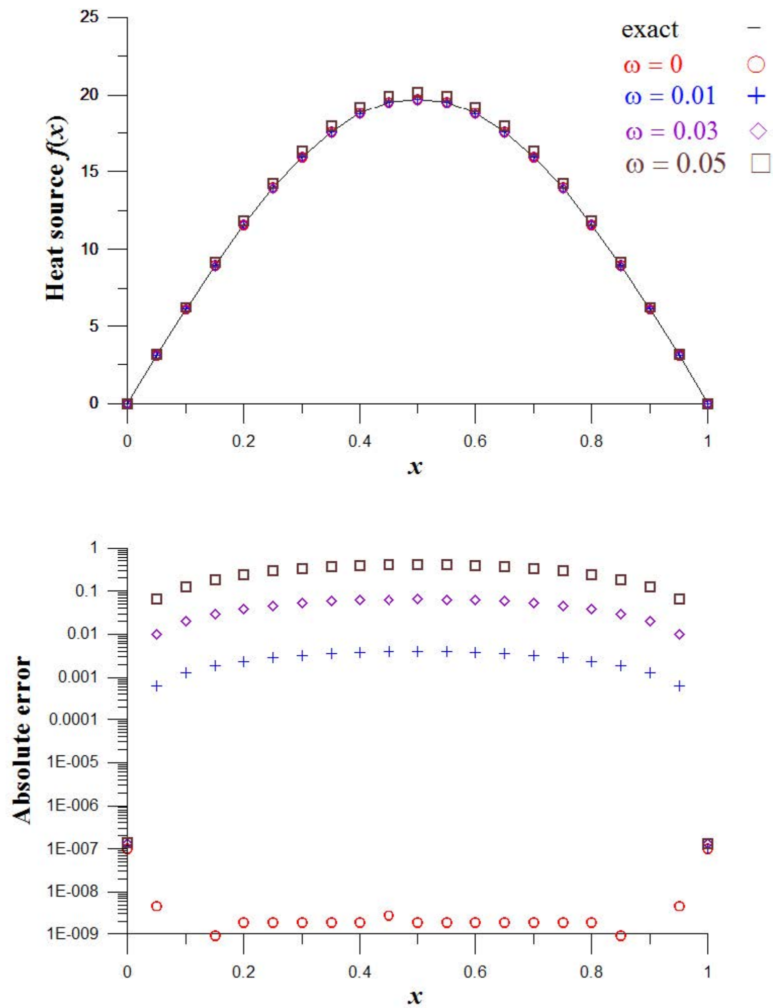


Figure 1: Comparisons of the exact solutions and numerical solutions for Example 1 with different levels of noise ω = 0, 0.01, 0.03, 0.05, and the corresponding numerical errors.

4.2 Example 2

Let us ponder another one-dimensional nonhomogeneous HCP:

$$u_t = u_{xx} + f,\ 0 < x < \ell, 0 < t < 1, \tag{19}$$

with the boundary conditions

$$u(0,t) = 2t + \sin(4\pi t),\ u(1,t) = 1 + 2t + \sin(4\pi t), \tag{20}$$

and the initial condition

$$u(x,0) = x^2. \tag{21}$$

The exact temperature and the heat source are given by

$$u(x,t) = x^2 + 2t + \sin(4\pi t), \tag{22}$$

$$f(x) = 4\pi\cos(4\pi t). \tag{23}$$

Under the following parameters: ℓ = 1, N = 11, Δx = 0.1, t = 1, and Δt = 1/30. Fig. 2 displays the numerical results and numerical errors with noises of ω = 0, 0.01, 0.03, 0.05, and the maximum error is about 0.36. The present results are also better than that calculated by Adrian and Lesnic (2005), of which the maximum error is about 1.5 (see Fig. 4 of the above cited paper), under a noise of ω= 0.05. To the authors' best knowledge, there has been no open literature that the numerical methods can calculate this inverse problem well as the DQM.

4.3 Example 3

The following another one-dimensional nonhomogeneous HCP is deliberated:

$$u_t = u_{xx} + f,\ 0 < x < \ell, 0 < t < 1, \tag{24}$$

with the boundary conditions

$$u(0,t) = 0,\ u(1,t) = [\sin(t) + t^2], \tag{25}$$

and the initial condition

$$u(x,0) = 6\sin(2\pi x). \tag{26}$$

The exact temperature and the heat source are given by

$$u(x,t) = 6\sin(2\pi x)e^{-(2\pi)^2 t} + [\sin(t) + t^2]x, \tag{27}$$

$$f(x) = x[\cos(t) + 2t]. \tag{28}$$

Under the following parameters: ℓ = 1, N = 31, Δx = 1/30, t = 1, and Δt = 0.025. Fig. 3 demonstrates the numerical results and numerical errors with noises of ω = 0, 0.01, 0.03, 0.05, and the maximum error is about 0.15 when ω is equal to 0.05. The accuracy as can be seen from Fig. 3(a) is rather good.

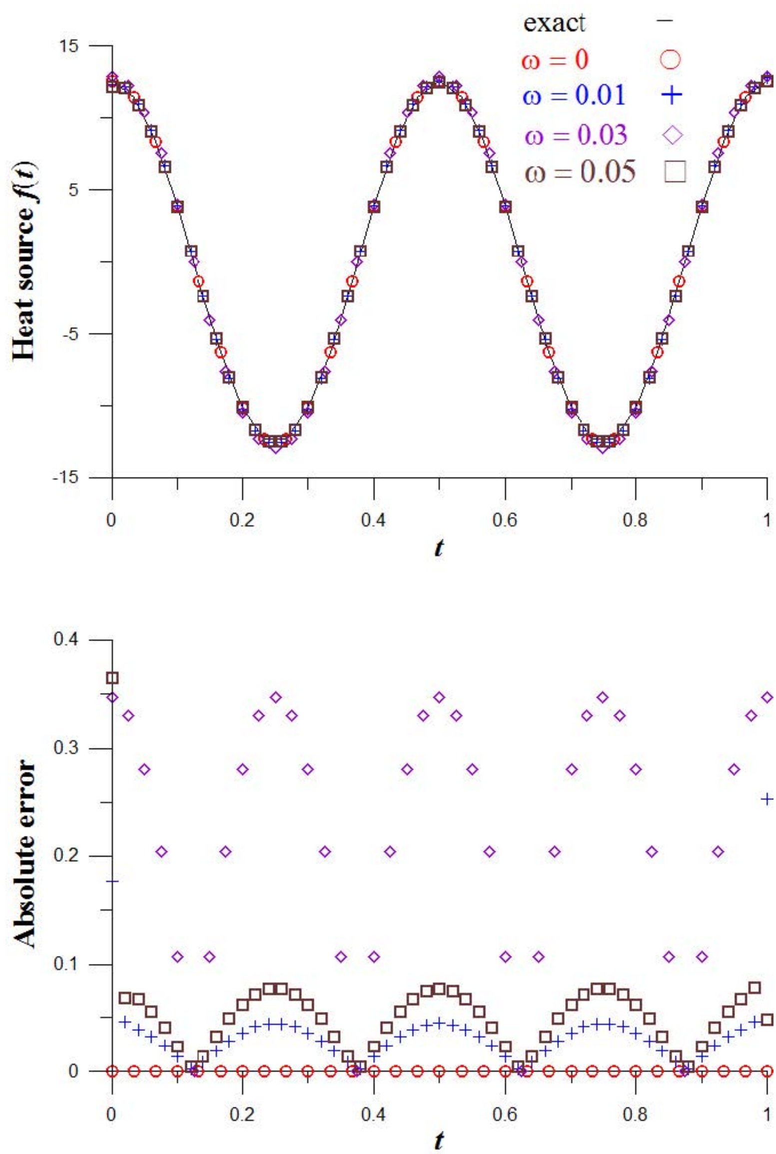


Figure 2: Comparisons of the exact solutions and numerical solutions for Example 2 with different levels of noise ω = 0, 0.01, 0.03, 0.05, and the corresponding numerical errors.

4.4 Example 4

We contemplate the one-dimensional nonlinear HCP is as follows:

$$u_t = u_{xx} + f + 2e^t \sin(x) - e^{4t} \sin^4(x), \quad 0 < x < \ell,\ 0 < t < 1, \tag{29}$$

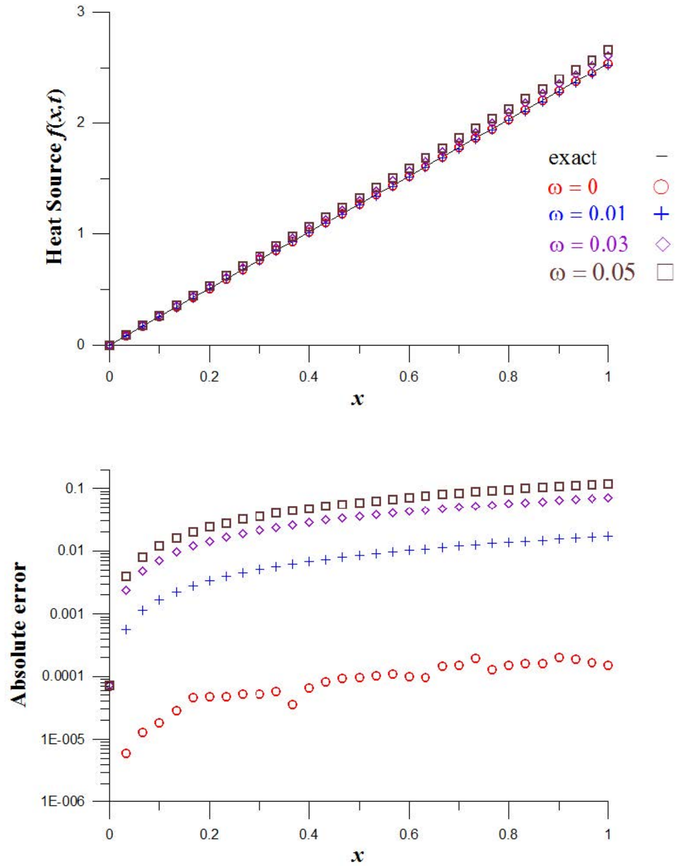


Figure 3: Comparisons of the exact solutions and numerical solutions for Example 3 with different levels of noise ω = 0, 0.01, 0.03, 0.05, and the corresponding numerical errors.

with the boundary conditions

$$u(0,t) = 0,\ u(1,t) = [\sin(t) + t^2)], \tag{30}$$

and the initial condition

$$u(x,0) = 6\sin(2\pi x). \tag{31}$$

The exact temperature and the heat source are given by

$$u(x,t) = e^t \sin(x), \tag{32}$$

$$f(x,t) = [e^t \sin(x)]^4. \quad (33)$$

Under the following parameters: $\ell = 1$, $N = 31$, $\Delta x = 1/30$, $t = 1$, and $\Delta t = 1/30$. Fig. 4 represents the numerical results and numerical errors with noises of $\omega = 0$, 0.01, 0.03, 0.05, and the maximum error is about 1.8 when ω is equal to 0.05. The accuracy as can be seen from Fig. 4(a) is rather good. It is remarkable that the present scheme is not sensitive to the noise.

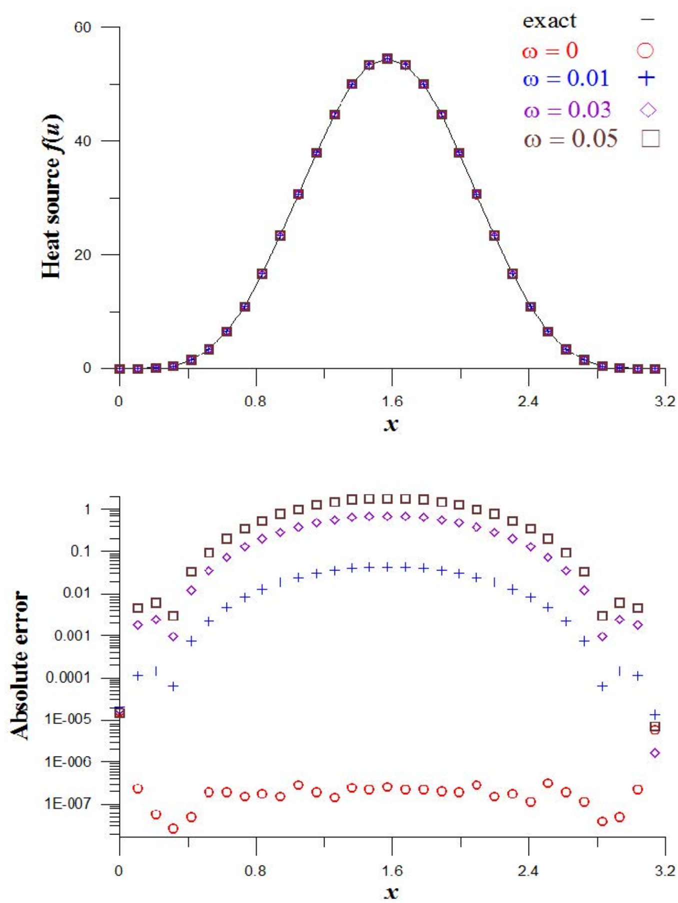


Figure 4: Comparisons of the exact solutions and numerical solutions for Example 4 with different levels of noise ω = 0, 0.01, 0.03, 0.05, and the corresponding numerical errors.

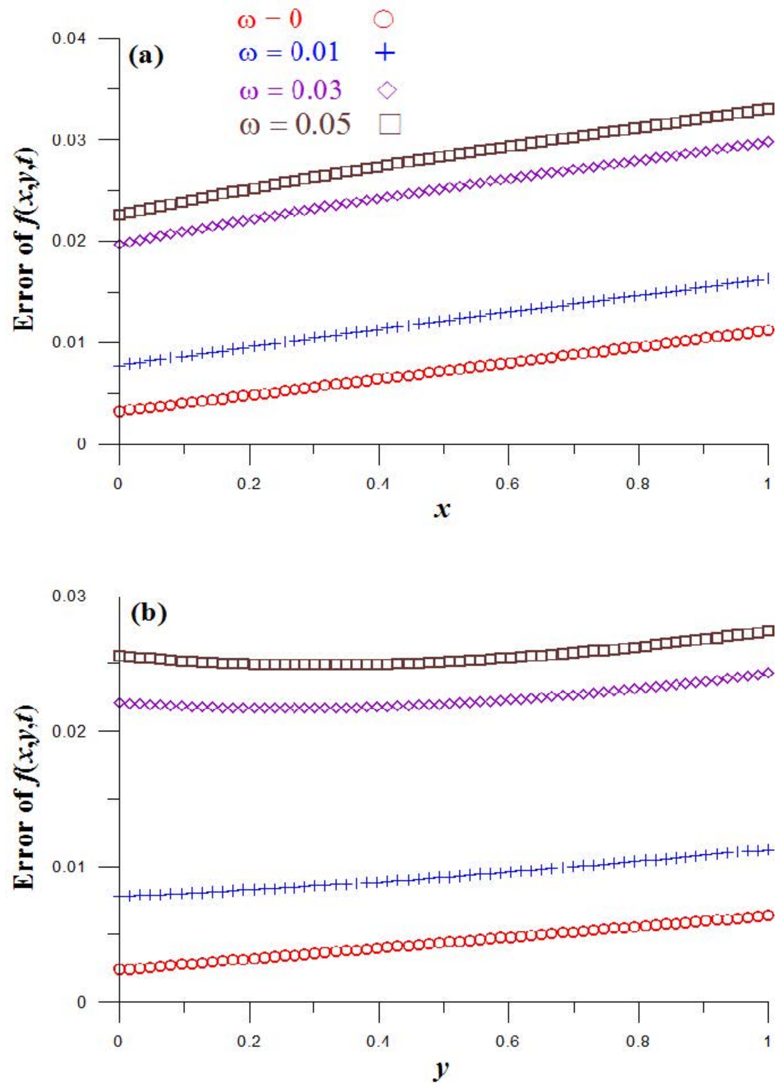


Figure 5: The numerical errors of DQM solutions with and without random noise effect for Example 5 are plotted in (a) with respect to x at fixed $y = 0.3016$, and in (b) with respect to y at fixed $x = 0.8095$.

4.5 Example 5

The following two-dimensional nonlinear HCP is pondered:

$$u_t = \nabla(\alpha(x,y)\nabla u) + f(x,y)g(t), \quad 0 < x < a,\ 0 < y < b,\ 0 < t < 1, \tag{34}$$

with the boundary conditions

$$u(0,y,t) = \frac{1}{2}ye^{-t}, \quad u(1,y,t) = (1+\frac{1}{2}y)e^{-t},$$

$$u(x,0,t) = xe^{-t}, \quad u(x,1,t) = (x+\frac{1}{2})e^{-t}, \tag{35}$$

and the initial condition

$$u(x,y,0) = (x+\frac{1}{2}y). \tag{36}$$

The exact temperature and the heat source are given by

$$u(x,y,t) = (x+\frac{1}{2}y)e^{-t}, \tag{37}$$

$$f(x,y) = [\frac{3}{2}\cos(x) - \frac{3}{2}\sin(x)]e^{-y} + x + \frac{y}{2}, \tag{38}$$

where

$$g(t) = -e^{-t}, \quad \alpha(x,y) = 1.5 + e^{-y}\sin(x). \tag{39}$$

A derivation according to the concept of DQM leads to

$$\begin{aligned} &f(x_i,y_j) = \\ &\left\{ \frac{(u_{i,j,g}-u_{i,j,g-1})}{\Delta t} - \left[\sum_{n=1}^{N} C_{i,n}^{[1]}\alpha(x_i,y_j)[\sum_{n=1}^{N} C_{i,n}^{[1]}u(x_n,y_j,t_g) + \sum_{m=1}^{M} D_{j,m}^{[1]}u(x_i,y_m,t_g)] \right.\right. \\ &+ \alpha(x_i,y_j)[\sum_{n=1}^{N} C_{i,n}^{[2]}u(x_n,y_j,t_g) + \sum_{n=1}^{N} C_{i,n}^{[1]} \sum_{m=1}^{M} D_{j,m}^{[1]}u(x_n,y_m,t_g)] \\ &+ \sum_{m=1}^{M} D_{j,m}^{[1]}\alpha(x_i,y_m)[\sum_{n=1}^{N} C_{i,n}^{[2]}u(x_n,y_j,t_g) + \sum_{m=1}^{M} D_{j,m}^{[1]}u(x_i,y_m,t_g)] \\ &\left.\left. + \alpha(x_i,y_j)[\sum_{m=1}^{M} D_{j,m}^{[1]} \sum_{n=1}^{N} C_{i,n}^{[1]}u(x_n,y_m,t_g) + \sum_{m=1}^{M} D_{j,m}^{[2]}u(x_i,y_m,t_g)] \right] \right\} / g(t_g). \end{aligned} \tag{40}$$

Under the following parameters: $a = b = 1$, $N = M = 64$, $\Delta x = \Delta y = 1/63$, $t = 1$, and $\Delta t = 1/63$. Fig. 5 shows the numerical results and numerical errors with noises of $\omega = 0, 0.01, 0.03, 0.05$. Besides, at the point $y = 0.3016$, the error is plotted with respect to x in Fig. 5(a), and at the point $x = 0.8095$, the error is plotted with respect to y in Fig. 5(b). The latter one is smaller than the former one because the point $x = 0.8095$ is near the boundary. Furthermore, the errors are smaller than that calculated by Coles and Murio (2001) as shown in Table 1 therein. For this difficult problem, the DQM proposed here is still good with a maximum error 0.027.

The exact solutions and numerical solutions are plotted in Figs. 6(a)-(c) sequentially. Even under the moderate noise, the numerical solution exhibited in Fig. 6(c) is a good approximation to the exact heat source as displayed in Fig. 6(a).

4.6 Example 6

Let us further consider the two-dimensional nonlinear HCP:

$$u_t = \nabla(\alpha(x,y)\nabla u) + f(x,y)g(t), \quad 0 < x < a,\ 0 < y < b,\ 0 < t < 1, \tag{41}$$

with the boundary conditions

$$u(0,y,t) = 0, \quad u(1,y,t) = e^{y-t}, \quad u(x,0,t) = xe^{-t}, \quad u(x,1,t) = xe^{1-t}, \tag{42}$$

and the initial condition

$$u(x,y,0) = xe^{y}. \tag{43}$$

The exact temperature and the heat source are given by

$$u(x,y,t) = xe^{y-t}, \tag{44}$$

$$f(x,y) = \frac{1}{4}e^{y}(19 - 4y - 4y^2 + 14x - 8xy + 8xy^2 - 3x^2 + 4x^2y + 4x^2y^2), \tag{45}$$

where

$$g(t) = -e^{-t}, \quad a(x,y) = -(x-1)(y-\frac{1}{2})^2 + 2. \tag{46}$$

Under the following parameters: $a = b = 1$, $N = M = 64$, $\Delta x = \Delta y = 1/63$, $t = 1$, and $\Delta t = 1/63$. Fig. 7 displays the numerical results and numerical errors with noises of $\omega = 0, 0.01, 0.03, 0.05$. In addition, at the point $y = 0.3016$, the error is plotted with respect to x in Fig. 7(a), and at the point $x = 0.8095$, the error is plotted with respect to y in Fig. 7(b). The latter one is smaller than the former one because the point $x = 0.8095$ is near the boundary. Furthermore, the errors are smaller than that calculated by Coles and Murio (2001) as shown in Table 1 therein. For this difficult problem, the DQM proposed here is still good with a maximum error 0.43.

The exact solutions and numerical solutions are drawn in Figs. 8(a)-(c) sequentially. Even under the moderate noise, the numerical solution indicated in Fig. 8(c) is a good approximation to the exact heat source as illustrated in Fig. 8(a).

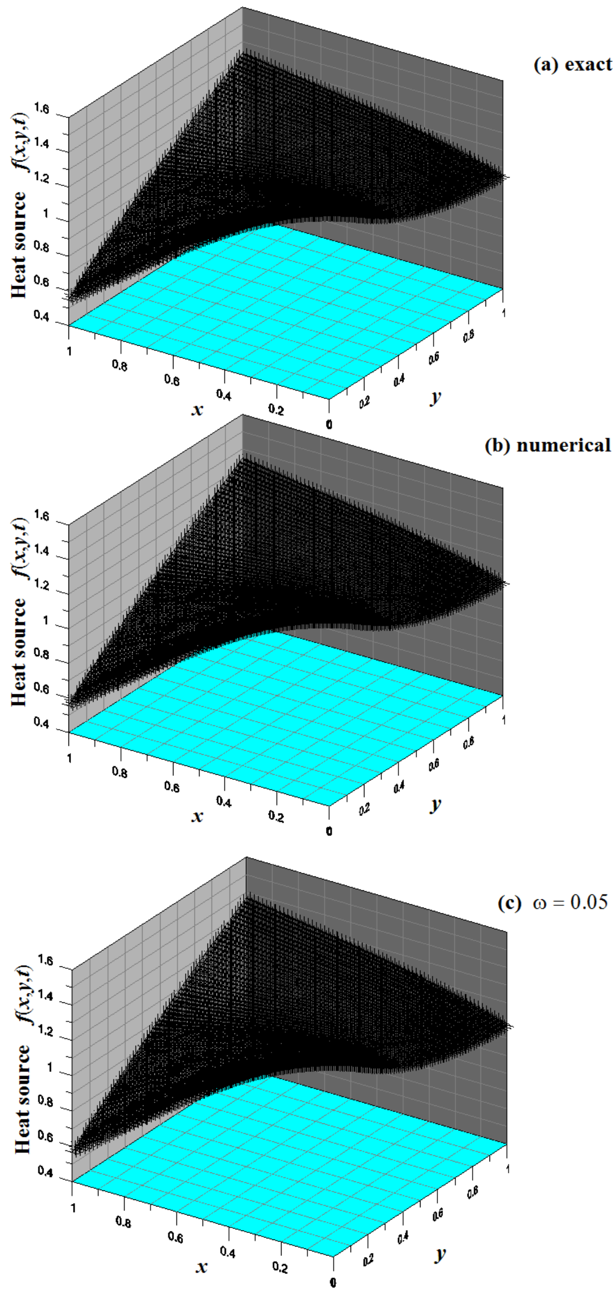
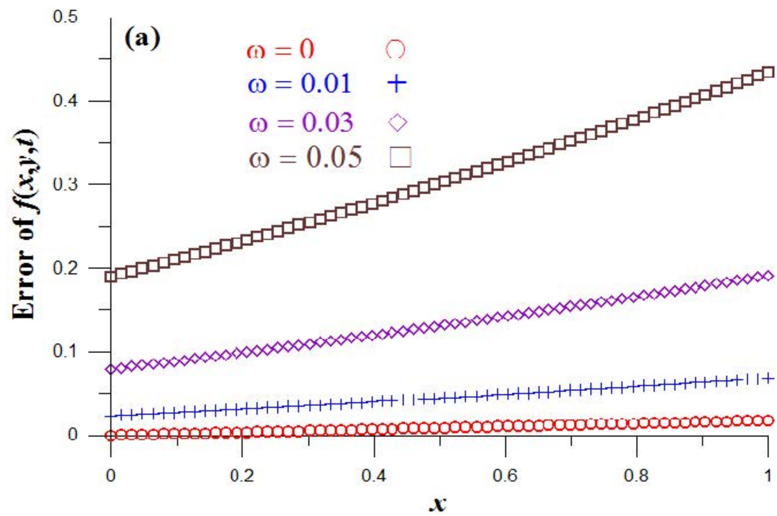


Figure 6: The exact solution for Example 5 of two-dimensional inverse problem is shown in (a), in (b) the DQM solution without random noise effect, and in (c) the DQM solution with random noise.



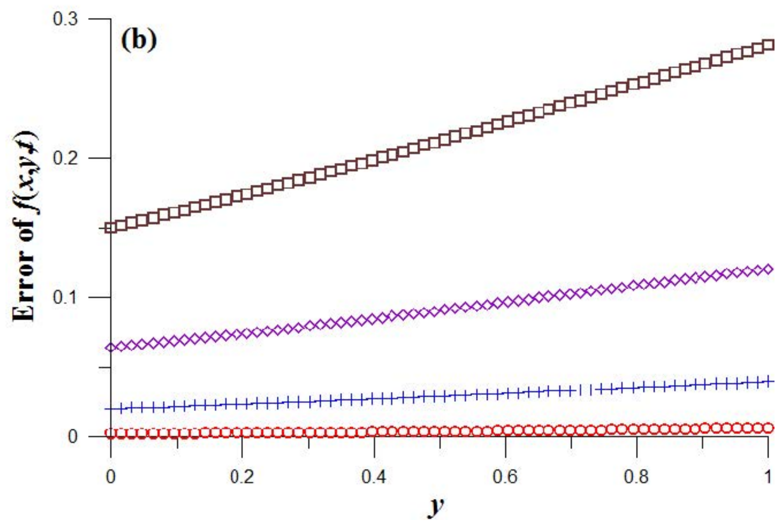


Figure 7: The numerical errors of DQM solutions with and without random noise effect for Example 6 are plotted in (a) with respect to x at fixed $y = 0.3016$, and in (b) with respect to y at fixed $x = 0.8095$.

4.7 Example 7

We deliberate a three-dimensional HCP:

$$u_t = \nabla(\alpha(x,y,z)\nabla u) + f(x,y,z)g(t), \quad 0 < x < a,\ 0 < y < b,\ 0 < z < c,\ 0 < t < 1,$$

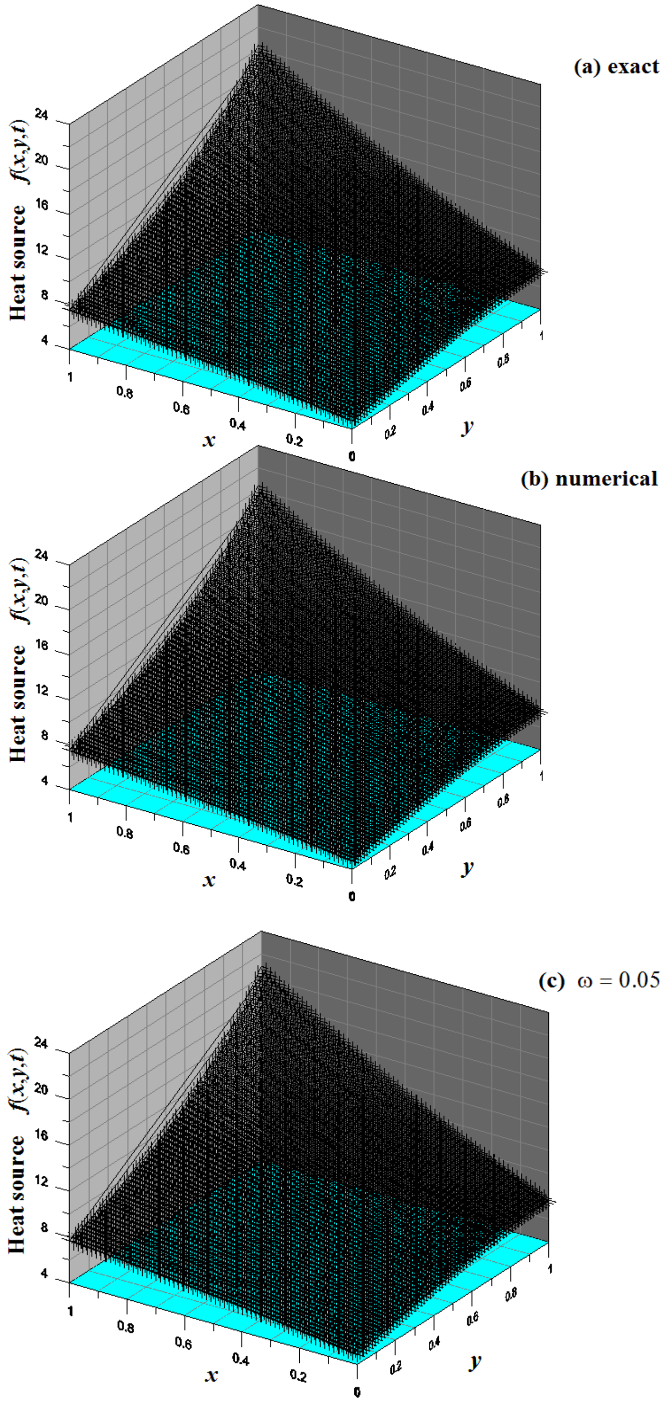


Figure 8: The exact solution for Example 6 of two-dimensional inverse problem is shown in (a), in (b) the DQM solution without random noise effect, and in (c) the DQM solution with random noise.

(47)

with the boundary conditions

$$u(0,y,z,t) = (\frac{1}{2}y + \frac{1}{2}z)e^{-t}, \quad u(1,y,z,t) = (1 + \frac{1}{2}y + \frac{1}{2}z)e^{-t},$$

$$u(x,0,z,t) = (x + \frac{1}{2}z)e^{-t}, \quad u(x,1,z,t) = (x + \frac{1}{2} + \frac{1}{2}z)e^{-t},$$

$$u(x,y,0,t) = (x + \frac{1}{2}y)e^{-t}, \quad u(x,y,1,t) = (x + \frac{1}{2}y + \frac{1}{2})e^{-t}, \tag{48}$$

and the initial condition

$$u(x,y,z,0) = [x + \frac{1}{2}y + \frac{1}{2}z]. \tag{49}$$

The exact temperature and the heat source are given by

$$u(x,y,z,t) = (x + \frac{1}{2}y + \frac{1}{2}z)e^{-t}, \tag{50}$$

$$f(x,y,z) = 2e^{-y}[\cos(x) - \sin(x)] + x + \frac{1}{2}y + \frac{1}{2}z + \frac{1}{2}, \tag{51}$$

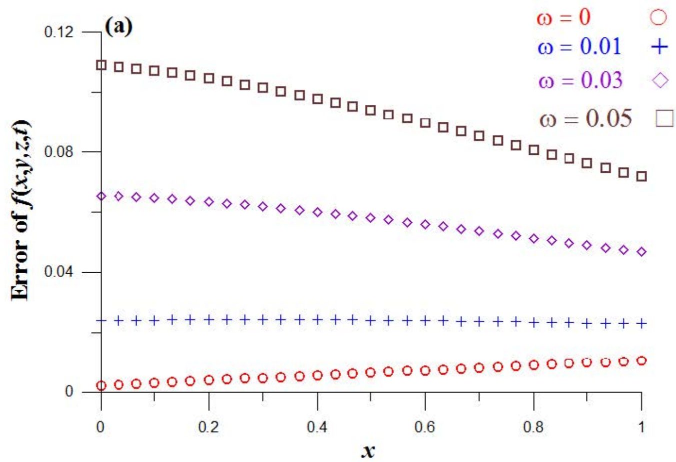
where

$$g(t) = -e^{-t}, \quad a(x,y,z) = 1.5 + e^{-y}\sin(x) + \frac{1}{4}z. \tag{52}$$

A derivation in accordance with the concept of DQM results in

$$
\begin{aligned}
f(x_i,y_j,z_k) = \Bigg\{ & \frac{(u_{i,j,k,g}-u_{i,j,k,g-1})}{\Delta t} - \left[\sum_{n=1}^{N} C_{i,n}^{[1]}\alpha(x_n,y_j,z_k)\right. \\
& \left[\sum_{n=1}^{N} C_{i,n}^{[1]}u(x_n,y_j,z_k,t_g)+\sum_{m=1}^{M} D_{j,m}^{[1]}u(x_i,y_m,z_k,t_g)+\sum_{h=1}^{H} E_{k,h}^{[1]}u(x,y_j,z_k,t_g)\right] \\
& +\alpha(x_i,y_j,z_k)\left[\sum_{n=1}^{N} C_{i,n}^{[2]}u(x_n,y_j,z_k,t_g)+\sum_{n=1}^{N} C_{i,n}^{[1]}\sum_{m=1}^{M} D_{j,m}^{[1]}u(x_n,y_m,z_k,t_g)\right. \\
& \left.+\sum_{n=1}^{N} C_{i,n}^{[1]}\sum_{h=1}^{H} E_{k,h}^{[1]}u(x_n,y_j,z_h,t_g)\right] \\
& +\sum_{m=1}^{M} D_{j,m}^{[1]}\alpha(x_i,y_m,z_k)\left[\sum_{n=1}^{N} C_{i,n}^{[2]}u(x_n,y_j,z_k,t_g)\right. \\
& \left.+\sum_{m=1}^{M} D_{j,m}^{[1]}u(x_i,y_m,z_k,t_g)+\sum_{h=1}^{H} E_{k,h}^{[1]}u(x,y_j,z_h,t_g)\right] \\
& +\alpha(x_i,y_j,z_k)\left[\sum_{m=1}^{M} D_{j,m}^{[1]}\sum_{n=1}^{N} C_{i,n}^{[1]}u(x_n,y_m,z_k,t_g)+\sum_{m=1}^{M} D_{j,m}^{[2]}u(x_i,y_m,z_k,t_g)\right. \\
& \left.+\sum_{m=1}^{M} D_{j,m}^{[1]}\sum_{h=1}^{H} E_{k,h}^{[1]}u(x_i,y_m,z_h,t_g)\right] \\
& \sum_{h=1}^{H} E_{k,h}^{[1]}\alpha(x_i,y_j,z_h)\left[\sum_{n=1}^{N} C_{i,n}^{[1]}u(x_n,y_j,z_k,t_g)+\sum_{m=1}^{M} D_{j,m}^{[1]}u(x_i,y_m,z_k,t_g)\right. \\
& \left.+\sum_{h=1}^{H} E_{k,h}^{[1]}u(x,y_j,z_h,t_g)\right] \\
& +\alpha(x_i,y_j,z_k)\left[\sum_{h=1}^{H} E_{k,h}^{[1]}\sum_{n=1}^{N} C_{i,n}^{[1]}u(x_n,y_j,z_h,t_g)+\sum_{h=1}^{H} E_{k,h}^{[1]}\sum_{m=1}^{M} D_{j,m}^{[1]}u(x_i,y_m,z_h,t_g)\right. \\
& \left.\left.\left.+\sum_{h=1}^{H} E_{k,h}^{[2]}u(x_i,y_j,z_h,t_g)\right]\right]\right]\Bigg\} / g(t_g).
\end{aligned} \tag{53}
$$

Under the following parameters: $a=b=c=1$, $N=M=H=31$, $\Delta x=\Delta y=\Delta z=1/30$, $t=1$, and $\Delta t=1/60$. Fig. 9 exhibits the numerical results and numerical errors with noises of $\omega=0, 0.01, 0.03, 0.05$. In addition, at fixed points $y=8/30$ and $z=8/30$,



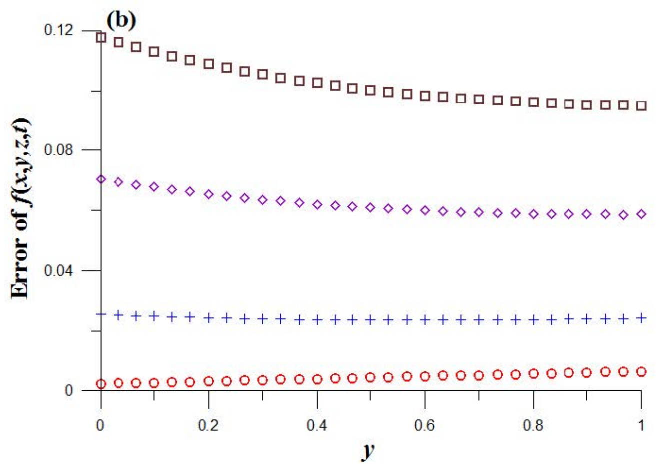


Figure 9: The numerical errors of DQM solutions with and without random noise effect for Example 7 are plotted in (a) with respect to x at fixed y = 8/30 and z = 8/30, (b) with respect to y at fixed x = 4/30 and z = 8/30, and (a) with respect to z at fixed x = 4/30 and y = 8/30.

the error is plotted with respect to x in Fig. 9(a), and at fixed points x = 4/30 and z = 8/30, the error is plotted with respect to y in Fig. 9(b), and at fixed points x = 4/30 and y = 8/30, the error is plotted with respect to z in Fig. 9(c). For this difficult problem, the DQM proposed here is still good with a maximum error 0.13. To the authors' best knowledge, there has been no open report that the numerical methods can calculate this inverse problem well as the DQM.

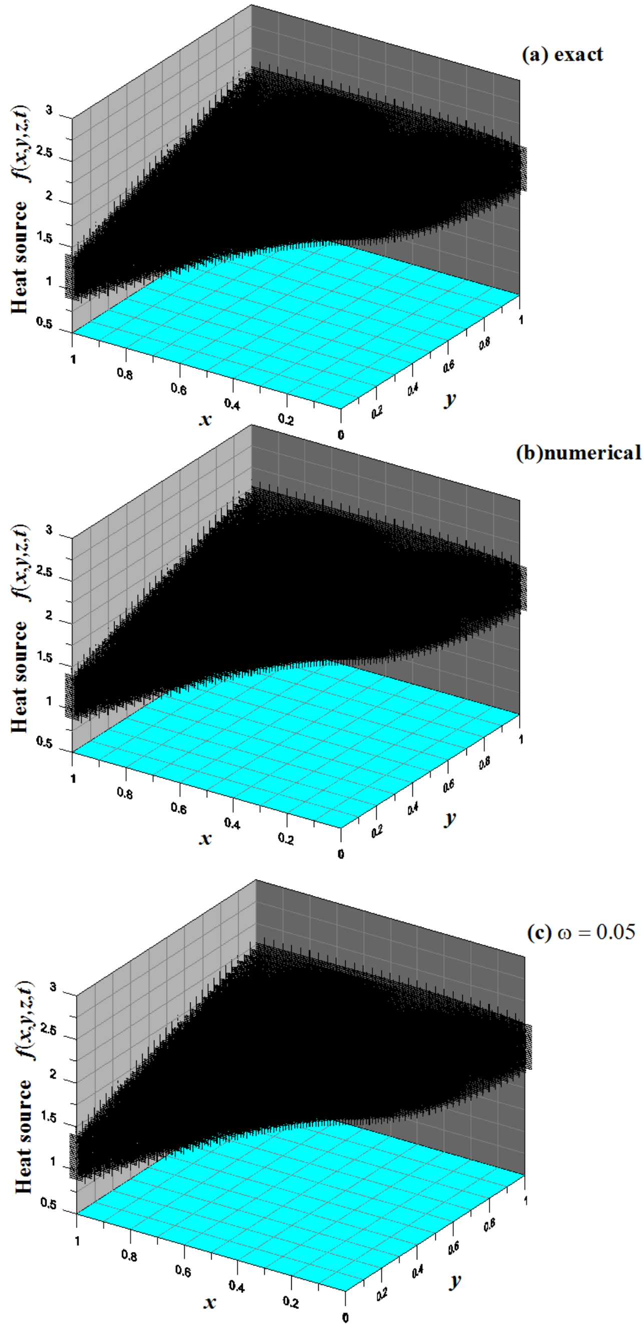


Figure 10: The exact solution for Example 7 of three-dimensional inverse problem is shown in (a), in (b) the DQM solution without random noise effect, and in (c) the DQM solution with random noise.

The exact solutions and numerical solutions are drawn in Figs. 10(a)-(c) sequentially. Even under the moderate noise, the numerical solution shown in Fig. 10(c) is a good approximation to the exact heat source as illustrated in Fig. 10(a).

5 Conclusions

In the paper, by employing the DQM, we can estimate the multi-dimensional inverse heat conduction problem of heat source very well with a high order accuracy. Seven numerical experiments of the inverse problem are worked out, which display that our proposed approach is applicable to the ill-posed problem. The numerical errors of our scheme are in the order of $O(10^{-2})$–$O(10^{-7})$. Moreover, those effects are very significant in the computations of three-dimensional problem. Therefore, it can be concluded that the DQM is stable, accurate, effective, and insensitive to the disturbance on exact temperature data.

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