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A Structural Reliability Analysis Method Based on Radial Basis Function

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Abstract: The first-order reliability method (FORM) is one of the most widely used structural reliability analysis techniques due to its simplicity and efficiency. However, direct using FORM seems disability to work well for complex problems, especially related to high-dimensional variables and computation intensive numerical models. To expand the applicability of the FORM for more practical engineering problems, a response surface (RS) approach based FORM is proposed for structural reliability analysis. The radial basis function (RBF) is employed to approximate the implicit limit-state functions combined with Latin Hypercube Sampling (LHS) strategy. To guarantee the numerical stability, the improved HL-RF (*i*HL-RF) algorithm is used to assess the reliability index and corresponding probability of failure based on the constructed RS model. The effectiveness of the proposed method is demonstrated through five numerical examples.

Keywords: structural reliability analysis, first-order reliability method, response surface method, radial basis function

1 Introduction

During the past three decades, much effort has been made to develop efficient methods for structural reliability analysis. Response surface method (RSM) is considered to be one of the most widely used methods with the ability to approximate the limit-state functions of large and complex structural systems. By integrating the RS approaches, various reliability analysis techniques such as first-order reliability method (FORM), second-order reliability method (SORM), Monte Carlo simulation (MCS) can be easily applied to evaluate the probability of failure for more complex practical engineering problems.

Bucher and Bourgund (1990) originally adopted a quadratic polynomial without

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cross terms and selected the axial experimental sampling method to efficiently construct the RS, based on which the probability of failure can be easily evaluated in combination with advanced MCS. Rajashekhar and Ellingwood (1993) made improvements to the Bucher's algorithm and formulated more iterations to satisfy a given convergence criteria. Gaytona et al (2003) proposed a RSM named CQ2RS (Complete Quadratic Response Surface with ReSampling) allowing to take into account the knowledge of the engineer on one hand and to reduce the cost of the reliability analysis using a statistical formulation of the RSM problem on the other hand. Kaymaza and McMahon (2005) proposed a response surface method based on weighted regression for structural reliability analysis instead of normal regression. Liu and Moses (1994) suggested a sequential RSM together with Monte Carlo importance sampling (MCIS). Lu et al (2007) applied an advanced RSM for mechanical reliability analysis. Kang et al (2010) proposed an efficient RSM applying a moving least squares (MLS) approximation for structural reliability analysis instead of the traditional least squares approximation generally used in the RSM. Gupta and Manohar (2004) introduced an improved RSM for the determination of failure probability and importance measures and also proposed global measures of sensitivity of failure probability with respect to the basic random variables. Zou et al (2008) employed the cross-validated MLS method to construct the RS of the indicator function in conjunction with the optimum symmetric LHS technique. Das and Zheng (2000) proposed a method with cumulative formation of the response surface function and its use in reliability analysis. Guan and Melchers (2001) estimated the effect of response surface parameter variation on structural reliability analysis. Gavin and Ya (2008) represented high-order limit state functions in the response surface method for structural reliability analysis. Zheng and Das (2000) developed an improved RSM and applied to the reliability analysis of a stiffened plated structure. SchuiBier et al (1989) proposed the efficient computational schemes to compute structural failure probabilities. Kim and Na (1996) proposed an improved sequential RSM by using the gradient projection method, the sampling points for RS approximation are selected to be close to the original failure surface. Bucher and Most (2008) presented a comparison of approximate response functions in structural reliability analysis. For the reliability estimation of complex structures, RS methodology has been suggested as a way to estimate the implicit limit state function. Typically the RS is constructed by using polynomial functions and fitted to the implicit function at a number of points. In recent years, a few approaches have been developed to apply RBF with implicit performance functions. Deng (2006) presents three RBF network (RBF) based reliability analysis methods, i.e. RBF based MCS, RBF based FORM, and RBF based SORM. Cheng et al (2007); Cheng et al (2008) proposed an artificial neural network (ANN)-based inverse FORM and ANN-based RSM in conjunction with the uniform design method for predicting failure probability.

However, the above-mentioned methods will still encounter a severe drawback in efficiency when more complicated structures with high-dimensional variables. The existing methods usually involve repeated deterministic response analyses of complex structures due to the variation of the basis variables, and therefore require a relatively long computation time as the number of random variables increases. Therefore, it is necessary to propose a structural reliability analysis approach based on RBF.

In this paper, radial basis function combined with FORM is proposed for structural reliability analysis. RBF is adopted to approximate the limit-state functions combined with Latin Hypercube Sampling (LHS), and the probability of failure is evaluated by conducting FORM on the created RS. The structure of this paper is illustrated as follows. Section 2 introduces description of first-order reliability method (FORM). Section 3 presents Radial basis function (RBF)-based structural reliability analysis. Five numerical examples are investigated in section 4. Section 5 draws some conclusions on the proposed method.

2 Description of FORM

Consider following limit state function with n uncertain parameter

$$g(\mathbf{X}) = 0,\ X_i = 1, 2, \cdots, n \tag{1}$$

where the system state is separated into two domains, the system is safe if $g(\mathbf{X})>0$, while failure if $g(\mathbf{X})<0$. The probability of failure is defined by integrating the joint probability density function $f_x(\mathbf{X})$ over the failure domain.

$$P_f = \Pr\{g(\mathbf{X}) < 0\} = \int_{g(\mathbf{X})<0} f_{\mathbf{X}}(\mathbf{X})\mathrm{d}\mathbf{X} \tag{2}$$

The limit state function $g(\mathbf{X})$ is usually a nonlinear function of $\mathbf{X}$, therefore the integration boundary is nonlinear. The number of random variables is usually high, multidimensional integration is involved. Because of these complexities, it is difficult or even impossible to directly solve Eq. (2). To easily evaluate the integral in Eq. (2), approximation methods have been developed, such as FORM, SORM, etc. Due to its simplicity and efficiency, FORM has used for engineering applications. By using FORM, the probability of failure can be approximated by

$$P_f = \Phi(-\beta) \tag{3}$$

where Φ is the standard normal distribution function, and the reliability index β is defined as the minimum distance from origin in the standard normal space to the limit state surface. Du (2007) The first step of using FORM is to transform the non-normal random variable X in X-space to normal variables U in U-space whose elements follow a standard normal distribution. The transformation is given by

$$U_i = \Phi^{-1}\left[F_{X_i}(X_i)\right], i = 1, 2...n \tag{4}$$

where F_{Xi} is the cumulative distribution function (CDF) of X_i, and Φ^{-1} is the inverse normal distribution function. Then the reliability index and the so-called most probable point (MPP) U* is identified with the following optimization model.

$$\begin{aligned} &\min \ \ \|\mathbf{U}\| \\ &\text{Subject to } g(\mathbf{U}) = 0 \end{aligned} \tag{5}$$

where $\|\bullet\|$ represents the normal. Du (2007) There are many optimization algorithms to solve Eq.(5), the *i*HL-RF method is commonly used to achieve the solution.

3 RBF-based structural reliability analysis

Direct using FORM may not work well for complex structural reliability problems. Thus, a RBF-based structural reliabiliy analysis technique is developed here. RBF is used to approximate the limit-state function and optimal Latin Hypercube Sampling strategy is employed to locate the samples in the design space.

3.1 Construction of RBF

Radial basis functions are developed for scattered multivariate data interpolation. Mullur and Messac (2006) The method uses linear combinations of a radially symmetric function based on Euclidean distance. The response functions can be approximated by using a RBF model. A RBF model be expressed as:

$$\tilde{g}(\mathbf{X}) = \sum_{i=1}^{n} w_i \phi(\|\mathbf{X} - \mathbf{X}_i\|) \tag{6}$$

where n is the number of sampling points, $\mathbf{X}$ is the vector of input variables, $\mathbf{X}_i$ is vector of input variables at the ith sampling point. RBFs are expressed in terms of the Euclidean distance $r = \|\mathbf{X} - \mathbf{X}_i\|$. The coefficients w_i, i=1,…,n, are unknown weighting coefficients to be determined. Gutmann (2001) Some of the most commonly used basis functions include: linear, cubic, thin plate spline, Gaussian, multi-quadric, inverse multi-quadric, etc.

In this study, Gaussian radial basis function is adopted, which can be expressed as

$$\phi(r) = e^{-\alpha r^2},\ r \geq 0;\ 0 < \alpha < 1 \tag{7}$$

Considering a set of n sampling points, the RBF model can be expressed in matrix form as

$$\tilde{G} = \Phi w \tag{8}$$

where

$$\Phi = \begin{bmatrix} \phi_1(r_1) & \phi_1(r_2) & \cdots & \phi_1(r_n) \\ \phi_2(r_1) & \phi_2(r_2) & \cdots & \phi_2(r_n) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_n(r_1) & \phi_n(r_2) & \cdots & \phi_n(r_n) \end{bmatrix} \tag{9}$$

The weighting coefficient vector w can be obtained by solving Eq.(8).

To construct RBF model, the LHS method is selected to locate the sampling points. The LHS method contains n sample points between upper bound and lower bound over m dimensions is a matrix of n rows and m columns. Each row corresponds to a sample point. The m columns are randomly permuted to yield sample that appears random overall, but is uniformly distributed if each dimension is viewed separately. The n values in each column are randomly selected one from each of the intervals. The design domain of sampling points is limited in $\mu_i - k\sigma_i \leq \mathrm{X}_i \leq \mu_i + k\sigma_i$, as illustrated in Fig.1 for two-dimensional problem, where k is "sampling coefficient". The sampling coefficient is selected by designer according to practical engineering problems.

3.2 Solution of the approximate reliability index

For the MPP search, there are many general optimization algorithms available to solve Eq.(5). Du (2007) The widely used method is the *i*HL-RF algorithm owing to its simplicity and efficiency. The *i*HL-RF is computationally efficient and globally convergent, meaning that it guarantees to converge to a local MPP from any starting point. The recursive algorithm of method for the minimum probability is summarized as follows.

In iteration k+1, the MPP is defined by

$$\mathbf{U}_{k+1} = \mathbf{U}_k + \alpha \mathbf{d}_k \tag{10}$$

where the search direction $\mathbf{d}_k$ is determined by

$$\mathbf{d}_k = \frac{\nabla g(\mathbf{U}_k)\mathbf{U}_k^T - g(\mathbf{U}_k)}{\|\nabla g(\mathbf{U}_k)\|^2} \nabla g(\mathbf{U}_k) - \mathbf{U}_k \tag{11}$$

where $\nabla g(U_k)$ represents the gradient of the state limit function $g(U)$ at U_k

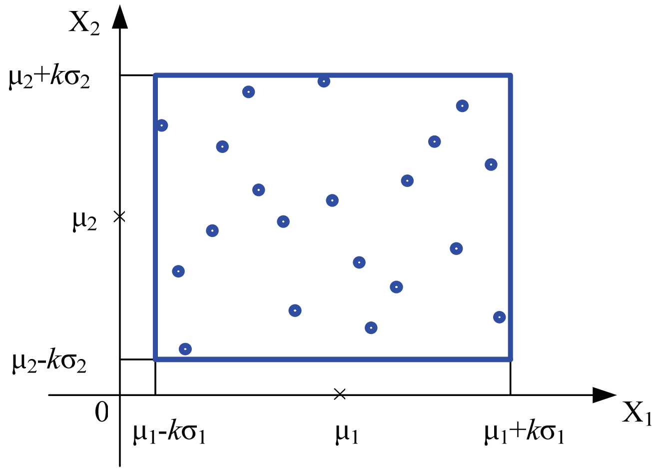

Figure 1: Sampling points for a two variable problem

The step size α is determined by minimizing the merit function defined by

$$m(\mathbf{U}) = \frac{1}{2}\|\mathbf{U}\| + c\,|g(\mathbf{U})| \tag{12}$$

In which the constant c should satisfy

$$c > \frac{\|\mathbf{U}\|}{\|\nabla g(\mathbf{U})\|} \tag{13}$$

To reduce the computational cost, in practice, the step size is computed finding a value α that the merit function is sufficiently reduced. The following rule is employed to find α such that

$$\alpha = \max_{h \in N}\left\{ b^h \left| m(\mathbf{U}_k + b^h \mathbf{d}_k) - m(\mathbf{U}_k) < 0 \right.\right\},\ b > 0 \tag{14}$$

In this algorithm b=0.5 and $c = \frac{2\|\mathbf{U}_k\|}{\|\nabla g(\mathbf{U}_k)\|} + 10$ are used.

In Eq. 14 indicates that α=b^h is the first integer.

3.3 Computational procedures of the proposed method

The calculation steps of the proposed method can be described as follows (Fig. 2):

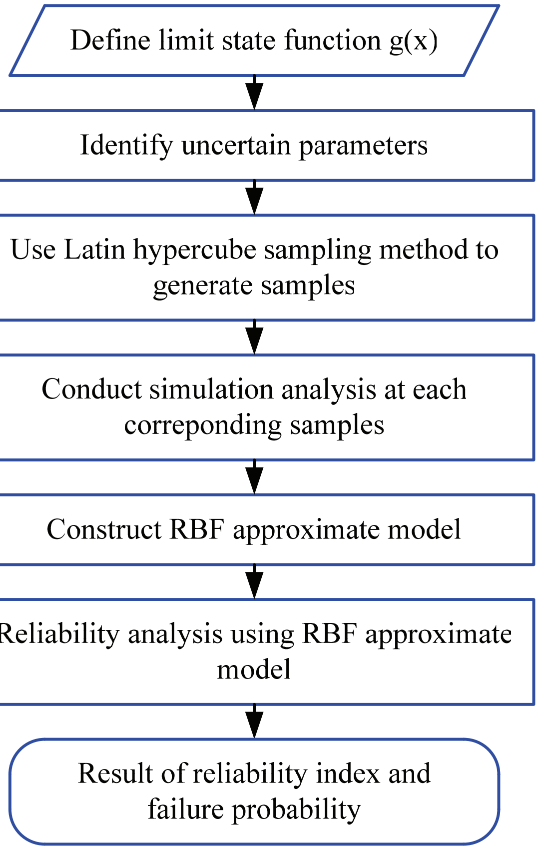

Figure 2: Flowchart of the proposed method

Step 1. Select random variables **X** and define the state function g(**X**) according to the engineering problem.

Step 2. Define the sampling space and generate sampling points by using Latin hypercube sampling method.

Step 3. Conduct simulation analysis at each corresponding sample and compute the corresponding value of the performance function.

Step 4. Construct radial basis function approximate model and calculate the weighting coefficient **w** vector using Eq.(9).

Step 5. Compute the distance β to this new design point from the origin, probability of failure by *i*HL-RF algorithm and check the convergence criterion.

4 Numerical examples

4.1 Example 1

Consider a highly non-linear limit state function discussed in Ref. Cheng et al (2009).

$$g(X) = X_1^2 + X_1^2 X_2 + X_3^3 + 18 \tag{15}$$

The statistics of the two random variables in this limit state function are listed in Tab.1. The exact reliability index and failure probability value is obtained using Adaptive MCS with 100000 samples are 2.535 and 0.00563, respectively. The results and relative error are shown in Tab.2. The reliability results are almost the same with the exact solution.

Table 1: Statistics of the random variables for example 1

Variable	Mean values	Standard deviation	Distribution
X_1	10.0	5.0	Normal
X_2	9.9	5.0	Normal

Table 2: Comparison of results of example 1

Method	Failure probability	Reliability index	Relative error of β
MCS	0.00563	2.535	—
Proposed method	0.00564	2.534	0.039%

4.2 Example 2

Consider a problem of Ref. Das and Zheng (2000). The limit state function is expressed

$$g(X) = X_2 X_3 X_4 - \frac{X_3^2 X_4^2 X_5}{X_6 X_7} - X_1 \tag{16}$$

where all the random variables are normal and mutually independent. The statistics are listed in Tab. 3. The result calculated by the proposed method is compared with directional simulation with 100000 samples yields P_f= 3.369×10^{-4}, and the corresponding reliability index β=3.4, which are regarded as the exact referenced solution. The results are shown in Tab. 4. It should be pointed out that the reliability results of the proposed method are almost the same with the exact solution.

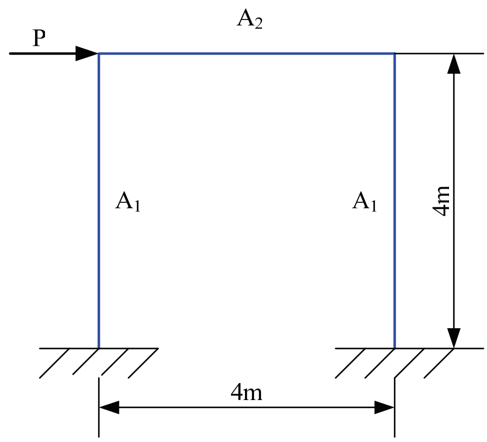

Figure 3: Linear portal frame of example 3

4.3 Example 3

This numerical example is a linear frame structure one story and one bay as shown in Fig.3. Cheng et al (2007) Different cross sectional areas A_i and horizontal load P are treated as independent random variables, their statistics are listed in Tab.5. The sectional moments of inertia expressed as $I_i = \alpha_i A_i^2$ ($\alpha_1 = 0.0833,\ \alpha_2 = 0.16670$). The Young's modulus is treated as deterministic $E = 2\times 10^6 kN/m^2$.

The limit state function is expressed

$$g(A_1, A_2, P) = 0.01 - u_{\max}(A_1, A_2, P) \tag{17}$$

where u_{max} denotes the max horizontal displacement as the function of basic random variables. The limit state function is implicit, and the structural response has to be calculated by using the FEM. The reliability index and failure probability calculated by the proposed method is compared with the MCS result with the exact solution $P_f = 0.2322\times 10^{-2}$ and its corresponding reliability index $\beta = 2.834$. The results and relative error are shown in Tab.6. It can be seen that the analysis results are all very close to the exact ones.

4.4 Example 4

This example, the proposed method has been applied to a truss structure (Fig. 4). Kim and Na (1996) All the random properties are summarized in Tab. 7, E is the elastic modulus and A is the section area.

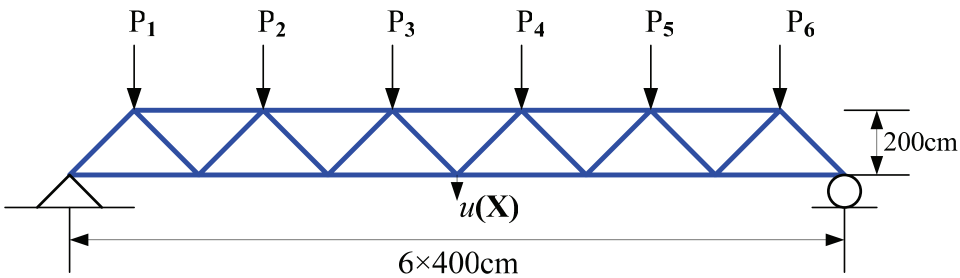

Figure 4: Truss type girder of example 4

Table 3: Statistics of the random variables for example 2

Variable	Mean values	Standard deviation	Distribution
X_1	0.01	0.003	Normal
X_2	0.3	0.015	Normal
X_3	360	36	Normal
X_4	226×10^{-6}	11.3×10^{-6}	Normal
X_5	0.5	0.05	Normal
X_6	0.12	0.006	Normal
X_7	40	6	Normal

Table 4: Comparison of results of example 2

Method	Failure probability	Reliability index	Relative error of β
MCS	0.03369%	3.40	—
Proposed method	0.03495%	3.39	0.294%

Table 5: Distributional properties of random variables in example 3

Variable	Mean	Standard deviation	Unit	Distribution
A_1	0.36	0.036	m^2	Lognormal
A_2	0.18	0.018	m^2	Lognormal
P	20.0	5.0	kN	Type I largest

The limit state function is defined by the center deflection of the truss-type girder.

$$g(X) = 11.0 - u(X) \tag{18}$$

Table 6: Comparison of analysis results of example 3

Method	Failure probability	Reliability index	Relative error of β
MCS	0.2322%	2.834	—
Proposed method	0.2292%	2.831	0.106%

Table 7: Distributional properties of random variables in example 4

Variable	Mean	Standard deviation	Unit	Distribution
E_1 of diagonal member	2100000	210000	kg/cm^2	Lognormal
A_1 of diagonal member	20	1	cm^2	Lognormal
E_2 of main member	2100000	210000	kg/cm^2	Lognormal
A_2 of main member	10	2	cm^2	Lognormal
P_1	5000	750	Kg	Type I largest
P_2	5000	750	Kg	Type I largest
P_3	5000	750	Kg	Type I largest
P_4	5000	750	Kg	Type I largest
P_5	5000	750	Kg	Type I largest
P_6	5000	750	Kg	Type I largest

Table 8: Comparison of analysis results of example 4

Method	Failure probability	Reliability index	Relative error of β
MCS	0.6350%	2.492	—
Proposed method	0.5544%	2.539	-1.85%

where $u(\mathbf{X})$ denotes the max displacement as the function of basic random variables. The limit state function of this problem is also implicit response function, and the structural response is computed by using the FEM. The reliability index and failure probability computed by the proposed is compared with the MCS result is obtained with 200000 simulations with the exact solution $P_f = 0.635 \times 10^{-2}$and its corresponding reliability index $\beta = 2.492$. The results and relative error of the proposed method are shown in Tab. 8. It is found that the reliability results are accurately close to the exact solution.

4.5 Example 5

A frame structure with 12 stories and 3 bays as shown in Fig. 5. Das and Zheng (2000); Deng (2006); Cheng et al (2007) Different cross sectional areas A_i and horizontal load P are treated as independent random variables, their statistics are listed in Tab. 9. The sectional moments of inertia expressed as $I_i = \alpha_i A_i^2$ ($\alpha_1 =$

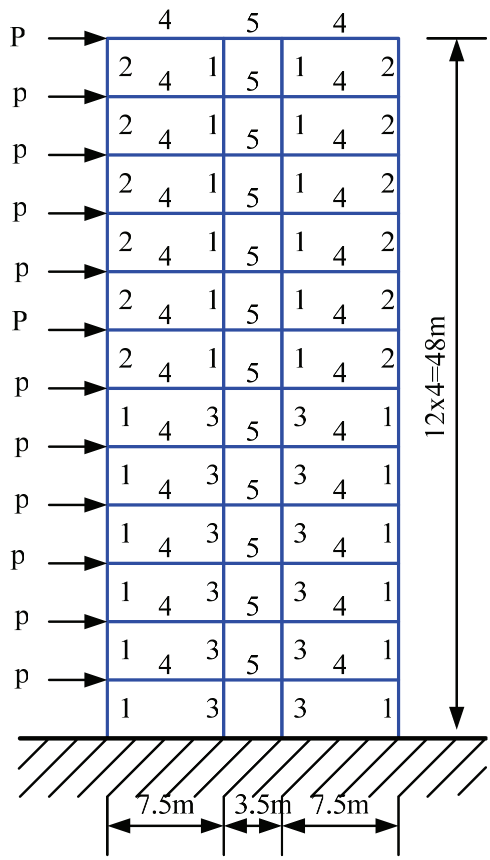

Figure 5: Linear portal frame structure of example 5

Table 9: Distributional properties of random variables in example 5

Variable	Mean	Standard deviation	Unit	Distribution
A_1	0.25	0.025	m^2	Lognormal
A_2	0.16	0.016	m^2	Lognormal
A_3	0.36	0.036	m^2	Lognormal
A_4	0.20	0.020	m^2	Lognormal
A_5	0.15	0.015	m^2	Lognormal
P	30.0	7.5	kN	Type I largest

Table 10: Comparison of analysis results of example 5

Method	Failure probability	Reliability index	Relative error β
MCS	7.5058%	1.4391	—
Proposed method	7.4637%	1.4421	-0.208%

$\alpha_2 = \alpha_3 = 0.0833,\ \alpha_4 = 0.2667,\ \alpha_5 = 0.2$). The Young's modulus is treated as deterministic $E = 2\times 10^7 kN/m^2$. The limit state function is defined as

$$g(A_1,A_2,A_3,A_4,A_5,P) = 0.096 - u_{\max}(A_1,A_2,A_3,A_4,A_5,P) \quad (19)$$

where u_{max} denotes the max horizontal displacement as the function of basic random variables. The limit state function is also implicit function, and the structural response is computed by using the FEM. The results of the proposed method is compared with MCS yields $P_f = 7.5058\times 10^{-2}$, and the corresponding reliability index $\beta = 1.4391$, which are regarded as the exact referenced solution. The results and relative error are shown in Tab. 10. It can be found that the result obtained by the proposed method is closer to the referenced solution with less computation time than of the MCS.

5 Conclusions

The main contribution of this paper is to propose a response surface (RS) based FORM for structural reliability analysis. The method involves using LHS method to generate samples, approximation of the limit state function by the RBF model and estimation of the failure probability using the FORM. In the proposed method, an RBF model is used to approximate the structural response function so that the number of deterministic response analyses can be dramatically reduced. It can thus prohibitively reduce the computation time. Comparisons were made with MCS method to evaluate the accuracy and computational efficiency of the proposed method. It was shown through the examples that the proposed method provides accurate results and is a computationally efficient approach for estimation of the probability of failure of structures. Compared with the MCS method, the proposed method is much more economical to achieve reasonable accuracy when dealing with problems where closed-form failure functions are not available or the estimated failure probability is extremely small. Five numerical examples illustrate the effectiveness of the proposed method.

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