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Stress Analysis of Elastic Roof with Boundary Element Formulations

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Abstract: Roof is one of the most important structures in coal mining engineering, which needs to be studied thoroughly at the theoretical level, while elastic roof is treated as one of the problems for elastic plates in this paper. The existing literatures on elastic plates have largely restricted to different engineering but all most minority for coal mining engineering. Based on the mechanical models of plane and bending stress for elastic roof, using the boundary integral equations which is obtained by the natural boundary reduction, this paper obtains the elastic roof's Airy stress functions of the problem of inner elastic plane roof and bending deflection respectively, as well as the analytical and numerical solutions to the each stress field functions. We also analyze the rules of different stress distribution for the two stress fields varying with the radius and the angle by comparison. The results of calculation show that, with the increasing of the radius and the angles of the elastic circular plate roof with a pair of tensile forces along its diameter on the boundary, the value of the stress meets uniformity; with the increasing of radius, the stress declines under a concentrated force.

Keywords: boundary element method; elastic roof; stress function; boundary integral formula; Airy stress function.

1 Introduction

In coal mining engineering, especially in longwall mining, the fracture of the roof would affect the distribution of the abutment pressure ahead of the longwall face and following the fracture of roof a series of phenomena would be happened around the face area. For instance, the convergence in the working area and the load acting on the support would be increased, and sometimes some kinetic phenomena may be

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occured [Miao et al. (2011)]. Because of this, the mining researchers and engineers always paid special attention to the research of pressure at roof.

In generally, rock stratum is usually regarded as plate in the working area. Miao et al. (2010) treated the roof of rock stratum as elastic plate. For the elastic plate, the stress solution in different engineering problems has been analyzed [León and París (1989), Tanaka et al. (1996), Lin et al. (1999), England (2009a,b), Banichuk et al. (2010), Peter and Lee (2011), Sakagami et al. (2011)], but so far, there is all most minority analysis in coal mining engineering, which needs to be studied thoroughly at the theoretical level.

The Boundary Element Method (BEM) [Hartmann (1989)] solves field problems by solving an equivalent source problem. In the case of electric fields, it solves for equivalent charge, while in the case of magnetic fields, it solves for equivalent currents. BEM also uses an integral formulation of Maxwell's Equations, which allows for very accurate field calculations. In theory, it is convenient to analyze the problems of elastic roof under various boundary conditions using the natural BEM.

The natural BEM [Yu (2002)] is a branch of a number of BEMs, based on a complex variable method, a method using a Fourier series, or a Green's function method to induce a Dirichlet boundary value problem as a differential equation into Poisson integration equation of the studied area or to induce Neumann boundary value problem of differential equation into a strong singular boundary integral equation [Yu (2002)]. The natural BEM is widely used to solve problems of a circular interior and exterior domain and other plane and engineering problems, as rock surrounding the roadway [Li et al. (2011), Ma et al. (2011)]. Yu and Du (2003) and Liu and Yu (2008) have investigated coupling methods between natural BEM and finite element methods. Based on the natural BEM on the boundary value problem of a bi-harmonic equation of a circular exterior domain, a boundary integration equation of the Airy stress function in polar coordinates is obtained.

This paper uses the surface forces on the boundary to calculate the stress function and its normal derivative, which are substituted into the integration equation, thus obtained the specific expression of a stress function under various boundary conditions, thus permits the analysis of stress and related deformation inside the elastic roof, and then the rules of different distribution for the two stress fields varying with the radius and the angles are analyzed.

2 The differential equation and boundary conditions of elastic circle plate

The differential equation for the plate bending based on the G. R. Kirchhoff assumptions is:

$$\Delta^2\phi = \frac{q}{D} = f \tag{1}$$

where, ϕ is the deflection of plate, q is surface distribution set degree of the lateral load acting on the plate, D is the bending rigidity of the plate. There are usually three boundary conditions for bending of plate, as following:

(1) $\phi|_\Gamma = \phi_0,\ \left.\frac{\partial\phi}{\partial n}\right|_\Gamma = \phi_n,$

(2) $\phi|_\Gamma = \phi_0, M\phi = m,$

(3) $T\phi = t, M\phi = m.$

where, T,M are the differential boundary operator, respectively, $T\phi, M\phi$ have the following expressions respectively under the condition of polar coordinate system:

$$\begin{cases} T\phi = \left[-\frac{\partial}{\partial r}\Delta\phi + (1-\mu)\frac{1}{r\partial\theta}\left(\frac{1}{r^2}\frac{\partial\phi}{\partial\theta} - \frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta}\right)\right]_\Gamma = \frac{V_r}{D} \\ M\phi = \left[\mu\Delta\phi + (1-\mu)\frac{\partial^2\phi}{\partial r^2}\right]_\Gamma = -\frac{M_r}{D} \end{cases} \tag{2}$$

where, μ is the Poisson's ratio, and V_r, M_r are denoted as the total radial distribution shear force and the radial distribution bending moment respectively.

3 Boundary integration formula and natural integration formula of biharmonic equation inner the elastic circle plate with natural boundary reduction

Assumption: Eq. (3) is the boundary value problem of biharmonic equation inner the elastic circle plate.

$$\begin{cases} \Delta^2\phi(r,\theta) = \frac{q(r,\theta)}{D} = f(r,\theta) & \text{inner the region } \Omega \\ (T\phi, M\phi) = (t,m) & \text{on the boundary } \Gamma \end{cases} \tag{3}$$

where, its region of integration is $\Omega = \int_0^{2\pi}\int_0^R f(r,\theta)\,drd\theta$, Yu (2002) using Green's function method and a natural boundary reduction, obtains the boundary integration formula of the stress function:

$$\phi(r,\theta) = A_r(\theta)*\phi_n(\theta) + B_r(\theta)*\phi_0(\theta) + \int_0^{2\pi}\int_0^R G(r,\theta;r',\theta')\,f(r',\theta')\,r'dr'd\theta' \tag{4}$$

where, $\phi_0(\theta), \phi_n(\theta)$ are denoted as the boundary stress function and its normal derivative, respectively, and the natural integration formula of the bending inner the elastic circle plate:

$$T\phi - \int_0^{2\pi}\int_0^{R} TG\left(r,\theta;r',\theta'\right) f\left(r',\theta'\right) r'dr'd\theta' = \\ -\frac{1+\mu}{R^3}\phi_0(\theta) + \frac{1}{2\pi R^3 \sin^2\frac{\theta}{2}} * \phi_0(\theta) + \frac{1+\mu}{R^2}\phi_n(\theta) + \frac{1}{2\pi R^2\sin^2\frac{\theta}{2}} * \phi_n(\theta) \quad (5)$$

$$M\phi - \int_0^{2\pi}\int_0^{R} MG\left(r,\theta;r',\theta'\right) f\left(r',\theta'\right) r'dr'd\theta' = \\ \frac{1+\mu}{R^2}\phi_0(\theta) + \frac{1}{2\pi R^2 \sin^2\frac{\theta}{2}} * \phi_0(\theta) + \frac{1+\mu}{R}\phi_n(\theta) - \frac{1}{2\pi R\sin^2\frac{\theta}{2}} * \phi_n(\theta) \quad (6)$$

where, "$*$" is the convol for variable θ, $G(r,\theta;r',\theta')$ is the Green's function of the biharmonic equation inner the elastic circle plate, and we have:

$$A_r(\theta) = -\frac{\left(R^2-r^2\right)^2}{4\pi R\left[R^2-2Rr\cos\theta+r^2\right]} \quad (7)$$

$$B_r(\theta) = -\frac{\left(R^2-r^2\right)^2\left[R-r\cos\theta\right]}{2\pi R\left[R^2-2Rr\cos\theta+r^2\right]^2} \quad (8)$$

$$G\left(r,\theta;r',\theta'\right) = \\ \frac{1}{16\pi}\left\{\left[r^2-2rr'\cos\left(\theta-\theta'\right)+r'^2\right]\ln\frac{R^2\left[r^2-2rr'\cos\left(\theta-\theta'\right)+r'^2\right]}{R^4-2rr'R^2\cos\left(\theta-\theta'\right)+r^2r'^2} + \frac{\left(R^2-r^2\right)\left(R^2-r'^2\right)}{R^2}\right\} \quad (9)$$

Collect the equations above, we obtain:

$$\phi(r,\theta) = \int_0^{2\pi} \frac{\left(R^2 - r^2\right)^2}{4\pi}$$
$$\left\{ \frac{[2R - 2r\cos(\theta - \theta')]\,\phi_0(\theta')}{[R^2 + r^2 - 2Rr\cos(\theta - \theta')]^2} - \frac{\phi_n(\theta')}{[R^2 + r^2 - 2Rr\cos(\theta - \theta')]} \right\} d\theta' +$$
$$\int_0^{2\pi}\int_0^{R} \frac{1}{16\pi} \left\{ \left[r^2 + r'^2 - 2rr'\cos\left(\theta - \theta'\right)\right] \ln \frac{R^2\left[r^2 + r'^2 - 2rr'\cos(\theta - \theta')\right]}{R^4 + r^2 r'^2 - 2rr'R^2\cos(\theta - \theta')} \right.$$
$$\left. + \frac{\left(R^2 - r^2\right)\left(R^2 - r'^2\right)}{R^2} \right\} dr'd\theta' \quad (10)$$

For the problem of elastic bending roof, Eq. (10) is the deflection $\phi(r,\theta)$ of boundary integration formula for a random point inner the elastic circle plate, while $f(r,\theta) = q(r,\theta)/D$.

For the problem of elastic inner roof, in generally, and compared with Eq. (1), we have $f(r,\theta) = 0$, thus we can obtain the Airy stress function $\phi(r,\theta)$ of Poisson integration formula for a random point inner the elastic circle plate, as Eq. (11).

$$\phi(r,\theta) = A_r(\theta) * \phi_n(\theta) + B_r(\theta) * \phi_0(\theta) =$$
$$\int_0^{2\pi} \frac{\left(R^2 - r^2\right)^2}{4\pi} \left\{ \frac{[2R - 2r\cos(\theta - \theta')]\,\phi_0(\theta')}{[R^2 + r^2 - 2Rr\cos(\theta - \theta')]^2} - \frac{\phi_n(\theta')}{[R^2 + r^2 - 2Rr\cos(\theta - \theta')]} \right\} d\theta' \quad (11)$$

4 Stress analysis inner elastic plane roof

Fig. 1 is a schematic diagram of elastic circular plate roof with a pair of tensile forces along its diameter on the boundary. Let the stress $P = 1$ and the radius $R = 1$. By analyzing, this is a problem of stress analysis inner elastic plane, we can obtain the stress function according to Eq. (11). We first need to determine the boundary stress function $\phi_0(\theta)$ and its normal derivative $\phi_n(\theta)$ according to the known surface force $\bar{X}$ and $\bar{Y}$ on the roadway.

Firstly, a base point A is selected on the roadway boundary [Love (1944)], namely

$$\phi_A = 0, \quad \left(\frac{\partial\phi}{\partial x}\right)_A = 0, \left(\frac{\partial\phi}{\partial y}\right)_A = 0. \quad (12)$$

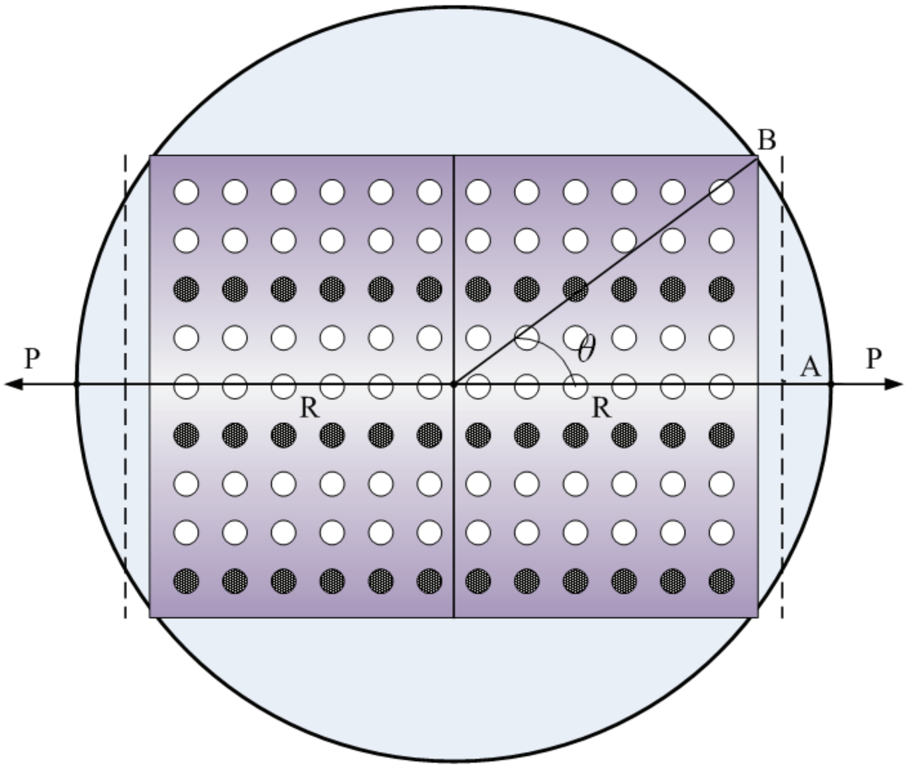

Figure 1: Elastic circular plate roof with a pair of tensile forces along its diameter on the boundary

Then, according to the known surface force $\bar{X}$ and $\bar{Y}$ on the roadway boundary, for a random point on the boundary, we have

$$\phi_B = \int_A^B (x - x_B)\bar{Y}\mathrm{d}s + \int_A^B (y_B - y)\bar{X}\mathrm{d}s \tag{13}$$

$$\left(\frac{\partial \phi}{\partial x}\right)_B = -\int_A^B \bar{Y}\mathrm{d}s \tag{14}$$

$$\left(\frac{\partial \phi}{\partial y}\right)_B = \int_A^B \bar{X}\mathrm{d}s \tag{15}$$

Here, we have,

$$\phi_0(\theta) = \phi_B = \begin{cases} PR\sin\theta & 0 \le \theta \le \pi \\ 0 & \pi \le \theta \le 2\pi \end{cases} \tag{16}$$

$$\left(\frac{\partial\phi}{\partial x}\right)_B = 0 \tag{17}$$

$$\left(\frac{\partial\phi}{\partial y}\right)_B = \begin{cases} P & 0 \le \theta \le \pi \\ 0 & \pi \le \theta \le 2\pi \end{cases} \tag{18}$$

$$\phi_n(\theta) = \left(\frac{\partial\phi}{\partial n}\right)_B = \left(\frac{\partial\phi}{\partial x}\right)_B \cos\theta + \left(\frac{\partial\phi}{\partial y}\right)_B \sin\theta = \begin{cases} P & 0 \le \theta \le \pi \\ 0 & \pi \le \theta \le 2\pi \end{cases} \tag{19}$$

Substituting Eqs. (16) and (19), $P = 1$ and $R = 1$ into Eq. (11), we have:

$$\begin{aligned}\phi(r,\theta) &= \frac{\left(1-r^2\right)^3}{4\pi}\int_0^{\pi}\frac{\sin\theta'}{\left[1+r^2-2r\cos(\theta-\theta')\right]^2}d\theta' \\ &= \frac{\left(1-r^2\right)^3}{4\pi}\int_{\theta-\pi}^{\theta}\frac{\sin(\theta-\theta')}{\left[1+r^2-2r\cos\theta'\right]^2}d\theta'\end{aligned} \tag{20}$$

thus the integral becomes:

$$\phi(r,\theta) = \frac{1}{4\pi}\left[2\pi r\sin\theta + 4r\sin\theta\arctan\left(\frac{2r\sin\theta}{1-r^2}\right) + 2 - 2r^2\right], \quad (0 \le r \le 1) \tag{21}$$

Based on the literature [Love (1944)],

$$\sigma_r = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}, \quad \sigma_\theta = \frac{\partial^2\phi}{\partial r^2}, \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right). \tag{22}$$

Thus, we can get the stress function inner the elastic plane roof:

$$\sigma_r(r,\theta) = -\frac{1}{\pi}\frac{\left(r^2-1\right)^2\left[\left(r^2+1\right)^2 - 4\cos^2\theta\right]}{\left(r^4+1\right)^2 + 4r^2\left(r^4+r^2+1\right)^2 - 8r^2\cos^2\theta\left[\left(r^2+1\right)^2 - 2r^2\cos^2\theta\right]} \tag{23}$$

$$\sigma_\theta(r,\theta) = -\frac{1}{\pi}\frac{\left(r^4+1\right)^2 + 4\left(r^6-r^2-1\right) - 4\cos^2\theta\left(2r^6-r^2-1\right)}{\left(r^4+1\right)^2 + 4r^2\left(r^4+r^2+1\right)^2 - 8r^2\cos^2\theta\left[\left(r^2+1\right)^2 - 2r^2\cos^2\theta\right]} \tag{24}$$

$$\tau_{r\theta}(r,\theta) = -\frac{1}{\pi}\frac{2\left(r^6-r^4-r^2-1\right)\sin 2\theta}{\left(r^4+1\right)^2+4r^2\left(r^4+r^2+1\right)^2-8r^2\cos^2\theta\left[\left(r^2+1\right)^2-2r^2\cos^2\theta\right]} \tag{25}$$

Fig. 2 is a three-dimensional plot to show the numeric solution of $\sigma_r(r,\theta)$ with the variation of r and θ. Fig. 3 shows that when θ is invariable, the numeric solution become 0 while $r = 0.8$ with r range from 0 to 1. It can be seen that: The values of $\sigma_r(r,\theta)$ are symmetry while $\theta = 0.5\pi, \pi, 1.5\pi$, which mean uniformity of the stress distribution.

Fig. 4 is a three-dimensional plot to show the numeric solution of $\sigma_\theta(r,\theta)$ with the variation of r and θ. Fig. 5 shows that while r is invariable, the numeric solution become cyclical variation with θ range from 0 to 2π. It can be seen that: The values of $\sigma_\theta(r,\theta)$ are symmetry while $\theta = 0.5\pi, \pi, 1.5\pi$, which mean uniformity of the stress distribution.

Fig. 6 is a three-dimensional plot to show the numeric solution of $\tau_{r\theta}(r,\theta)$ with the variation of r and θ, which has a uniformity of the stress distribution, too.

5 Stress analysis on elastic bending roof

Fig. 7 is a schematic diagram of elastic clamped circular plate roof under a concentrated force P at the point $K(r_k,\theta_k)$, which is a non-axisymmetric bending problem.

We can solve the bending problem of different lateral loads on the circular plate roof. Because the boundary of circular plate roof is fixed, thus both the boundary stress function and its normal derivative equal to zero, namely, $\phi_0(\theta) = 0, \phi_n(\theta) = 0$. we obtain Eq. (26) according to Eq. (4).

$$\phi(r,\theta) = \int_0^{2\pi}\int_0^R G\left(r,\theta;r',\theta'\right) f\left(r',\theta'\right) r' dr' d\theta' \tag{26}$$

where, $f(r,\theta) = q(r,\theta)/D$, and the key point is apply singular generalized functions to express surface distribution set degree $q(r,\theta)$ of the lateral load acting on the plate. Here, the concentrated force q can be expressed by Dirac Function δ:

$$q(r,\theta) = \frac{P}{r}\delta\langle r-r_k\rangle\,\delta\langle\theta-\theta_k\rangle \tag{27}$$

Substituting Eq. (27) into Eq. (26), we have:

$$\phi(r,\theta) = \int_0^{2\pi}\int_0^R G\left(r,\theta;r',\theta'\right)\frac{\frac{P}{r}\delta\langle r-r_k\rangle\,\delta\langle\theta-\theta_k\rangle}{D} r' dr' d\theta' = \frac{P}{D}G(r,\theta;r_k,\theta_k)$$

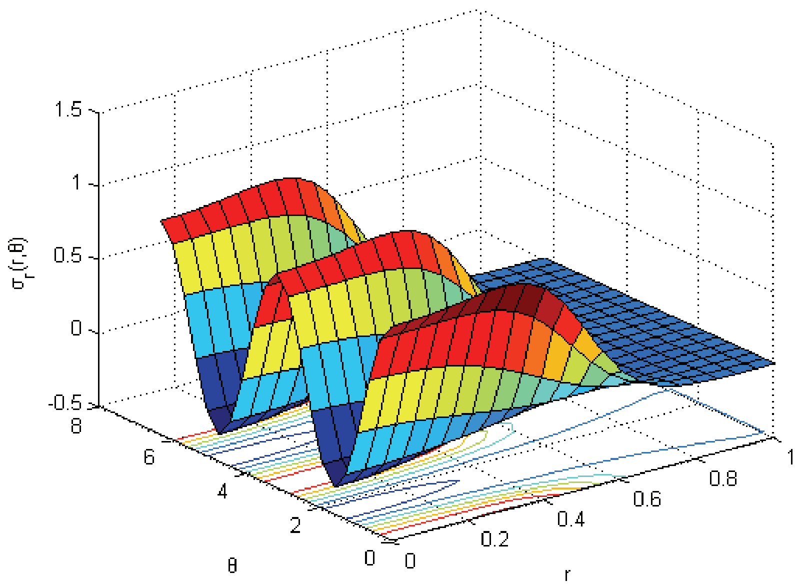

Figure 2: Numeric solution of $\sigma_r(r,\theta)$

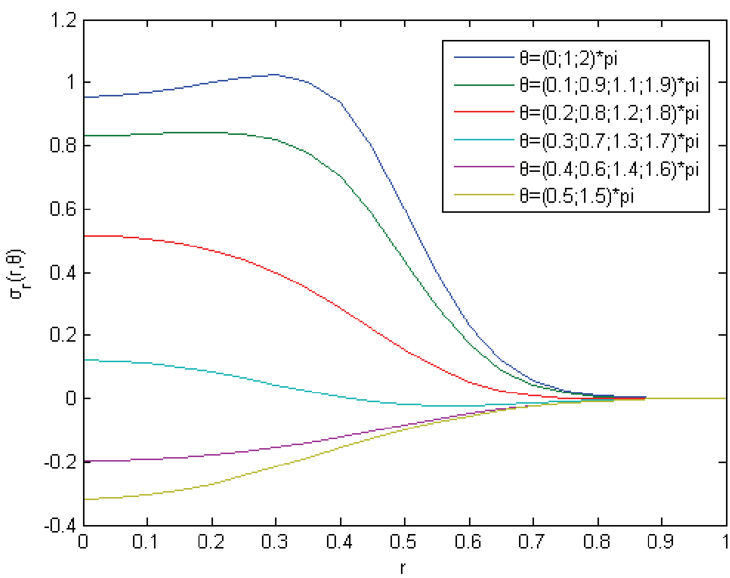

Figure 3: Variation curve for $\sigma_r(r,\theta)$ with the increase of r while θ is invariable

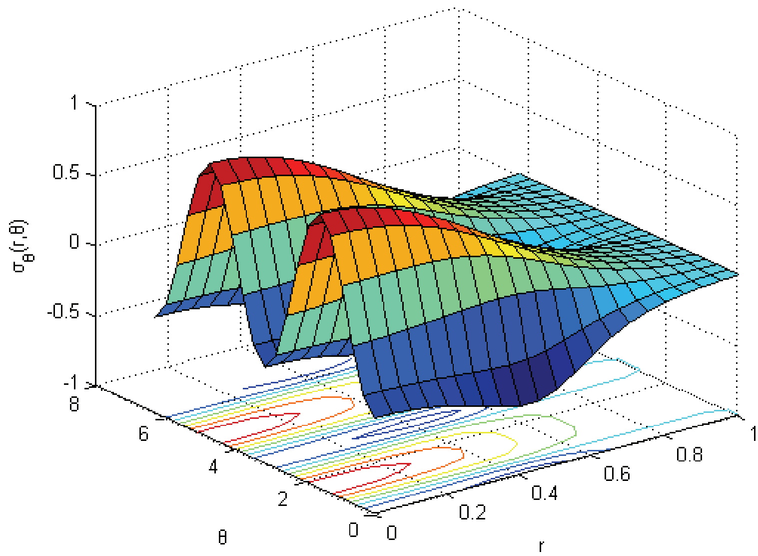

Figure 4: Numeric solution of $\sigma_\theta\left(r,\theta\right)$

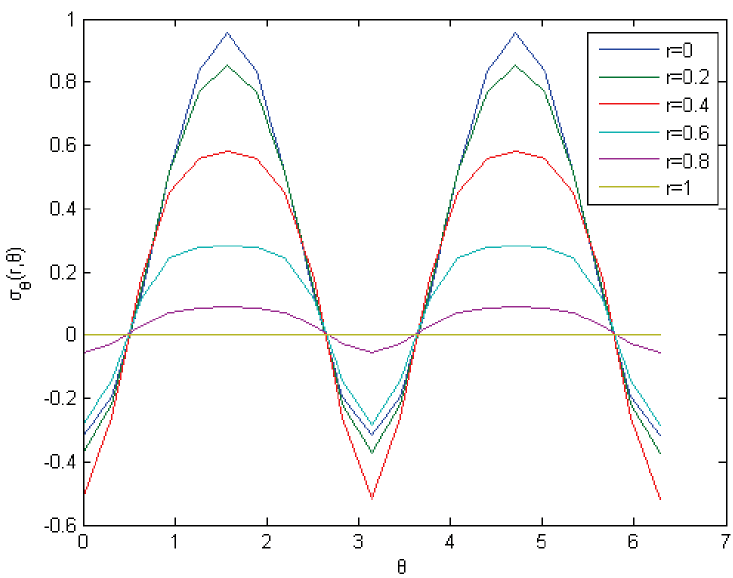

Figure 5: Variation curve for $\sigma_\theta\left(r,\theta\right)$ with the increase of θ while r is invariable

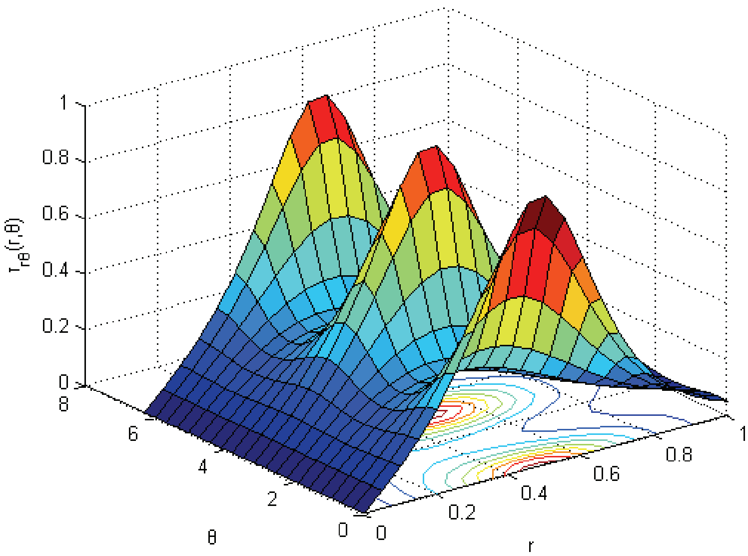

Figure 6: Numeric solution of $\tau_{r\theta}(r, \theta)$

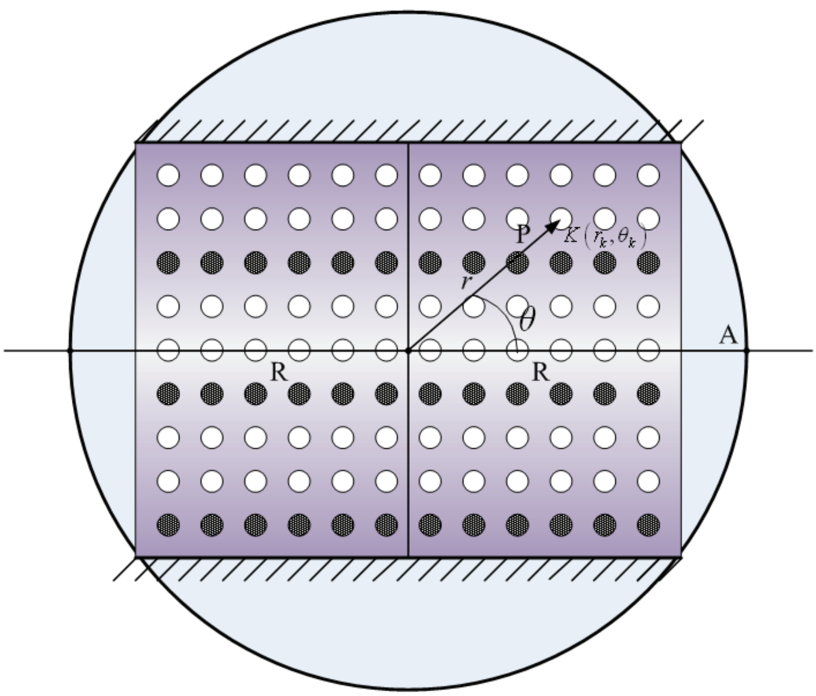

Figure 7: Elastic clamped circular plate roof under a concentrated force

(28)

It can be seen that, for elastic clamped circular plate roof under a concentrated force, we can obtain the analytic solution of the concentrated force at the point $K(r_k,\theta_k)$ based on the Green's Function $G(r,\theta;r',\theta')$, without piecewise integral.

$$\phi(r,\theta)=\frac{P}{D}G(r,\theta;r_k,\theta_k)$$
$$=\frac{P}{D}\int_0^{2\pi}\int_0^{R}\frac{1}{16\pi}\left\{\begin{array}{l}\left[r^2-2rr'\cos(\theta-\theta')+r'^2\right]\ln\frac{R^2\left[r^2-2rr'\cos(\theta-\theta')+r'^2\right]}{R^4-2rr'R^2\cos(\theta-\theta')+r^2r'^2}\\+\frac{(R^2-r^2)(R^2-r'^2)}{R^2}\end{array}\right\}dr'd\theta' \quad (29)$$

Especially, we can obtain the deflection solution of the concentrated force at the center of the elastic clamped circular plate roof.

$$\phi(r,\theta)=\frac{P}{D}G(r,\theta;0,0)|_{R=1}=\frac{P}{16\pi D}\left(2r^2\ln r-r^2+1\right) \quad (30)$$

Fig. 8 shows the numeric solution of the deflection solution of the concentrated force at the center, which shows that with an increase of r, make a decrease for ϕ, $r\in(0,1],\phi\in[0,0.02)$.

6 Conclusions

(1) In this paper, the problems of stress analysis inner elastic plane and bending for the plane can be treated as boundary problem of biharmonic equation inner the elastic circle plate. We obtain the Poisson integration formula based on a natural boundary reduction, thus we get the analytic solution for the stress function of the problem of inner elastic plane roof and bending deflection. The detailed examples have show the advantages to solve the problem by BEM, and offered a similarity solution.

(2) We have analyzed rules of the distribution of the stress function inner elastic plane roof, while r is invariable, the numeric solution become cyclical variation with θ range from 0 to 2π. For the problem of bending deflection, with an increase of r, make a decrease for ϕ.

(3)We have developed the analytical solution of stress inner elastic plane roof and bending deflection. The solutions are exactly equal to the answers given in various sources of the literature.

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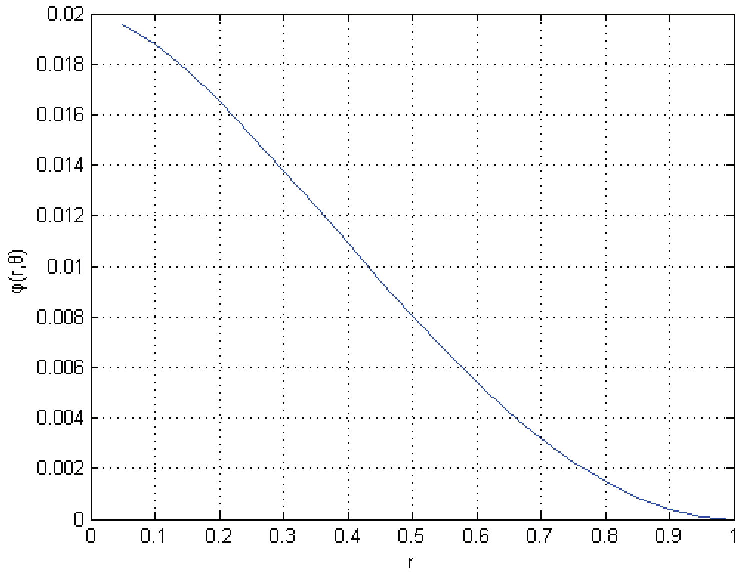

Figure 8: Numeric solution of the deflection solution of the concentrated force at the center

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