# Porous Media Analysis by Modified MLPG Formulations 

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#### Abstract

This work proposes a modified procedure, based on analytical integrations, to analyse poroelastic models discretized by time-domain Meshless Local Petrov-Galerkin formulations. In this context, Taylor series expansions of the incognita fields are considered, and the related integrals of the meshless formulations are solved analytically, rendering a so called modified methodology. The work is based on the $\mathbf{u}-p$ formulation and the incognita fields of the coupled analysis in focus are the solid skeleton displacements and the interstitial fluid pore pressures. Independent spatial discretization is considered for each phase of the model, rendering a more flexible and efficient methodology. The Moving Least Squares approximation is employed for the spatial variation of the displacement and porepressure fields and two variants of the meshless local Petrov-Galerkin formulation are discussed here, which are based on the use of Heaviside or Gaussian weight test functions. Modified expressions to properly compute the shape function derivatives are also considered. At the end of the paper, numerical examples illustrate the performance and potentialities of the proposed techniques.


Keywords: Meshless Local Petrov-Galerkin; Moving Least Squares; Analytical Integration; Shape Function Derivatives; Poroelastodynamics; Independent Phase Discretization.

## 1 Introduction

The numerical investigation of processes in porous media is attracting attention because of its significance in a great number of practical engineering problems. Since the coupled differential equations are generally difficult to solve exactly, it appears that numerical approaches have to be adopted to attain solutions. Despite

[^0]the universality and great success of the finite and boundary element methods in their applicability even to multi-field problems, there are some restrictions leading to exclusion of the finite elements with equal order interpolation for pressure and displacements in poroelastic problems (Lewis and Schrefler, 1998; Zienkiewicz et al., 1999; Li et al., 2003; Soares, 2008).
On the other hand, element free formulations (Belytschko, 1994; Atluri and Shen, 2002, Liu, 2003) have become popular because of their high adaptivity, low cost to prepare input data, elimination of remeshing as well as large distortions of element etc. The accuracy of derivatives of field variables in various meshless approximations is decreasing with increasing the order of the derivatives, especially near the boundary of the analyzed domain. The utilization of weak formulations decreases the order of the derivatives involved, but background cells are often required for integration over the analysed domain. This inappropriateness can be avoided by developing local weak formulations (Atluri and Shen, 2002). According to the selection of the test function, various variants of the meshless local Petrov-Galerkin (MLPG) can be employed. The shape and size of sub-domains are arbitrary, but a proper selection of the computational parameters (such as the size of the integration local sub-domains, the size of the influence domain and the shape parameters of the weight functions used in meshless approximations) with respect to the spatial distribution of the nodal points is an important matter and it is done usually in numerical experiments. Moreover, in dynamic problems, certain correlation between the spatial and temporal discretizations is required.
The main objection against weak formulations with meshless approximations of field variables is the prolongation of the computational time due to procedural evaluation of the shape functions and their derivatives at each integration point, since the related integrals are not available in closed form. Thus, the computational efficiency is much better in strong than in weak meshless formulations. To overcome this handicap of the weak formulation, Sladek et al. (2008; 2009; 2010a; 2010b) proposed to utilize Taylor series expansion of the integrands and analytical integration over local sub-domains. Then, the evaluation of the shape functions and their derivatives is focused on nodal points, like in strong formulations. Truncated Taylor series expansions (considering only lower order terms) are allowed by using reduced size of the sub-domains. However, the dependence of the formulation on higher order derivatives of the shape function may decrease the accuracy of the procedure or render unstable analyses for hyperbolic problems. Soares et al. (2011) proposed a modified computation of the derivatives of the shape function, adding a correctional term to the standard derivatives expressions.
The present work is organized as follows: first, the governing poroelastodynamic equations are presented and the standard MLPG is developed, considering two for-
mulations based on different test functions. The derived semi-discretized governing equations are given by the set of ordinary differential equations (ODE) for evolution of nodal unknowns. The coefficients of the ODE are given by boundary and/or domain integrals involving the shape functions and their derivatives. Then, the modified MLPG is introduced and the analytical expressions for the integrals, as well as the modified computation of the shape function derivatives, are described. For the temporal discretization, the generalized Newmark method is considered as a one-step finite difference approximation. At the end of the paper, numerical examples are discussed, illustrating the performance and potentialities of the proposed methodologies. In the MLPG formulations, the adoption of different discretizations for each phase of the model is trivial, allowing appropriate numerical simulations of impermeable and incompressible soil models. In the Appendix, the basic aspects of the MLS are presented, describing the standard computation of the shape functions and their derivatives.

## 2 Governing equations

The present work is focused on the $\mathbf{u}-p$ formulation, as presented by Zienkiewicz et al. $(1984,1990)$. In this case, the governing equations of the poroelastodynamic model can be written as:

$$
\begin{align*}
& \sigma_{i j, j}-\rho_{m} \ddot{u}_{i}+\rho_{m} b_{i}=0  \tag{1}\\
& \alpha \dot{\varepsilon}_{i i}-\left(\kappa p,_{i}\right)_{, i}+(1 / Q) \dot{p}-a=0 \tag{2}
\end{align*}
$$

where equation (1) stands for the balance of momentum of the mixture and equation (2) is a combination of the balance of mass and momentum for the interstitial fluid. In equation (1), $\sigma_{i j}(X, t)$ is the total Cauchy stress (usual indicial notation for Cartesian axes is considered); the effective stress is defined as $\sigma_{i j}^{\prime}=\sigma_{i j}+\alpha \delta_{i j} p$, where $\alpha(X)$ accounts for slight strain changes, $p(X, t)$ stands for interstitial fluid porepressure and $\delta_{i j}$ represents the Kronecker delta ( $\delta_{i j}=0$ if $i \neq j$ and $\delta_{i j}=1$ if $i=j$ ). Further on in equation (1), $u_{i}(X, t)$ stands for the solid matrix displacement and $b_{i}(X, t)$ for the body force distribution. Inferior commas and overdots indicate partial space $\left(u_{j, i}=\partial u_{j} / \partial x_{i}\right)$ and time $\left(\dot{u}_{i}=\partial u_{i} / \partial t\right)$ derivatives, respectively. $\rho_{m}=v \rho_{f}+(1-v) \rho_{s}$ stands for the mass density of the mixture, where $\rho_{s}(X)$ and $\rho_{f}(X)$ are the mass density of the solid and fluid phase, respectively, and $v(X)$ is the porosity of the medium. In equation (2), $\varepsilon_{i j}(X, t)$ represents the strain tensor and $\kappa(X)$ defines the permeability coefficients, according to the D'Arcy seepage law. $a(X, t)$ stands for domain source terms and the mixture parameter $Q(X)$ is defined by $(1 / Q)=\mu / K_{f}+(\alpha-\mu) / K_{s}$, where the bulk moduli of the solid and fluid phase are represented by $K_{s}(X)$ and $K_{f}(X)$, respectively. In the present
work, linear kinematical and constitutive relations are focused and they are represented by $\varepsilon_{i j}=\frac{1}{2}\left(u_{i, j}+u_{j, i}\right)$ and $\sigma_{i j}^{\prime}=D_{i j k l} \varepsilon_{k l}=\lambda \delta_{i j} \varepsilon_{k k}+2 \mu \varepsilon_{i j}$, respectively, where $D_{i j k l}=\lambda \delta_{i j} \delta_{k l}+\mu\left(\delta_{i k} \delta_{j l}+\delta_{i l} \delta_{j k}\right)$ is the elastic constitutive tensor and $\lambda(X)$ and $\mu(X)$ are the Lamé's coefficients of the medium.
Equations (1)-(2), accompanied by appropriate initial ( $u_{i}=\bar{u}_{i 0}, \dot{u}_{i}=\dot{\bar{u}}_{i 0}$ and $p=\bar{p}_{0}$ ) and boundary conditions $\left(u_{i}=\bar{u}_{i}\right.$ or $\tau_{i}=\sigma_{i j} n_{j}=\bar{\tau}_{i}$ and $p=\bar{p}$ or $q=p_{, j} n_{j}=$ $\bar{q}$, where the prescribed values are indicated by over bars and $q$ and $\tau_{i}$ represent fluxes and tractions acting along the boundary whose unit outward normal vector components are represented by $n_{i}$ ), define the model to be solved by the MLPG formulations proposed here.

## 3 Standard MLPG formulations

Instead of writing the global weak-form for the governing equations described in the previous section, the MLPG method constructs a weak-form over local fictitious sub-domains, such as $\Omega_{s}$, which is a small region taken for each node inside the global domain. The local sub-domains overlap each other and its geometrical shape and size can be arbitrary, covering the whole global domain $\Omega$. The local weakform of the governing equations can be written as

$$
\begin{align*}
& \int_{\partial \Omega_{s}} \varphi_{i k} \sigma_{i j} n_{j} d \Gamma-\int_{\Omega_{s}} \varphi_{i k, j} \sigma_{i j} d \Omega+\int_{\Omega_{s}} \varphi_{i k}\left(\rho_{m} b_{i}-\rho_{m} \ddot{u}_{i}\right) d \Omega=0  \tag{3}\\
& \int_{\partial \Omega_{s}} \varphi \kappa p_{, i} n_{i} d \Gamma-\int_{\Omega_{s}} \varphi_{, i} \kappa p_{, i} d \Omega+\int_{\Omega_{s}} \varphi\left(a-(1 / Q) \dot{p}-\alpha \dot{\varepsilon}_{i i}\right) d \Omega=0 \tag{4}
\end{align*}
$$

where $\varphi$ and $\varphi_{i k}$ are test functions. In equations (3) and (4), $\partial \Omega_{s}$ is the boundary of the local sub-domain, which consists of three parts, in general: $\partial \Omega_{s}=L_{s} \cup \Gamma_{s 1} \cup$ $\Gamma_{s 2}$. Here, $L_{s}$ is the local boundary that is totally inside the global domain, $\Gamma_{s 2}$ is the part of the local boundary which coincides with the global natural boundary, i.e., $\Gamma_{s 2}=\partial \Omega_{s} \cap \Gamma_{2}$ (where $\Gamma_{2}$ stands for the natural boundary, i.e., $\Gamma_{2} \equiv \Gamma_{q}$ or $\left.\Gamma_{2} \equiv \Gamma_{\tau}\right)$ and, similarly, $\Gamma_{s 1}$ is the part of the local boundary that coincides with the global essential boundary, i.e., $\Gamma_{s 1}=\partial \Omega_{s} \cap \Gamma_{1}$ (where $\Gamma_{1}$ stands for the essential boundary, i.e., $\Gamma_{1} \equiv \Gamma_{p}$ or $\Gamma_{1} \equiv \Gamma_{u}$ ).
Equations (3)-(4) can be simplified according to the chosen test function. By specifying the local sub-domain as the circle $\Omega^{c}$ centred at node $X^{c}$ and by choosing a test function which provides null spatial derivatives in each sub-domain, such as the Heaviside step function $\varphi_{i k}=\delta_{i k} \varphi=\delta_{i k}\left(1-H\left(d^{c}-r^{c}\right)\right.$ ) (see the Appendix for the definition of $d^{c}$ and $r^{c}$ ), the local weak-form (3)-(4) can then be converted into
the following local boundary-domain integral equations:

$$
\begin{align*}
& \sum_{a=1}^{N}\left(\int_{\Omega^{c}} \rho_{m} \phi_{u}^{a} d \Omega\right) \ddot{\hat{\mathbf{u}}}^{a}-\sum_{a=1}^{N}\left(\int_{L^{c}+\Gamma_{u}^{c}} \mathbf{N D S}_{u}^{a} d \Gamma\right) \hat{\mathbf{u}}^{a}+\sum_{a=1}^{N}\left(\int_{L^{c}+\Gamma_{u}^{c}} \mathbf{n} \alpha \phi_{p}^{a} d \Gamma\right) \hat{p}^{a}= \\
& =\int_{\Gamma_{\tau}^{c}} \bar{\tau} d \Gamma+\int_{\Omega^{c}} \rho_{m} \mathbf{b} d \Omega \tag{5}
\end{align*}
$$

$$
\begin{align*}
& \sum_{a=1}^{N}\left(\int_{\Omega^{c}}(1 / Q) \phi_{p}^{a} d \Omega\right) \dot{\hat{p}}^{a}-\sum_{a=1}^{N}\left(\int_{L^{c}+\Gamma_{p}^{c}} \mathbf{n}^{T} \kappa \mathbf{s}_{p}^{a T} d \Gamma\right) \hat{p}^{a}+\sum_{a=1}^{N}\left(\int_{\Omega^{c}} \alpha \mathbf{s}_{u}^{a} d \Omega\right) \dot{\hat{\mathbf{u}}}^{a}= \\
& =\int_{\Gamma_{q}^{c}} \kappa \bar{q} d \Gamma+\int_{\Omega^{c}} a d \Omega \tag{6}
\end{align*}
$$

where, for the development of the above equations, definition (A1) is employed and Voigt notation (i.e., $\sigma=\left[\sigma_{11}, \sigma_{22}, \sigma_{12}\right]^{T}, \tau=\left[\tau_{1}, \tau_{2}\right]^{T}, \mathbf{u}=\left[u_{1}, u_{2}\right]^{T}$ etc.) is considered.
On the other hand, by choosing a test function with vanishing value on the local boundary that is totally inside the global domain, such as the Gaussian weight function $\varphi_{i k}=\delta_{i k} \varphi=\delta_{i k} w^{c}$ (see equation (A4) for the definition of $w^{c}$ ), one can rewrite equations (3)-(4) into the following local integral equations:

$$
\begin{align*}
& \sum_{a=1}^{N}\left(\int_{\Omega^{c}} w_{u}^{c} \rho_{m} \phi_{u}^{a} d \Omega\right) \ddot{\hat{\mathbf{u}}}^{a}+\sum_{a=1}^{N}\left(\int_{\Omega^{c}} \mathbf{W}_{u}^{c} \mathbf{D} \mathbf{S}_{u}^{a} d \Omega-\int_{\Gamma_{u}^{c}} w_{u}^{c} \mathbf{N D} \mathbf{S}_{u}^{a} d \Gamma\right) \hat{\mathbf{u}}^{a}+ \\
& -\sum_{a=1}^{N}\left(\int_{\Omega^{c}} \mathbf{w}_{u}^{c} \alpha \phi_{p}^{a} d \Omega-\int_{\Gamma_{u}^{c}} w_{u}^{c} \mathbf{n} \alpha \phi_{p}^{a} d \Gamma\right) \hat{p}^{a}=\int_{\Gamma_{\tau}^{c}} w_{u}^{c} \bar{\tau} d \Gamma+\int_{\Omega^{c}} w_{u}^{c} \rho_{m} \mathbf{b} d \Omega  \tag{7}\\
& \sum_{a=1}^{N}\left(\int_{\Omega^{c}} w_{p}^{c}(1 / Q) \phi_{p}^{a} d \Omega\right) \dot{\hat{p}}^{a}+\sum_{a=1}^{N}\left(\int_{\Omega^{c}} \mathbf{w}_{p}^{c T} \kappa \mathbf{s}_{p}^{a T} d \Omega-\int_{\Gamma_{p}^{c}} w_{p}^{c} \mathbf{n}^{T} \kappa \mathbf{s}_{p}^{a T} d \Gamma\right) \hat{p}^{a}+ \\
& +\sum_{a=1}^{N}\left(\int_{\Omega^{c}} w_{p}^{c} \alpha \mathbf{s}_{u}^{a} d \Omega\right) \dot{\mathbf{u}}^{a}=\int_{\Gamma_{q}^{c}} w_{p}^{c} \kappa \bar{q} d \Gamma+\int_{\Omega^{c}} w_{p}^{c} a d \Omega \tag{8}
\end{align*}
$$

where matrices $\mathbf{N}, \mathbf{n}, \mathbf{W}^{c}, \mathbf{w}^{c}, \mathbf{S}^{a}$ and $\mathbf{s}^{a}$ are specified as:

$$
\begin{align*}
& \mathbf{N}=\left[\begin{array}{ccc}
n_{1} & 0 & n_{2} \\
0 & n_{2} & n_{1}
\end{array}\right]  \tag{9a}\\
& \mathbf{n}=\left[\begin{array}{ll}
n_{1} & n_{2}
\end{array}\right]^{T} \tag{9b}
\end{align*}
$$

$\mathbf{W}^{c}=\left[\begin{array}{ccc}w_{, 1}^{c} & 0 & w_{, 2}^{c} \\ 0 & w_{, 2}^{c} & w_{, 1}^{c}\end{array}\right]$
$\mathbf{w}^{c}=\left[\begin{array}{ll}w_{, 1}^{c} & w_{, 2}^{c}\end{array}\right]^{T}$
$\mathbf{S}^{a}=\left[\begin{array}{ccc}\phi_{, 1}^{a} & 0 & \phi_{, 2}^{a} \\ 0 & \phi_{, 2}^{a} & \phi_{, 1}^{a}\end{array}\right]^{T}$
$\mathbf{s}^{a}=\left[\begin{array}{ll}\phi_{, 1}^{a} & \phi_{, 2}^{a}\end{array}\right]$
and $\mathbf{D}$ is the linear elastic constitutive matrix. In equations (5)-(8), subscripts $u$ and $p$ are relative to solid and fluid phase discretizations, respectively.
By collecting all nodal unknown fictitious values $\hat{p}^{a}$ and $\hat{\mathbf{u}}^{a}$ into vectors $\hat{\mathbf{P}}$ and $\hat{\mathbf{U}}$, respectively, the above systems of the discretized equations can be rewritten into matrix form, as follows:
$\mathbf{M} \ddot{\hat{\mathbf{U}}}+\mathbf{K} \hat{\mathbf{U}}-\mathbf{Q} \hat{\mathbf{P}}=\mathbf{F}$
$\mathbf{C} \dot{\hat{\mathbf{P}}}+\mathbf{H} \hat{\mathbf{P}}+\mathbf{G} \dot{\hat{\mathbf{U}}}=\mathbf{R}$
where $\mathbf{M}$ (mass matrix) and $\mathbf{C}$ (compressibility matrix) are evaluated taking into account the first integral term on the l.h.s. of equations (5) and (6) or (7) and (8), respectively; $\mathbf{K}$ (stiffness matrix) and $\mathbf{H}$ (permeability matrix) are computed considering the second term on the l.h.s. of equations (5) and (6) or (7) and (8), respectively; $\mathbf{Q}$ and $\mathbf{G}$ (coupling matrices) are calculated considering the third term on the l.h.s. of equations (5) and (6) or (7) and (8), respectively; and $\mathbf{F}$ and $\mathbf{R}$ (load nodal vectors) are evaluated considering the terms on the r.h.s. of equations (5) and (6) or (7) and (8), respectively. If $X$ is a Dirichlet node (i.e., $X \in \Gamma_{u}$ or $X \in \Gamma_{p}$ ), the related equation in the system of equations (10)-(11) is replaced by the definition $\sum_{a=1}^{N} \phi_{u}^{a}(X) \hat{u}_{i}^{a}(t)=\bar{u}_{i}(X, t)$ or $\sum_{a=1}^{N} \phi_{p}^{a}(X) \hat{p}^{a}(t)=\bar{p}(X, t)$, thus exactly enforcing the Dirichlet conditions over the boundary.
In standard MLPG formulations, matrices $\mathbf{M}, \mathbf{C}, \mathbf{K}, \mathbf{H}, \mathbf{Q}$ and $\mathbf{G}$ are computed taking into account numerical integration techniques, such as Gauss-Legendre quadrature rules etc. In this work, these procedures are not applied and a modified computation of these matrices is carried out, as described in the next section.

## 4 Modified MLPG formulations

In the modified MLPG formulation, the integrals related to the computation of the system matrices are evaluated analytically, taking into account a Taylor series expansion of the related integrands (Sladek et al., 2008; 2009; 2010a; 2010b). Applying a Taylor series expansion, a continuous function $f(X)$ can be approximated in
the vicinity of point $X_{0}$ as follows (sum over greek indices is assumed implicitly):
$f(X)=\sum_{m=0}^{M} \frac{1}{m!}\left(\bar{x}_{\beta} \frac{\partial}{\partial x_{\beta}}\right)^{m} f\left(X_{0}\right)$
where $\bar{x}_{\beta}=x_{\beta}-x_{0 \beta}$ and $m$ is related to the order of the approximation.
Taking into account first order approximations (i.e., $m=1$ ), equations (5)-(6) can be rewritten as:

$$
\begin{align*}
& \sum_{a=1}^{N}\left(\int_{\Omega^{c}} \rho_{m} d \Omega \phi_{u}^{a}\left(X_{0}^{c}\right)+\int_{\Omega^{c}} \rho_{m} \bar{x}_{\beta} d \Omega \phi_{u, \beta}^{a}\left(X_{0}^{c}\right)\right) \ddot{\hat{\mathbf{u}}}^{a}+ \\
& -\sum_{a=1}^{N}\left(\int_{L^{c}+\Gamma_{u}^{c}} \mathbf{N D} d \Gamma \mathbf{S}_{u}^{a}\left(X_{0}^{c}\right)+\int_{L^{c}+\Gamma_{u}^{c}} \mathbf{N D} \bar{x}_{\beta} d \Gamma \mathbf{S}_{u, \beta}^{a}\left(X_{0}^{c}\right)\right) \hat{\mathbf{u}}^{a}+ \\
& +\sum_{a=1}^{N}\left(\int_{L^{c}+\Gamma_{u}^{c}} \mathbf{n} \alpha d \Gamma \phi_{p}^{a}\left(X_{0}^{c}\right)+\int_{L^{c}+\Gamma_{u}^{c}} \mathbf{n} \alpha \bar{x}_{\beta} d \Gamma \phi_{p, \beta}^{a}\left(X_{0}^{c}\right)\right) \hat{p}^{a}=\int_{\Gamma_{\tau}^{c}} \bar{\tau} d \Gamma+\int_{\Omega^{c}} \rho_{m} \mathbf{b} d \Omega \tag{13}
\end{align*}
$$

$$
\begin{align*}
& \sum_{a=1}^{N}\left(\int_{\Omega^{c}}(1 / Q) d \Omega \phi_{p}^{a}\left(X_{0}^{c}\right)+\int_{\Omega^{c}}(1 / Q) \bar{x}_{\beta} d \Omega \phi_{p, \beta}^{a}\left(X_{0}^{c}\right)\right) \dot{\hat{p}}^{a}+ \\
& -\sum_{a=1}^{N}\left(\int_{L^{c}+\Gamma_{p}^{c}} \mathbf{n}^{T} \kappa d \Gamma \mathbf{s}_{p}^{a T}\left(X_{0}^{c}\right)+\int_{L^{c}+\Gamma_{p}^{c}} \mathbf{n}^{T} \kappa \bar{x}_{\beta} d \Gamma \mathbf{s}_{p}^{a T}, \beta\right.  \tag{14}\\
& \left.+\sum_{a=1}^{N}\left(X_{\Omega^{c}}^{c}\right)\right) \hat{p}^{a}+ \\
& \left.\alpha d \Omega \mathbf{s}_{u}^{a}\left(X_{0}^{c}\right)+\int_{\Omega^{c}} \alpha \bar{x}_{\beta} d \Omega \mathbf{s}_{u, \beta}^{a}\left(X_{0}^{c}\right)\right) \dot{\hat{\mathbf{u}}}^{a}=\int_{\Gamma_{q}^{c}} \kappa \bar{q} d \Gamma+\int_{\Omega^{c}} a d \Omega
\end{align*}
$$

as well as equations (7)-(8) can be rewritten as:

$$
\begin{align*}
& \sum_{a=1}^{N}\left(\int_{\Omega^{c}} w_{u}^{c} \rho_{m} d \Omega \phi_{u}^{a}\left(X_{0}^{c}\right)+\int_{\Omega^{c}} w_{u}^{c} \rho_{m} \bar{x}_{\beta} d \Omega \phi_{u, \beta}^{a}\left(X_{0}^{c}\right)\right) \ddot{\hat{\mathbf{u}}}^{a}+ \\
& +\sum_{a=1}^{N}\binom{\int_{\Omega^{c}} \mathbf{W}_{u}^{c} \mathbf{D} d \Omega \mathbf{S}_{u}^{a}\left(X_{0}^{c}\right)-\int_{\Gamma_{u}^{c}} w_{u}^{c} \mathbf{N D} d \Gamma \mathbf{S}_{u}^{a}\left(X_{0}^{c}\right)+}{+\int_{\Omega^{c}} \mathbf{W}_{u}^{c} \mathbf{D} \bar{x}_{\beta} d \Omega \mathbf{S}_{u, \beta}^{a}\left(X_{0}^{c}\right)-\int_{\Gamma_{u}^{c}} w_{u}^{c} \mathbf{N D} \bar{x}_{\beta} d \Gamma \mathbf{S}_{u, \beta}^{a}\left(X_{0}^{c}\right)} \hat{\mathbf{u}}^{a}+  \tag{15}\\
& -\sum_{a=1}^{N}\binom{\int_{\Omega^{c}} \mathbf{w}_{u}^{c} \alpha d \Omega \phi_{p}^{a}\left(X_{0}^{c}\right)-\int_{\Gamma_{u}^{c}} w_{u}^{c} \mathbf{n} \alpha d \Gamma \phi_{p}^{a}\left(X_{0}^{c}\right)+}{+\int_{\Omega^{c}} \mathbf{w}_{u}^{c} \alpha \bar{x}_{\beta} d \Omega \phi_{p, \beta}^{a}\left(X_{0}^{c}\right)-\int_{\Gamma_{u}^{c}} w_{u}^{c} \mathbf{n} \alpha \bar{x}_{\beta} d \Gamma \phi_{p, \beta}^{a}\left(X_{0}^{c}\right)} \hat{p}^{a}= \\
& =\int_{\Gamma_{\tau}^{c}} w_{u}^{c} \bar{\tau} d \Gamma+\int_{\Omega^{c}} w_{u}^{c} \rho_{m} \mathbf{b} d \Omega
\end{align*}
$$

$$
\begin{align*}
& \sum_{a=1}^{N}\left(\int_{\Omega^{c}} w_{p}^{c}(1 / Q) d \Omega \phi_{p}^{a}\left(X_{0}^{c}\right)+\int_{\Omega^{c}} w_{p}^{c}(1 / Q) \bar{x}_{\beta} d \Omega \phi_{p, \beta}^{a}\left(X_{0}^{c}\right)\right) \dot{\hat{p}}^{a}+ \\
& +\sum_{a=1}^{N}\binom{\int_{\Omega^{c}} \mathbf{w}_{p}^{c T} \kappa d \Omega \mathbf{s}_{p}^{a T}\left(X_{0}^{c}\right)-\int_{\Gamma_{p}^{c}} w_{p}^{c} \mathbf{n}^{T} \kappa d \Gamma \mathbf{s}_{p}^{a T}\left(X_{0}^{c}\right)+}{+\int_{\Omega^{c}} \mathbf{w}_{p}^{c T} \kappa \bar{x}_{\beta} d \Omega \mathbf{s}_{p, \beta}^{a T}\left(X_{0}^{c}\right)-\int_{\Gamma_{p}^{c}} w_{p}^{c} \mathbf{n}^{T} \kappa \bar{x}_{\beta} d \Gamma_{p}^{a T}{ }_{, \beta}\left(X_{0}^{c}\right)} \hat{p}^{a}+ \\
& +\sum_{a=1}^{N}\left(\int_{\Omega^{c}} w_{p}^{c} \alpha d \Omega \mathbf{s}_{u}^{a}\left(X_{0}^{c}\right)+\int_{\Omega^{c}} w_{p}^{c} \alpha \bar{x}_{\beta} d \Omega \mathbf{s}_{u, \beta}^{a}\left(X_{0}^{c}\right)\right) \dot{\hat{\mathbf{u}}}^{a}=\int_{\Gamma_{q}^{c}} w_{p}^{c} \kappa \bar{q} d \Gamma+\int_{\Omega^{c}} w_{p}^{c} a d \Omega \tag{16}
\end{align*}
$$

in which the integrals can be computed analytically.
Taking into account homogeneous media (just to simplify) and circular integration sub-domains, the related integrals are given by:

$$
\begin{align*}
& \int_{L^{c}} n_{1} d \Gamma=\left(\sin \theta_{B}^{c}-\sin \theta_{A}^{c}\right) r^{c}  \tag{17a}\\
& \int_{L^{c}} n_{2} d \Gamma=\left(\cos \theta_{A}^{c}-\cos \theta_{B}^{c}\right) r^{c}  \tag{17b}\\
& \int_{L^{c}} \bar{x}_{1} n_{1} d \Gamma=\frac{1}{2}\left(\cos \theta_{B}^{c} \sin \theta_{B}^{c}-\cos \theta_{A}^{c} \sin \theta_{A}^{c}+\theta_{B}^{c}-\theta_{A}^{c}\right) r^{c 2}  \tag{17c}\\
& \int_{L^{c}} \bar{x}_{1} n_{2} d \Gamma=\frac{1}{2}\left(\cos ^{2} \theta_{A}^{c}-\cos ^{2} \theta_{B}^{c}\right) r^{c 2}  \tag{17d}\\
& \int_{L^{c}} \bar{x}_{2} n_{1} d \Gamma=\frac{1}{2}\left(\cos ^{2} \theta_{A}^{c}-\cos ^{2} \theta_{B}^{c}\right) r^{c 2}  \tag{17e}\\
& \int_{L^{c}} \bar{x}_{2} n_{2} d \Gamma=\frac{1}{2}\left(\cos \theta_{A}^{c} \sin \theta_{A}^{c}-\cos \theta_{B}^{c} \sin \theta_{B}^{c}+\theta_{B}^{c}-\theta_{A}^{c}\right) r^{c 2} \tag{17f}
\end{align*}
$$

$$
\begin{equation*}
\int_{\Omega^{c}} d \Omega=\frac{1}{2}\left(\theta_{B}^{c}-\theta_{A}^{c}\right) r^{c 2} \tag{18a}
\end{equation*}
$$

$$
\begin{equation*}
\int_{\Omega^{c}} \bar{x}_{1} d \Omega=\frac{1}{3}\left(\sin \theta_{B}^{c}-\sin \theta_{A}^{c}\right) r^{c 3} \tag{18b}
\end{equation*}
$$

$$
\begin{equation*}
\int_{\Omega^{c}} \bar{x}_{2} d \Omega=\frac{1}{3}\left(\cos \theta_{A}^{c}-\cos \theta_{B}^{c}\right) r^{c 3} \tag{18c}
\end{equation*}
$$

$$
\begin{align*}
& \int_{\Omega^{c}} w_{, 1}^{c} d \Omega=\frac{c^{c} \varepsilon\left[r^{c} / c^{c}\right]-2 r^{c} \exp \left[-\left(r^{c} / c^{c}\right)^{2}\right]}{1-\exp \left[-\left(r^{c} / c^{c}\right)^{2}\right]} \frac{1}{2}\left(\sin \theta_{A}^{c}-\sin \theta_{B}^{c}\right) \\
& \int_{\Omega^{c}} w_{, 2}^{c} d \Omega=\frac{c^{c} \varepsilon\left[r^{c} / c^{c}\right]-2 r^{c} \exp \left[-\left(r^{c} / c^{c}\right)^{2}\right]}{1-\exp \left[-\left(r^{c} / c^{c}\right)^{2}\right]} \frac{1}{2}\left(\cos \theta_{B}^{c}-\cos \theta_{A}^{c}\right) \\
& \int_{\Omega^{c}} \bar{x}_{1} w_{, 1}^{c} d \Omega= \\
& \quad=\frac{c^{c 2}-\left(c^{c 2}+r^{c 2}\right) \exp \left[-\left(r^{c} / c^{c}\right)^{2}\right]}{1-\exp \left[-\left(r^{c} / c^{c}\right)^{2}\right]} \frac{1}{2}\left(\cos \theta_{A}^{c} \sin \theta_{A}^{c}-\cos \theta_{B}^{c} \sin \theta_{B}^{c}+\theta_{A}^{c}-\theta_{B}^{c}\right) \tag{19c}
\end{align*}
$$

$$
\int_{\Omega^{c}} \bar{x}_{1} w_{, 2}^{c} d \Omega=\frac{c^{c 2}-\left(c^{c 2}+r^{c 2}\right) \exp \left[-\left(r^{c} / c^{c}\right)^{2}\right]}{1-\exp \left[-\left(r^{c} / c^{c}\right)^{2}\right]} \frac{1}{2}\left(\cos ^{2} \theta_{B}^{c}-\cos ^{2} \theta_{A}^{c}\right)
$$

$$
\int_{\Omega^{c}} \bar{x}_{2} w_{, 1}^{c} d \Omega=\frac{c^{c 2}-\left(c^{c 2}+r^{c 2}\right) \exp \left[-\left(r^{c} / c^{c}\right)^{2}\right]}{1-\exp \left[-\left(r^{c} / c^{c}\right)^{2}\right]} \frac{1}{2}\left(\cos ^{2} \theta_{B}^{c}-\cos ^{2} \theta_{A}^{c}\right)
$$

$$
\int_{\Omega^{c}} \bar{x}_{2} w_{, 2}^{c} d \Omega=
$$

$$
\begin{equation*}
=\frac{c^{c 2}-\left(c^{c 2}+r^{c 2}\right) \exp \left[-\left(r^{c} / c^{c}\right)^{2}\right]}{1-\exp \left[-\left(r^{c} / c^{c}\right)^{2}\right]} \frac{1}{2}\left(\cos \theta_{B}^{c} \sin \theta_{B}^{c}-\cos \theta_{A}^{c} \sin \theta_{A}^{c}+\theta_{A}^{c}-\theta_{B}^{c}\right) \tag{19f}
\end{equation*}
$$

$$
\begin{equation*}
\int_{\Omega^{c}} w^{c} d \Omega=\frac{c^{c 2}-\left(c^{c 2}+r^{c 2}\right) \exp \left[-\left(r^{c} / c^{c}\right)^{2}\right]}{1-\exp \left[-\left(r^{c} / c^{c}\right)^{2}\right]} \frac{1}{2}\left(\theta_{B}^{c}-\theta_{A}^{c}\right) \tag{20a}
\end{equation*}
$$

$$
\begin{equation*}
\int_{\Omega^{c}} \bar{x}_{1} w^{c} d \Omega=\frac{3 c^{c} \varepsilon\left[r^{c} / c^{c}\right]-\left(4 r^{c 3}+6 r^{c} c^{c 2}\right) \exp \left[-\left(r^{c} / c^{c}\right)^{2}\right]}{1-\exp \left[-\left(r^{c} / c^{c}\right)^{2}\right]} \frac{1}{12}\left(\sin \theta_{B}^{c}-\sin \theta_{A}^{c}\right) \tag{20b}
\end{equation*}
$$

$$
\int_{\Omega^{c}} \bar{x}_{2} w^{c} d \Omega=\frac{3 c^{c} \varepsilon\left[r^{c} / c^{c}\right]-\left(4 r^{c 3}+6 r^{c} c^{c 2}\right) \exp \left[-\left(r^{c} / c^{c}\right)^{2}\right]}{1-\exp \left[-\left(r^{c} / c^{c}\right)^{2}\right]} \frac{1}{12}\left(\cos \theta_{A}^{c}-\cos \theta_{B}^{c}\right)
$$

where the sub-domains $\Omega^{c}$ are defined by the angles $\theta_{A}^{c}$ (initial) and $\theta_{B}^{c}$ (final) as well as by the radius $r^{c}$. Equations (17) and (18) are related to the integrals of the matrices presented in equations (13)-(14), whereas equations (19) and (20) are related to the integrals of the matrices presented in equations (15)-(16). Since relation (A1) is being employed to exactly enforce the Dirichlet conditions over $\Gamma_{u}$ and $\Gamma_{p}$, the integrals over $\Gamma_{u}$ and $\Gamma_{p}$ that appear in equations (13)-(16) will rarely occur and they can be computed numerically, whenever necessary.
The function $\varepsilon$ that appears in equations (19a-b) and (20b-c) is defined as:
$\varepsilon(x)=\sqrt{\pi} \int_{0}^{x} \exp \left[-\xi^{2}\right] d \xi$
which is computed numerically here.
Once equations (17)-(20) are presented, and the expressions for the derivatives of the approximation functions $\phi^{a}$ are available as described by equations (A6) and (A9), the matrices related to equations (13)-(14) can be computed as follows (disregarding the integrals along $\Gamma_{u}$ and $\Gamma_{p}$ ):

$$
\begin{align*}
\mathbf{K}_{c a}= & -\left[\begin{array}{ccc}
e q \cdot(17 \mathrm{a}) & 0 & e q \cdot(17 \mathrm{~b}) \\
0 & e q \cdot(17 \mathrm{~b}) & e q \cdot(17 \mathrm{a})
\end{array}\right]_{u} \mathbf{D}\left[\begin{array}{cc}
\phi_{u, 1}^{a}\left(X_{0}^{c}\right) & 0 \\
0 & \phi_{u, 2}^{a}\left(X_{0}^{c}\right) \\
\phi_{u, 2}^{a}\left(X_{0}^{c}\right) & \phi_{u, 1}^{a}\left(X_{0}^{c}\right)
\end{array}\right]+ \\
& -\left[\begin{array}{ccc}
e q \cdot(17 \mathrm{c}) & 0 & e q \cdot(17 \mathrm{~d}) \\
0 & e q \cdot(17 \mathrm{~d}) & e q \cdot(17 \mathrm{c})
\end{array}\right]_{u} \mathbf{D}\left[\begin{array}{cc}
\phi_{u, 11}^{a}\left(X_{0}^{c}\right) & 0 \\
0 & \phi_{u, 21}^{a}\left(X_{0}^{c}\right) \\
\phi_{u, 21}^{a}\left(X_{0}^{c}\right) & \phi_{u, 11}^{a}\left(X_{0}^{c}\right)
\end{array}\right]+  \tag{22a}\\
& -\left[\begin{array}{ccc}
e q \cdot(17 \mathrm{e}) & 0 & e q \cdot(17 \mathrm{f}) \\
0 & e q \cdot(17 \mathrm{f}) & e q \cdot(17 \mathrm{e})
\end{array}\right]_{u} \mathbf{D}\left[\begin{array}{cc}
\phi_{u, 12}^{a}\left(X_{0}^{c}\right) & 0 \\
0 & \phi_{u, 22}^{a}\left(X_{0}^{c}\right) \\
\phi_{u, 22}^{a}\left(X_{0}^{c}\right) & \phi_{u, 12}^{a}\left(X_{0}^{c}\right)
\end{array}\right] \\
\mathbf{M}_{c a} & =\left[\begin{array}{cc}
e q \cdot(18 \mathrm{a}) & 0 \\
0 & e q \cdot(18 \mathrm{a})
\end{array}\right]_{u} \rho_{m} \phi_{u}^{a}\left(X_{0}^{c}\right)+ \\
& +\left[\begin{array}{cc}
e q \cdot(18 \mathrm{~b}) & 0 \\
0 & e q \cdot(18 \mathrm{~b})
\end{array}\right]_{u} \rho_{m} \phi_{u, 1}^{a}\left(X_{0}^{c}\right)+\left[\begin{array}{cc}
e q \cdot(18 \mathrm{c}) & 0 \\
0 & e q \cdot(18 \mathrm{c})
\end{array}\right]_{u} \rho_{m} \phi_{u, 2}^{a}\left(X_{0}^{c}\right) \tag{22b}
\end{align*}
$$

$$
\mathbf{Q}_{c a}=-\left[\begin{array}{l}
e q .(17 a) \\
e q .(17 b)
\end{array}\right]_{u} \alpha \phi_{p}^{a}\left(X_{0}^{c}\right)-\left[\begin{array}{l}
e q .(17 c) \\
e q \cdot(17 d)
\end{array}\right]_{u} \alpha \phi_{p, 1}^{a}\left(X_{0}^{c}\right)-\left[\begin{array}{l}
e q .(17 e) \\
e q .(17 f)
\end{array}\right]_{u} \alpha \phi_{p, 2}^{a}\left(X_{0}^{c}\right)
$$

$$
\mathbf{G}_{c a}=[e q \cdot(18 a)]_{p} \alpha\left[\phi_{u, 1}^{a}\left(X_{0}^{c}\right) \quad \phi_{u, 2}^{a}\left(X_{0}^{c}\right)\right]+
$$

$$
\begin{equation*}
+[e q \cdot(18 b)]_{p} \alpha\left[\phi_{u, 11}^{a}\left(X_{0}^{c}\right) \quad \phi_{u, 21}^{a}\left(X_{0}^{c}\right)\right]+[e q .(18 c)]_{p} \alpha\left[\phi_{u, 12}^{a}\left(X_{0}^{c}\right) \quad \phi_{u, 22}^{a}\left(X_{0}^{c}\right)\right] \tag{22f}
\end{equation*}
$$

and the matrices related to equations (15)-(16) can be computed as follows (also disregarding the integrals along $\Gamma_{u}$ and $\Gamma_{p}$ ):

$$
\begin{align*}
\mathbf{K}_{c a} & =\left[\begin{array}{ccc}
e q \cdot(19 \mathrm{a}) & 0 & e q \cdot(19 \mathrm{~b}) \\
0 & e q \cdot(19 \mathrm{~b}) & e q \cdot(19 \mathrm{a})
\end{array}\right]_{u} \mathbf{D}\left[\begin{array}{cc}
\phi_{u, 1}^{a}\left(X_{0}^{c}\right) & 0 \\
0 & \phi_{u, 2}^{a}\left(X_{0}^{c}\right) \\
\phi_{u, 2}^{a}\left(X_{0}^{c}\right) & \phi_{u, 1}^{a}\left(X_{0}^{c}\right)
\end{array}\right]+ \\
& +\left[\begin{array}{ccc}
e q \cdot(19 \mathrm{c}) & 0 & e q \cdot(19 \mathrm{~d}) \\
0 & e q \cdot(19 \mathrm{~d}) & e q \cdot(19 \mathrm{c})
\end{array}\right]_{u} \mathbf{D}\left[\begin{array}{cc}
\phi_{u, 11}^{a}\left(X_{0}^{c}\right) & 0 \\
0 & \phi_{u, 21}^{a}\left(X_{0}^{c}\right) \\
\phi_{u, 21}^{a}\left(X_{0}^{c}\right) & \phi_{u, 11}^{a}\left(X_{0}^{c}\right)
\end{array}\right]+  \tag{23a}\\
& +\left[\begin{array}{ccc}
e q \cdot(19 \mathrm{e}) & 0 & e q \cdot(19 \mathrm{f}) \\
0 & e q \cdot(19 \mathrm{f}) & e q \cdot(19 \mathrm{e})
\end{array}\right]_{u} \mathbf{D}\left[\begin{array}{cc}
\phi_{u, 12}^{a}\left(X_{0}^{c}\right) & 0 \\
0 & \phi_{u, 22}^{a}\left(X_{0}^{c}\right) \\
\phi_{u, 22}^{a}\left(X_{0}^{c}\right) & \phi_{u, 12}^{a}\left(X_{0}^{c}\right)
\end{array}\right] \\
\mathbf{M}_{c a} & =\left[\begin{array}{cc}
e q \cdot(20 \mathrm{a}) & 0 \\
0 & e q \cdot(20 \mathrm{a})
\end{array}\right]_{u} \rho_{m} \phi_{u}^{a}\left(X_{0}^{c}\right)+ \\
& +\left[\begin{array}{cc}
e q \cdot(20 \mathrm{~b}) & 0 \\
0 & e q \cdot(20 \mathrm{~b})
\end{array}\right]_{u} \rho_{m} \phi_{u, 1}^{a}\left(X_{0}^{c}\right)+\left[\begin{array}{cc}
e q \cdot(20 \mathrm{c}) & 0 \\
0 & e q \cdot(20 \mathrm{c})
\end{array}\right]_{u} \rho_{m} \phi_{u, 2}^{a}\left(X_{0}^{c}\right) \tag{23b}
\end{align*}
$$

$$
\mathbf{Q}_{c a}=\left[\begin{array}{l}
e q .(19 a) \\
e q \cdot(19 b)
\end{array}\right]_{u} \alpha \phi_{p}^{a}\left(X_{0}^{c}\right)+\left[\begin{array}{l}
e q \cdot(19 c) \\
e q \cdot(19 d)
\end{array}\right]_{u} \alpha \phi_{p, 1}^{a}\left(X_{0}^{c}\right)+\left[\begin{array}{l}
e q \cdot(19 e) \\
e q \cdot(19 f)
\end{array}\right]_{u} \alpha \phi_{p, 2}^{a}\left(X_{0}^{c}\right)
$$

$$
\begin{align*}
& \mathbf{H}_{c a}=-\left[\begin{array}{ll}
e q .(17 \mathrm{a}) & e q .(17 \mathrm{~b})
\end{array}\right]_{p} \kappa\left[\begin{array}{l}
\phi_{p, 1}^{a}\left(X_{0}^{c}\right) \\
\phi_{p, 2}^{a}\left(X_{0}^{c}\right)
\end{array}\right]+ \\
& -\left[\begin{array}{ll}
e q .(17 \mathrm{c}) & e q .(17 \mathrm{~d})]_{p} \kappa\left[\begin{array}{l}
\phi_{p, 11}^{a}\left(X_{0}^{c}\right) \\
\phi_{p, 21}^{a}\left(X_{0}^{c}\right)
\end{array}\right]+
\end{array}\right.  \tag{22d}\\
& -\left[\begin{array}{ll}
\text { eq.(17e) } & \left.e q .(17 \mathrm{f})]_{p} \kappa\left[\begin{array}{l}
\phi_{p, 12}^{a}\left(X_{0}^{c}\right) \\
\phi_{p, 22}^{a}\left(X_{0}^{c}\right)
\end{array}\right], ~\right]
\end{array}\right. \\
& \begin{aligned}
\mathbf{C}_{c a} & =[e q \cdot(18 \mathrm{a})]_{p}(1 / Q) \phi_{p}^{a}\left(X_{0}^{c}\right)+ \\
& +[e q \cdot(18 \mathrm{~b})]_{p}(1 / Q) \phi_{p, 1}^{a}\left(X_{0}^{c}\right)+[e q \cdot(18 \mathrm{c})]_{p}(1 / Q) \phi_{p, 2}^{a}\left(X_{0}^{c}\right)
\end{aligned} \tag{22e}
\end{align*}
$$

$$
\begin{align*}
& \mathbf{H}_{c a}=\left[\begin{array}{ll}
\text { eq.(19a) } & \text { eq.(19b) }]_{p} \kappa\left[\begin{array}{l}
\phi_{p, 1}^{a}\left(X_{0}^{c}\right) \\
\phi_{p, 2}^{a}\left(X_{0}^{c}\right)
\end{array}\right]+, ~+~+~
\end{array}\right. \\
& +[\text { eq.(19c) } \quad \text { eq.(19d) }]_{p} \kappa\left[\begin{array}{c}
\phi_{p, 11}^{a}\left(X_{0}^{c}\right) \\
\phi_{p, 21}^{a}\left(X_{0}^{c}\right)
\end{array}\right]+  \tag{23d}\\
& +[\text { eq.(19e) } \quad \text { eq.(19f) }]_{p} \kappa\left[\begin{array}{l}
\phi_{p, 12}^{a}\left(X_{0}^{c}\right) \\
\phi_{p, 22}^{a}\left(X_{0}^{c}\right)
\end{array}\right] \\
& \mathbf{C}_{c a}=[e q .(20 \mathrm{a})]_{p}(1 / Q) \phi_{p}^{a}\left(X_{0}^{c}\right)+  \tag{23e}\\
& +[e q \cdot(20 \mathrm{~b})]_{p}(1 / Q) \phi_{p, 1}^{a}\left(X_{0}^{c}\right)+[e q \cdot(20 \mathrm{c})]_{p}(1 / Q) \phi_{p, 2}^{a}\left(X_{0}^{c}\right)
\end{align*}
$$

$$
\begin{align*}
\mathbf{G}_{c a} & =[e q \cdot(20 a)]_{p} \alpha\left[\phi_{u, 1}^{a}\left(X_{0}^{c}\right)\right. \\
& \left.\phi_{u, 2}^{a}\left(X_{0}^{c}\right)\right]+  \tag{23f}\\
& +[e q \cdot(20 b)]_{p} \alpha\left[\phi_{u, 11}^{a}\left(X_{0}^{c}\right)\right. \\
\left.\phi_{u, 21}^{a}\left(X_{0}^{c}\right)\right]+[e q \cdot(20 c)]_{p} \alpha\left[\phi_{u, 12}^{a}\left(X_{0}^{c}\right)\right. & \left.\phi_{u, 22}^{a}\left(X_{0}^{c}\right)\right]
\end{align*}
$$

where, once again, subscripts $u$ and $p$ are relative to the solid and fluid phase discretizations, respectively.
Expressions (22) and (23) are very computationally efficient, providing more attractive procedures than those based on standard numerical integration techniques, which may be quite computationally demanding, especially if domain integrals are involved. A low order approximation in the Taylor series extension (as the considered first order approximation) is reasonable in this kind of formulations, since the local fictitious sub-domains $\Omega^{c}$ can be considered of very reduced size. However, the dependence of the formulation on higher order derivatives of the shape function may decrease the accuracy of the procedure or, even worse, render unstable analyses.

In order to avoid this drawback, which can be quite destructive considering hyperbolic models, Soares et al. (2011) proposed a modified computation of the derivatives of the shape function, adding a correctional term to the standard derivatives expressions. In this case, the modified derivatives are computed as follows:
$\bar{\phi}_{, i}^{a}(\mathbf{x})=\phi_{, i}^{a}(\mathbf{x})-s_{i} \phi^{a}(\mathbf{x})$
$\bar{\phi}_{, i j}^{a}(\mathbf{x})=\phi_{, i j}^{a}(\mathbf{x})-s_{i j} \phi^{a}(\mathbf{x})$
where the correcting terms $s_{i}$ and $s_{i j}$ are given by:
$s_{i}=\sum_{a=1}^{N} \phi_{, i}^{a}(\mathbf{x})$
$s_{i j}=\sum_{a=1}^{N} \phi_{, i j}^{a}(\mathbf{x})$
Taking into account equations (24)-(25), it is ensured that $\sum_{a=1}^{N} \bar{\phi}_{, i}^{a}(\mathbf{x})=0$ and $\sum_{a=1}^{N} \bar{\phi}_{, i j}^{a}(\mathbf{x})=$ 0 (one should keep in mind that $\sum_{a=1}^{N} \phi^{a}(\mathbf{x})=1$ ), which improves the accuracy and stability of the method, avoiding the destructive influence of accumulative residual errors in hyperbolic models.

## 5 Temporal discretization

For the temporal discretization, the following one-step finite difference approximations are considered (generalized Newmark method):
$\ddot{\hat{\mathbf{U}}}^{n}=\left(1 /\left(\gamma_{2} \Delta t^{2}\right)\right)\left(\hat{\mathbf{U}}^{n}-\hat{\mathbf{U}}^{n-1}\right)-\left(1 /\left(\gamma_{2} \Delta t\right)\right) \dot{\hat{\mathbf{U}}}^{n-1}+\left(1-1 /\left(2 \gamma_{2}\right)\right) \ddot{\hat{\mathbf{U}}}^{n-1}$
$\dot{\hat{\mathbf{U}}}^{n}=\left(\gamma_{1} /\left(\gamma_{2} \Delta t\right)\right)\left(\hat{\mathbf{U}}^{n}-\hat{\mathbf{U}}^{n-1}\right)+\left(1-\gamma_{1} / \gamma_{2}\right) \dot{\hat{\mathbf{U}}}^{n-1}+\Delta t\left(1-\gamma_{1} /\left(2 \gamma_{2}\right)\right) \ddot{\hat{\mathbf{U}}}^{n-1}$
$\dot{\hat{\mathbf{P}}}^{n}=\left(1 /\left(\gamma_{3} \Delta t\right)\right)\left(\hat{\mathbf{P}}^{n}-\hat{\mathbf{P}}^{n-1}\right)+\left(1-1 / \gamma_{3}\right) \dot{\hat{\mathbf{P}}}^{n-1}$
where $\Delta t$ is the selected time-step and $\hat{\mathbf{U}}^{n}$ stands for a numerical approximation of $\hat{\mathbf{U}}\left(t_{n}\right)$. For an unconditionally stable scheme, the relations $\gamma_{1} \geq 0.5, \gamma_{2} \geq 0.5 \gamma_{1}$ and $\gamma_{3} \geq 0.5$ must hold in equations (26), where $\gamma_{1}, \gamma_{2}$ and $\gamma_{3}$ are the parameters of the time integration method.
By introducing relations (26) into equations (10)-(11), the following final system of equations is obtained:

$$
\left[\begin{array}{cc}
\left(1 /\left(\gamma_{2} \Delta t^{2}\right)\right) \mathbf{M}+\mathbf{K} & -\mathbf{Q}  \tag{27}\\
\left(\gamma_{1} /\left(\gamma_{2} \Delta t\right)\right) \mathbf{G} & \left(1 /\left(\gamma_{3} \Delta t\right)\right) \mathbf{C}+\mathbf{H}
\end{array}\right]\left[\begin{array}{c}
\hat{\mathbf{U}}^{n} \\
\hat{\mathbf{P}}^{n}
\end{array}\right]=\left[\begin{array}{c}
\overline{\mathbf{F}}^{n} \\
\overline{\mathbf{R}}^{n}
\end{array}\right]
$$

where the r.h.s. of equation (27) is defined by:

$$
\begin{align*}
\overline{\mathbf{F}}^{n}= & \mathbf{F}^{n}+\mathbf{M}\left(\left(1 /\left(\gamma_{2} \Delta t^{2}\right)\right) \hat{\mathbf{U}}^{n-1}+\left(1 /\left(\gamma_{2} \Delta t\right)\right) \dot{\hat{\mathbf{U}}}^{n-1}+\left(1 /\left(2 \gamma_{2}\right)-1\right) \ddot{\hat{\mathbf{U}}}^{n-1}\right)  \tag{28a}\\
\overline{\mathbf{R}}^{n} & =\mathbf{R}^{n}+\mathbf{C}\left(\left(1 /\left(\gamma_{3} \Delta t\right)\right) \hat{\mathbf{P}}^{n-1}+\left(1 / \gamma_{3}-1\right) \mathbf{P}^{n-1}\right)+ \\
& +\mathbf{G}\left(\left(\gamma_{1} /\left(\gamma_{2} \Delta t\right)\right) \hat{\mathbf{U}}^{n-1}+\left(\gamma_{1} / \gamma_{2}-1\right) \hat{\mathbf{U}}^{n-1}+\Delta t\left(\gamma_{1} /\left(2 \gamma_{2}\right)-1\right) \ddot{\hat{\mathbf{U}}}^{n-1}\right) \tag{28b}
\end{align*}
$$

Solving the algebraic system of equations (27) at each time-step of the analysis enables the computation of the solid skeleton displacement and interstitial fluid pore-pressure time-histories.

## 6 Numerical applications

Two numerical applications are considered here, illustrating the discussed methodologies. In the first application, the simulation of a one-dimensional problem is focused, and a soil column is analysed taking into account static and dynamic behaviour. In the second application, a two-dimensional soil strip is considered. The results obtained by the proposed MLPG formulations are compared with analytical answers, whenever possible, and with results provided by the Finite Element Method (FEM).

The following nomenclature is adopted here, concerning the meshless formulations in focus: (i) "MLPG1" denotes the MLPG formulation employing Heaviside test functions; (ii) "MLPG2" denotes the MLPG formulation that employs the MLS weight functions as the test functions. Moreover, the following nomenclature is also considered: (i) "Numerical" denotes the standard MLPG formulation, as presented in section 3 (with the computation of the matrices taking into account GaussLegendre quadrature); (ii) "Analytical" denotes the modified MLPG formulation, as presented in section 4 , taking into account the modified computation of the shape function derivatives (equations (24)-(25)).

Along the following analyses, the radii of the influence domain and of the local subdomain are set to $\theta_{x} d_{i}^{3}$ and $\theta_{s} d_{i}^{1}$, respectively; where $d_{i}^{3}$ and $d_{i}^{1}$ are the distances to the third and first nearest points from node $i$, respectively. In all the applications that follow, $\theta_{x}=4.0$ is selected. Considering the MLPG1, $\theta_{s}=0.6$ is adopted, whereas, for the MLPG2, $\theta_{s}=1.0$ is employed along this section.

### 6.1 Soil column

In this first example, a soil column is analysed (Biot, 1940; de Boer et al., 1993; Soares, 2010a; 2010b). A sketch of the model is depicted in Fig.1. The top surface of the column is considered drained and uniformly loaded. The other surfaces of the model are undrained and have null normal displacements prescribed. The physical properties of the soil (which is incompressible) are: $v=0.3$ (Poisson); $E=14515880 \mathrm{~N} / \mathrm{m}^{2}$ (Young Modulus); $\rho_{s}=2000 \mathrm{~kg} / \mathrm{m}^{3}$ (mass density solid phase); $\rho_{f}=1000 \mathrm{~kg} / \mathrm{m}^{3}$ (mass density - fluid phase); $v=0.33$ (porosity); $\kappa=10^{-6} \mathrm{~m}^{4} / N s$ (permeability). The geometry of the column is defined by: $\mathrm{H}=10 \mathrm{~m}$.

For this configuration, exact answers for the vertical displacements at the top of the column are available, considering static and dynamic behaviour, and they are


Figure 1: Sketch of the soil column model.
expressed by:
$u_{e}(t)=\frac{8 A \mathrm{H}}{\pi^{2} c_{1}} \cdot \sum_{n=1}^{\infty}\left[\frac{1}{(2 n+1)^{2}}\left(1-\exp \left[-\left(\frac{(2 n+1) \pi}{2 \mathrm{H}}\right)^{2} c_{2} t\right]\right)\right]$
$u_{e}(t)=\frac{A}{c_{1} \sqrt{c_{3}}} \cdot \int_{0}^{t} \exp \left[-\frac{c_{2} \tau}{2 c_{3}}\right] I_{0}\left[\frac{c_{2} \tau}{2 c_{3}}\right] H[\tau] f(t-\tau) d \tau$
where equation (29a) is relative to the consolidation analysis of a column of height H (Biot, 1940) and equation (29b) is related to the poroelastodynamic analysis of a column of infinite height (de Boer et al., 1993) (in this case, numerical analyses are carried out just until the wave fronts reflected from the bottom of the column of height H do not interfere into the answers at the top of the column). The constants described in equations (29) are defined by: $c_{1}=E(1-v) /((1+v)(1-2 v)) ; c_{2}=$ $1 /\left(\kappa c_{1}\right) ; c_{3}=\left(\rho_{f}(1-v)^{2}+\rho_{s} v^{2}\right) /\left(v^{2} c_{1}\right)$. Still in equations (29), $A$ stands for the load amplitude and $f(t)$ for its time variation (adopted constant here) and $I_{0}$ stands for the modified Bessel function of the first kind and zero order.

Three spatial/temporal discretizations are considered for the numerical analyses, taking into account different refinement levels. These discretizations are defined by: (i) discretization $1-\Delta h=1.00 \mathrm{~m}$ and $\Delta t=0.2 s$ (static analysis) or $\Delta t=0.0016 s$ (dynamic analysis); (ii) discretization $2-\Delta h=0.50 \mathrm{~m}$ and $\Delta t=0.1 \mathrm{~s}$ (static analysis) or $\Delta t=0.0008 s$ (dynamic analysis); and (iii) discretization $3-\Delta h=0.25 m$ and $\Delta t$
$=0.05 s$ (static analysis) or $\Delta t=0.0004 s$ (dynamic analysis); where $\Delta h$ stands for a characteristic distance between the nodes in the spatial discretization. For the static analyses in focus, the algorithm described along the current work is considered, adopting null mass matrices.

(a)

(c)

(e)

(b)

(d)

(f)

Figure 2: Vertical displacements at point A for the static soil column: (a) discretization 1 and MLPG1; (b) discretization 1 and MLPG2; (c) discretization 2 and MLPG1; (d) discretization 2 and MLPG2; (e) discretization 3 and MLPG1; (f) discretization 3 and MLPG2.


Figure 3: Vertical displacements at point A for the dynamic soil column: (a) discretization 1 and MLPG1; (b) discretization 1 and MLPG2; (c) discretization 2 and MLPG1; (d) discretization 2 and MLPG2; (e) discretization 3 and MLPG1; (f) discretization 3 and MLPG2.

Fig. 2 shows the vertical displacement time-histories obtained at the top of the column by the MLPG1 and MLPG2, considering static analyses. Analytical time histories are also depicted in the figure (equation (29a)), for comparison. In Fig.3, analogous results are presented considering poroelastodynamic analyses.


Figure 4: Analysis of convergence and accuracy (error vs. discretization) considering static analyses: (a) MLPG1; (b) MLPG2.


Figure 5: Analysis of convergence and accuracy (error vs. discretization) considering dynamic analyses: (a) MLPG1; (b) MLPG2.

Convergence analyses are presented in Figs. 4 and 5. In this case, the following expression is considered for the error evaluation:
$\zeta(\mathbf{x})=\left[\sum_{j=1}^{N_{t}}\left(u\left(t_{j}\right)-u_{e}\left(t_{j}\right)\right)^{2} / \sum_{j=1}^{N_{t}}\left(u_{e}\left(t_{j}\right)\right)^{2}\right]^{1 / 2}$
where $u$ and $u_{e}$ stand for the computed and the exact answers, respectively, and $N_{t}$ is the total number of time-steps in the analysis.
As one can observe in the figures, convergence is achieved and modified MLPG formulations provides more accurate results than standard MLPG formulations, for both the static and dynamic cases. The efficiency of the proposed technique is also remarkable. In Tab.1, the relations between the CPU times (Analytical/Numerical)

Table 1: CPU time relation between the modified and the standard (Analyti$\mathrm{cal} /$ Numerical) MLPG formulations.

| Discretization | MLPG1 | MLPG2 |
| :---: | :---: | :---: |
| 1 | 0.1167 | 0.0919 |
| 2 | 0.0548 | 0.0454 |
| 3 | 0.0381 | 0.0317 |

related to the computation of the involved matrices are presented, taking into account the MLPG1 and the MLPG2. As one can observe, the modified approach is dozens of times computationally faster than the standard procedure, rendering an extremely attractive technique.


Figure 6: Sketch of the soil strip model.

### 6.2 Soil Strip

In this second example, a two-dimensional soil strip is analysed (Li et al., 2003; Soares, 2010a; 2010b). A sketch of the model is depicted in Fig.6. The geometry of the strip is defined by $a=5 m, b=10 m$ and $c=1 m$. The symmetry of the model is taken into account and 961 nodes are employed to spatially discretize the solid phase and 441 nodes are employed to spatially discretize the fluid phase. An important feature of meshless techniques is that they easily allow the adoption of different phase discretizations (Soares, 2010a; 2010b), a task which may be quite complex considering some mesh-based formulations, such as the Finite Element


Figure 7: Vertical displacements at point A for the compressible and permeable soil strip: (a) MLPG1; (b) MLPG2.


Figure 8: Vertical displacements at point A for the incompressible and impermeable soil strip: (a) MLPG1; (b) MLPG2.

Method. The soil strip is loaded as indicted in Fig. 6 and the adopted time-step is $\Delta t=5 \cdot 10^{-4} s$. The soil is compressible (fluid phase) and permeable: $v=0.2$; $E=10^{7} \mathrm{~N} / \mathrm{m}^{2} ; \rho_{s}=2538.5 \mathrm{~kg} / \mathrm{m}^{3} ; \rho_{f}=1000 \mathrm{~kg} / \mathrm{m}^{3} ; v=0.35 ; \kappa=10^{-7} \mathrm{~m}^{4} / \mathrm{Ns}$ and $K_{f}=3.3 \cdot 10^{9} \mathrm{~N} / \mathrm{m}^{2}$.
Vertical displacements at point A (see Fig.6) are depicted in Fig.7, considering the MLPG1 and MLPG2; FEM results are also plotted in the figure, for comparison. As can be observed, the results provided by the MLPG formulations are in good agreement among themselves and with those provided by the FEM. The adoption of different spatial discretizations for the solid and fluid phases is a very appropriate procedure for pore-mechanic analyses: not only it is more flexible and renders a smaller system of coupled equations (providing more efficient analyses), maintaining the good accuracy of the results, but also it allows the numerical simulation
of impermeable and incompressible media, as is described in Fig.8.
Fig. 8 depicts the vertical displacement time-history results at point A considering the soil as impermeable and incompressible (i.e., $\kappa=0$ and $K_{s}=K_{f}=\infty$ ). In a finite element context, for instance, in the limit of zero compressibility of water and soil grains and zero permeability, the functions used to interpolate displacements and pressures must fulfil either the Babuska-Brezzi conditions (Babuska, 1973; Brezzi, 1974) or the simpler patch test proposed by Zienkiewicz et al. (1986): these requirements exclude the use of elements with equal order interpolation for pressures and displacements, for which spurious oscillations may appear. Thus, considering mesh-based formulations, this kind of analysis may become quite complex. On the other hand, taking into account MLPG formulations, the adoption of different discretizations for each phase of the model is trivial, allowing appropriate numerical simulations of impermeable and incompressible soil models.

## 7 Conclusions

In this work, time-domain analyses of porous media, taking into account modified MLPG formulations, are discussed. The $\mathbf{u}-p$ formulation is focused and the coupled systems of equations that arise are characterized by incognita vectors whose entries are solid skeleton displacement and fluid interstitial pore-pressure fictitious nodal values. Two modified MLPG formulations (using Heaviside functions and Gaussian weights for the test functions) are discussed, taking into account the introduction of Taylor series expansions and analytical integrations over local sub-domains. Both modified formulations provided very good results, rendering accurate and very efficient analyses.
The expressions for the modified mass, compressibility, stiffness, permeability and coupling matrices, which define the system of equations governing the coupled model, are presented along the paper taking into account the two MLPG formulations. It is important to observe that these expressions are discussed regarding different spatial discretizations for each phase of the model, an approach which is quite hard to be considered taking into account mesh-based formulations. Independent discretizations for each phase of the model allow more flexible and effective analyses, also enabling the simulation of impermeable and incompressible porous media.
Both static and dynamic analyses are carried out in the paper. For the hyperbolic model (poroelastodynamics), the modified methodology becomes more sensitive to the computation of the shape function derivatives. In order to improve the robustness of the technique, easy-to-implement modified expressions for the derivatives are also presented, providing more stable numerical procedures.

As described by the numerical results presented in section 6, the proposed formulation is accurate and extremely efficient, promoting itself as a step forward in the direction of transforming meshless formulations into more competitive techniques, greatly reducing their computational costs (which is one of their main drawbacks), without damaging their high accuracy features.

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## Appendix - Moving Least Squares approximation

Consider a sub-domain $\Omega_{x}$, the neighbourhood of a point $\mathbf{x}$ and denoted as the domain of definition of the MLS approximation for the trial function at $\mathbf{x}$, which is located in the problem domain $\Omega$. To approximate the distribution of function $\psi$ in $\Omega_{x}$ (which may represent a solid skeleton displacement or fluid interstitial porepressure field), over a number of randomly located nodes, the MLS approximation $\psi^{h}$ of $\psi$, can be defined by (Lancaster and Salkauskas, 1981; Atluri and Shen, 2002):
$\psi^{h}(\mathbf{x}, t)=\sum_{a=1}^{N} \phi^{a}(\mathbf{x}) \hat{\psi}^{a}(t)$
where $\hat{\psi}$ is the fictitious nodal value of $\psi(N$ is the number of points in the subdomain $\Omega_{x}$ ) and the shape function $\phi^{a}(\mathbf{x})$ is defined by (sum over Greek indices is assumed implicitly):
$\phi^{a}(\mathbf{x})=p_{\alpha}(\mathbf{x}) A_{\alpha \beta}^{-1}(\mathbf{x}) B_{\beta a}(\mathbf{x})$
where
$A_{\alpha \beta}(\mathbf{x})=\sum_{a=1}^{N} w^{a}(\mathbf{x}) p_{\alpha}\left(\mathbf{x}^{a}\right) p_{\beta}\left(\mathbf{x}^{a}\right)$
$B_{\alpha a}(\mathbf{x})=w^{a}(\mathbf{x}) p_{\alpha}\left(\mathbf{x}^{a}\right)$
and $p_{\alpha}(\mathbf{x})$ stands for a term of a complete monomial basis of order $m . w^{a}(\mathbf{x})$ is the weight function associated with node $a$. The Gaussian weight function is adopted here, and it is given by:
$w^{a}(\mathbf{x})=\frac{\exp \left[-\left(d^{a} / c^{a}\right)^{2}\right]-\exp \left[-\left(r^{a} / c^{a}\right)^{2}\right]}{1-\exp \left[-\left(r^{a} / c^{a}\right)^{2}\right.}\left(1-H\left[d^{a}-r^{a}\right]\right)$
where $d^{a}=\left\|\mathbf{x}-\mathbf{x}^{a}\right\|$ is the distance between the sampling point $\mathbf{x}$ and node $\mathbf{x}^{a}, c^{a}$ is a constant controlling the shape of the weight function and $r^{a}$ is the radius of the circular support of the weight function. The Heaviside unit step function is defined as $H[z]=1$ for $z>0$ and $H[z]=0$ for $z \leq 0$. The size of the weight function support should be large enough to have a sufficient number of nodes covered in the domain of definition to ensure the regularity of matrix $\mathbf{A}$.
First derivatives can be computed as indicated below, taking into account the MLS:
$\psi_{, j}^{h}(\mathbf{x}, t)=\sum_{a=1}^{N} \phi_{, j}^{a}(\mathbf{x}) \hat{\psi}^{a}(t)$
where
$\phi_{, i}^{a}(\mathbf{x})=p_{\alpha, i}(\mathbf{x}) A_{\alpha \beta}^{-1}(\mathbf{x}) B_{\beta a}(\mathbf{x})+p_{\alpha}(\mathbf{x})\left(A_{\alpha \beta, i}^{-1}(\mathbf{x}) B_{\beta a}(\mathbf{x})+A_{\alpha \beta}^{-1}(\mathbf{x}) B_{\beta a, i}(\mathbf{x})\right)$
and
$A_{\alpha \beta, i}(\mathbf{x})=\sum_{a=1}^{N} w_{, i}^{a}(\mathbf{x}) p_{\alpha}\left(\mathbf{x}^{a}\right) p_{\beta}\left(\mathbf{x}^{a}\right)$
$B_{\alpha a, i}(\mathbf{x})=w_{, i}^{a}(\mathbf{x}) p_{\alpha}\left(\mathbf{x}^{a}\right)$
$\left.A_{\alpha \beta, i}^{-1}(\mathbf{x})=-A_{\alpha \gamma}^{-1}(\mathbf{x}) A_{\gamma \lambda, i}^{( } \mathbf{x}\right) A_{\lambda \beta}^{-1}(\mathbf{x})$
Second derivatives are computed as follows:
$\psi_{, j k}^{h}(\mathbf{x}, t)=\sum_{a=1}^{N} \phi_{, j k}^{a}(\mathbf{x}) \hat{\psi}^{a}(t)$
where

$$
\begin{align*}
& \phi_{, i j}^{a}(\mathbf{x})=p_{\alpha, i j}(\mathbf{x}) A_{\alpha \beta}^{-1}(\mathbf{x}) B_{\beta a}(\mathbf{x})+ \\
& +p_{\alpha, i}(\mathbf{x})\left(A_{\alpha \beta, j}^{-1}(\mathbf{x}) B_{\beta a}(\mathbf{x})+A_{\alpha \beta}^{-1}(\mathbf{x}) B_{\beta a, j}(\mathbf{x})\right)+ \\
& +p_{\alpha, j}(\mathbf{x})\left(A_{\alpha \beta, i}^{-1}(\mathbf{x}) B_{\beta a}(\mathbf{x})+A_{\alpha \beta}^{-1}(\mathbf{x}) B_{\beta a, i}(\mathbf{x})\right)+ \\
& +p_{\alpha}(\mathbf{x})\left(A_{\alpha \beta, i j}^{-1}(\mathbf{x}) B_{\beta a}(\mathbf{x})+A_{\alpha \beta, i}^{-1}(\mathbf{x}) B_{\beta a, j}(\mathbf{x})+A_{\alpha \beta, j}^{-1}(\mathbf{x}) B_{\beta a, i}(\mathbf{x})+A_{\alpha \beta}^{-1}(\mathbf{x}) B_{\beta a, i j}(\mathbf{x})\right) \tag{A9}
\end{align*}
$$

and

$$
\begin{align*}
& A_{\alpha \beta, i j}(\mathbf{x})=\sum_{a=1}^{N} w_{, i j}^{a}(\mathbf{x}) p_{\alpha}\left(\mathbf{x}^{a}\right) p_{\beta}\left(\mathbf{x}^{a}\right)  \tag{A10a}\\
& B_{\alpha a, i j}(\mathbf{x})=w_{, i j}^{a}(\mathbf{x}) p_{\alpha}\left(\mathbf{x}^{a}\right) \tag{A10b}
\end{align*}
$$

$A_{\alpha \beta, i j}^{-1}(\mathbf{x})=$
$\left.\left.\left.-A_{\alpha \gamma, j}^{-1}(\mathbf{x}) A_{\gamma \lambda, i}^{( } \mathbf{x}\right) A_{\lambda \beta}^{-1}(\mathbf{x})-A_{\alpha \gamma}^{-1}(\mathbf{x}) A_{\gamma \lambda, i j}^{( } \mathbf{x}\right) A_{\lambda \beta}^{-1}(\mathbf{x})-A_{\alpha \gamma}^{-1}(\mathbf{x}) A_{\gamma \lambda, i}^{( } \mathbf{x}\right) A_{\lambda \beta, j}^{-1}(\mathbf{x})$

The derivatives of the weight function, which appear in equations (A7) and (A10), are given by:
$w_{, i}^{a}(\mathbf{x})=\frac{\left(d^{a} / c^{a}\right)^{2} \exp \left[-\left(d^{a} / c^{a}\right)^{2}\right]}{1-\exp \left[-\left(r^{a} / c^{a}\right)^{2}\right.} \frac{\left(-2 d_{i}^{a}\right)}{\left(d^{a}\right)^{2}}\left(1-H\left[d^{a}-r^{a}\right]\right)$

$$
\begin{equation*}
w_{, i j}^{a}(\mathbf{x})=\frac{\left(d^{a} / c^{a}\right)^{2} \exp \left[-\left(d^{a} / c^{a}\right)^{2}\right]}{1-\exp \left[-\left(r^{a} / c^{a}\right)^{2}\right.}\left(\frac{\left(-2 d_{i}^{a}\right)\left(-2 d_{j}^{a}\right)}{\left(c^{a} d^{a}\right)^{2}}-\frac{2 \delta_{i j}}{\left(d^{a}\right)^{2}}\right)\left(1-H\left[d^{a}-r^{a}\right]\right) \tag{A11b}
\end{equation*}
$$


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