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A New Interval Comparison Relation and Application in Interval Number Programming for Uncertain Problems

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Abstract: For optimization or decision-making problems with *interval* uncertainty, the *interval comparison relation* plays a very important role, as only based on it a better or best decision can be determined. In this paper, a new kind of *interval comparison relation* termed as *reliability-based possibility degree of interval* is proposed to give quantitative evaluations on "how much better" of one interval than another, which is more suitable for engineering reliability analysis and numerical computation than the existing relations. In the new relation, the range of the comparing values is extended into the whole real number field, and the precise comparison is made possible for any pairs of intervals on the real line. Furthermore, the suggested interval comparison relation is applied to the *interval number programming*, and two kinds of transformation models are developed for both of the linear and nonlinear interval number programming problems, based on which the uncertain optimization problems can be changed into traditional deterministic optimization problems. Two numerical examples are finally investigated to demonstrate the effectiveness of the two transformation models.

Keywords: Reliability; Interval comparison; Interval number programming; Possibility degree of interval; Uncertainty modeling

1 Introduction

Uncertainty concerned with material properties, loads, boundary conditions and so on widely exists in practical engineering problems. Probability method (Charnes and Cooper, 1959; Kall, 1982; Gyeong-Mi, 2005; Abbas and Bellahcene, 2006) and fuzzy method (Delgado et al., 1989; Luhandjula, 1989; Slowinski, 1986; Liu

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and Iwamura, 2001) are commonly employed to quantify the uncertainty, based on precise probability distributions and membership functions of the uncertain parameters, which are not always easy to achieve because of lacking enough uncertainty information for engineering problems. However, it is always not difficult to obtain the variation bounds for a practical parameter, based on only a small amount of uncertainty information or engineering experience. Thus the *interval method* makes it possible to quantify the uncertainty in many practical problems without enough uncertainty information, and has been attracting more and more attentions.

In the formulation of realistic problems, *interval numbers* representing the uncertain parameters may occur in the constraints of an optimization problem or in the determination of the best alternative of a decision-making problem (Molai and Khorram, 2007), which makes the issue of interval comparison very important. Research on interval comparison has been aimed towards finding a complete and robust methodology which can give a quantitative or qualitative ranking for any two real interval attributes, and so far a quantity of such relations have been developed. According to reference (Sengupta and Pal, 2009), interval comparison relations can be divided into two classes, namely *preference-based interval comparison relation* (P-ICR) and *value-based interval comparison relation* (V-ICR). The former can be used to judge whether one interval is better or worse than another qualitatively, while the latter can give a specific value to represent the extent that one interval is better or worse than another.

For the P-ICR, the foremost work can be traced to references (Moore, 1966, 1979), in which two partial order relations were defined on intervals. The first one is an extension of '$<$' on the real line, and the other one is an extension of the concept of set inclusion '$\subset$'. However, these two relations can not be used to rank the partially or fully nested intervals. As an improvement, Ishibuchi and Tanaka (Ishibuchi and Tanaka, 1990) proposed two new relations, among which one is based on preference to both of the lower bound and upper bound of interval and the other is based on preference to both of the midpoint and width of interval. Though nested intervals can be compared now, there still exist several cases that these two relations fail. Subsequently, Ishibuchi and Tanaka's relations were integrated into a uniform formulation using a sort of bi-parametric cut of the comparing intervals, and whereby a kind of new interval comparison relation was constructed (Chanas and Kuchta, 1996a, 1996b). The above mentioned approaches all belong to the partial order relation, and they can only answer "which one is better" among two intervals, instead of "how much better". Nevertheless, for most practical applications the V-ICR identifying "how much better" of one interval than another might be more important. Especially for uncertain optimization problems, only after knowing the specific extents of interval comparison, the constraints with interval parameters can be successfully changed to deterministic ones that we can handle.

Most of the current V-ICRs are based on the probability method, namely the compared intervals are assumed as two random numbers with uniform distributions, and the probability that one random number is larger or smaller than another is interpreted as the extent that one interval is better or worse than another. For the best knowledge of the authors, reference (Nakahara et al., 1992) seems the first attempt introducing the probability method into the interval comparison. Subsequently, some different schemes for V-ICR were developed also based on the probability method (Sevastjanov and Rog, 1997; Sevastjanov and Venberg, 1998; Wadman et al., 1994; Kundu, 1997, 1998; Sevastjanov et al., 2001, 2002; Yager et al., 2001). Recently, an approach which can derive the results of comparison as a probability interval was proposed based on the Dempster-Shafer theory of evidence (Sevastjanov, 2004). A two-objective interval comparison technique was developed by taking into account the probability of supremacy of one interval over the other one and relation of compared widths of intervals (Sevastjanov and Rog, 2006). A possibility degree of interval number has been suggested based on the probability method and applied to the nonlinear interval number programming (Jiang et al., 2008), which does not contradict the Kundu' fuzzy relation (Kundu, 1997) but gives an explicit and easier format to work with. Except the probability method, the other ways are also employed to construct the V-ICR among which we can only cite a few works (Sengupta and Pal, 2000, 2001; Wang et al., 2005; Sun and Yao, 2008; Facchinetti et al., 1998; Liu and Da, 1999). Two relations were defined to compare and order any two intervals on the real line in terms of values for both of optimistic and pessimistic decision-makers (Sengupta and Pal, 2000), and they were further adopted to deal with the interval inequality constraints in the linear interval number programming (Sengupta and Pal, 2001). A degree of one interval utility being greater than another one was defined as the degree of preference (Wang et al., 2005). An index entitled as possibility degree function was proposed to compare two intervals, which behaves especially well when used to deal with equi-centered or closely-centered intervals (Sun and Yao, 2008). Two equivalent possibility degree formulas were proposed to compare intervals, whose value represents a specific extent that one interval is greater than another (Facchinetti et al., 1998; Liu and Da, 1999). Based on the Tseng and Klein's work (Tseng and Klein, 1989)), a satisfaction function is defined for interval comparison (Molai and Khorram, 2007).

There exist several limitations for the above mentioned V-ICRs, which form severe obstacles that make the interval method unable to play a bigger role in dealing with the practical uncertain problems as it merits. Firstly, for nearly all of the current V-ICRs the range of the values is limited within the scope of [0, 1], in which 0 and 1 represent that one interval is absolutely larger or smaller than another one. In theory, these relations can only work well when their values are not equal to 0 or 1, and once these two bound points are reached their comparing function will be weakened. For two active intervals, their comparing values will be fixed at 1 or 0 no matter how you move one of them on the real line, provided that two intervals are kept fully separated. However, for practical engineering problems different relative positions of two parameter intervals generally indicate different reliability, whereas the current V-ICRs can not grasp this important characteristic. As a matter of fact, lacking the ability of reliability analysis has become a most severe defect of the current V-ICRs, which limits a wider engineering application of the interval method. Secondly, the current V-ICRs are not well suited to the numerical computation also because of the above two bound points of 0 and 1. When applied to uncertain optimization or decision-making problems, these two bound points are likely to form inflection points and bring about a problem of non-differentiability for the considered function, which is generally a serious difficulty for numerical realization.

This paper aims to construct a new kind of V-ICR which can overcome the above limitations existing in the current V-ICRs, and furthermore develop corresponding interval number programming methods based on this relation. Four major parts are included in the following text. Firstly, three equivalent V-ICRs termed as *possibility degree of interval* are introduced, and their limitations are further investigated. Secondly, a new kind of V-ICR named reliability-based possibility degree of interval (RPDI) is proposed, and its fine properties in the aspects of reliability analysis and numerical computation are also analyzed. Thirdly, the RPDI is applied to the interval number programming and whereby several transformation models are obtained for both of the linear and nonlinear uncertain optimization problems. Finally, two numerical examples are presented to demonstrate the effectiveness of the suggested interval number programming methods.

2 Three equivalent V-ICRs

Firstly, the definition of the interval number X^I is given as follows, which represents a closed bounded set of real numbers (Moore, 1979):

$$X^I = \left[X^L, X^R\right] = \left\{X \left| X^L \le X \le X^R \right.\right\} \tag{1}$$

where the superscripts L and R represent lower and upper bounds of interval, respectively. The interval X^I can be also rewritten:

$$X^I = X^c + [-1, +1] X^w \tag{2}$$

where the superscripts c and w represent midpoint and radius of interval, respectively:

$$X^c = \frac{X^L + X^R}{2}, \quad X^w = \frac{X^R - X^L}{2} \tag{3}$$

For the above intervals, they should be compared or ranked using the interval comparison relation, rather than only based on their real values like we usually do for real number comparison. For two intervals A^I and B^I, three V-ICRs termed as possibility degree of interval were proposed (Facchinetti et al., 1998; Liu and Da, 1999; Xu and Da, 2003):

Relation 1 (Facchinetti et al., 1998)

$$p(A^I \leq B^I) = \min\left\{\max\left\{\frac{B^R - A^L}{2A^w + 2B^w}, 0\right\}, 1\right\} \tag{4}$$

Relation 2 (Liu and Da, 1999)

$$p(A^I \leq B^I) = \frac{\max\left\{0, 2A^w + 2B^w - \max\left\{A^R - B^L, 0\right\}\right\}}{2A^w + 2B^w} \tag{5}$$

Relation 2 (Xu and Da, 2003)

$$p(A^I \leq B^I) = \frac{\min\left\{2A^w + 2B^w, \max\left\{B^R - A^L, 0\right\}\right\}}{2A^w + 2B^w} \tag{6}$$

where the value of $p(A^I \leq B^I)$ represents a specific possibility that interval A^I is smaller than B^I (or B^I is larger than A^I). The above three relations have been proven equivalent (Xu and Da, 2003), and actually the recently developed relation (Sun and Yao, 2008) is also an equivalent form of these relations. For $p(A^I \leq B^I)$, the following properties can be concluded:

1. $0 \leq p(A^I \leq B^I) \leq 1$;

2. If $A^R \leq B^L$, then $p(A^I \leq B^I) = 1$, and it represents that A^I is absolutely less than B^I. On the real line, A^I is completely on the left of B^I.

3. If $A^L \geq B^R$, then $p(A^I \leq B^I) = 0$, and it represents that A^I is absolutely greater than B^I. On the real line, A^I is completely on the right of B^I.

4. (Complementarity) If $p(A^I \leq B^I) = q$, then $p(A^I \leq B^I) = 1 - q$, where $q \in [0, 1]$.

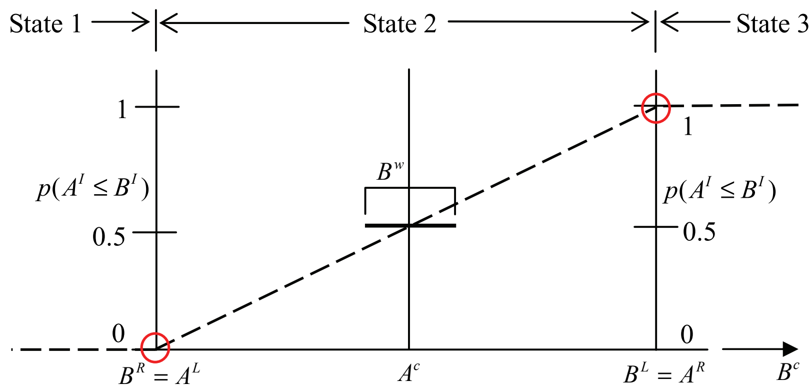

Figure 1: Variation pattern of possibility degree of interval by only changing B^c

5. $p(A^I \leq B^I) = \frac{1}{2}$,only when $A^L + A^R = B^L + B^R$ (equi-centered).

6. (Transitivity) For three intervals A^I, B^I and C^I, if $p(A^I \leq B^I) \geq q$ and $p(B^I \leq C^I) \geq q$, then $p(A^I \leq C^I) \geq q$, where $q \in [0,1]$.

Figure 1 is provided to illustrate the variation pattern of $p(A^I \leq B^I)$, in which A^c , A^w and B^w are all fixed, and only the midpoint B^c can be changed. The vertical axis denotes possibility degree of interval. Moving interval B^I along the horizontal axis, the values of $p(A^I \leq B^I)$ (dash line) take on three variation states, namely constant 0, monotonously increasing from 0 to 1, and constant 1, and the inflection points occur at two cases of $B^R = A^L$ and $B^L = A^R$. In the state 2, two intervals are partially or fully overlapped, and with moving of interval B^I towards the right side of interval A^I the possibility that A^I is less than B^I obviously becomes larger and correspondingly the value of $p(A^I \leq B^I)$ also has an increasing trend.

Comparing with the probability-based V-ICRs, the above relations have a simpler mathematical expression, and can be applied to engineering problems more conveniently. However, they still have some significant drawbacks, which erect barriers for their wider applications. It also should be noticed that these limitations which will be given in the following section exist not only in the above three V-ICRs but also in all of the current V-ICRs, though they are investigated only corresponding to the forgoing possibility degree of interval.

3 Limitations of the possibility degree of interval

Firstly, as shown in Fig. 1, we can find that the possibility degree of interval can only work well for cases that intervals are partially or fully overlapped (state 2), while nearly loses comparing function for completely separated intervals (states 1 and 3). For two intervals which are kept separated, we will always obtain a same value 0 or 1 using the above possibility degree, regardless of their relative positions. However, in practical engineering problems, different relative positions of parameter intervals generally indicate different reliability of a structure or system. Here, as an example, the possibility degree of interval is applied to the strength analysis of an uncertain structure, in which A^I and B^I represent the structural stress and strength, respectively. The stress A^I is expected to be less than the strength B^I to achieve a safe structure. The following cases are then investigated:

$$
\begin{aligned}
&\text{Case 1 } A^I = [150\text{Mpa},\ 170\text{Mpa}]\ B^I = [200\text{Mpa},\ 220\text{Mpa}]\\
&\text{Case 2 } A^I = [120\text{Mpa},\ 140\text{Mpa}]\ B^I = [200\text{Mpa},\ 220\text{Mpa}]\\
&\text{Case 3 } A^I = [195\text{Mpa},\ 215\text{Mpa}]\ B^I = [200\text{Mpa},\ 220\text{Mpa}]\\
&\text{Case 4 } A^I = [185\text{Mpa},\ 205\text{Mpa}]\ B^I = [200\text{Mpa},\ 220\text{Mpa}]\\
&\text{Case 5 } A^I = [260\text{Mpa},\ 280\text{Mpa}]\ B^I = [200\text{Mpa},\ 220\text{Mpa}]\\
&\text{Case 6 } A^I = [230\text{Mpa},\ 250\text{Mpa}]\ B^I = [200\text{Mpa},\ 220\text{Mpa}]
\end{aligned}
\tag{7}
$$

Firstly, cases 1 and 2 are analyzed, and the relative positions of two intervals in these two cases are illustrated in Fig. 2. It can be found that the stress interval A^I is completely on the left side of the strength interval B^I in both two cases and furthermore the distance of two intervals in case 2 is relatively larger than case 1. Thus, in case 2 the stress has a relatively smaller possibility to exceed the strength, and whereby possess a greater reliability than case 1. Similarly, the structural reliability of cases 4 and 6 is better than cases 3 and 5, respectively. Nevertheless, if using the possibility degree of interval $p(A^I \le B^I)$ to analyze the above 6 cases, we obtain a same value 1 for cases 1 and 2, and a same value 0 for cases 5 and 6, but different values 0.625 and 0.875 for cases 3 and 4. Obviously, the possibility degree of interval only successfully reflects the real reliability information in values for cases 3 and 4 (overlapped intervals), as through its values case 4 can be judged to be more reliable than case 3 which is in accordance with our foregoing analysis. However for completely separated intervals, such as cases 1 and 2, or cases 5 and 6, we can not judge which one is more reliable among two cases based on the values of $p(A^I \le B^I)$, as a same value 1 or 0 is obtained for two cases. Thus the above possibility degree of interval can not be used as an effective mathematical tool for reliability analysis, while for practical engineering problems the reliability analysis

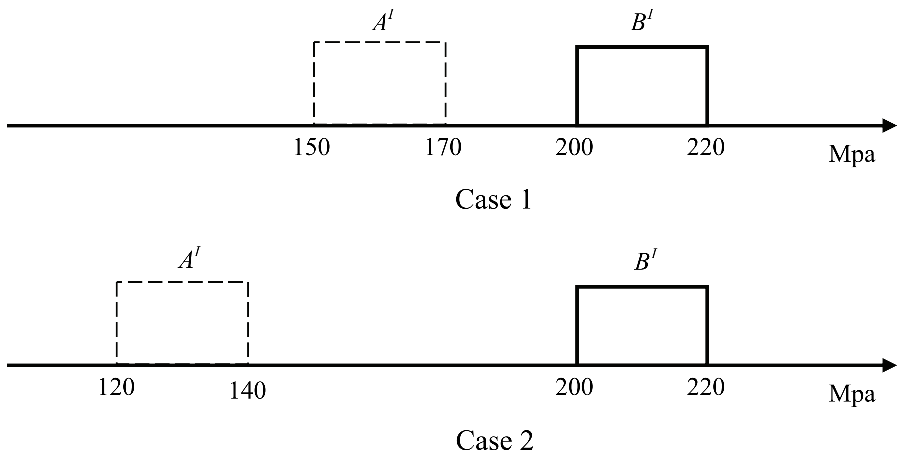

Figure 2: Relative positions of stress interval A^I and strength interval B^I

is generally extremely important. To a certain extent, the current V-ICRs including the possibility degree of interval remains more at mathematical level rather than application level. To develop a kind of more effective V-ICR for practical engineering analysis, the reliability characteristic should be considered and well reflected in the relation.

Secondly, as shown in Fig. 1, there exists two inflection points of 0 and 1 for the above possibility degree of interval, and it generally lead to non-differentiability for concerned functions. As an example, a simple interval inequality with a variable x in the left-side interval is investigated:

$$[x+1,\ x+2] \leq [8,10] \tag{8}$$

The possibility degree of interval can be computed for the above function:

$$p([x+1,\ x+2] \leq [8,10]) = \begin{cases} 1, & x \leq 6 \\ \frac{9-x}{3}, & 6 < x \leq 9 \\ 0, & x > 9 \end{cases} \tag{9}$$

It can be found that the obtained possibility degree is non-differentiable at two points of $x = 6$ and $x = 9$, though the bounds of the left-side interval are both continuous and differentiable with respect to the variable x. Therefore, if the possibility degree of interval is applied to uncertain optimization or decision-making problems, the traditional gradient-based computational techniques may lose efficiency and the numerical realization will become more difficult.

4 A reliability-based possibility degree of interval

In this section, a new kind of V-ICR, termed as reliability-based possibility degree of interval (RPDI) will be proposed, which can overcome the above defects of current V-ICRs. For the possibility degree of interval defined by Eqs. (4), (5) and (6), if we extend the variation characteristics of state 2 to both of states 1 and 3, then the RPDI can be formulated for intervals A^I and B^I:

$$p_r(A^I \leq B^I) = \frac{B^R - A^L}{2A^w + 2B^w} \tag{10}$$

where the symbol p_r is used to denote RPDI, which also represents a specific possibility that interval A^I is less than B^I (or B^I is greater than A^I). When interval A^I is degenerated into a real number A or B^I is degenerated into a real number B, the RPDI is still applicable and it can be rewritten:

$$p_r(A \leq B^I) = \frac{B^R - A}{2B^w},\ p_r(A^I \leq B) = \frac{B - A^L}{2A^w} \tag{11}$$

For $p_r(A^I \leq B^I)$, the following properties can be concluded:

1. $-\infty \leq p_r(A^I \leq B^I) \leq +\infty$;
2. If $A^R \leq B^L$, then $p_r(A^I \leq B^I) \geq 1$. On the real line, A^I is completely on the left of B^I.
3. If $A^L \geq B^R$, then $p_r(A^I \leq B^I) \leq 0$. On the real line, A^I is completely on the right of B^I.
4. (Complementarity) If $p_r(A^I \leq B^I) = q$, then $p_r(A^I \leq B^I) = 1 - q$, where $q \in [-\infty, \infty]$.
5. $p_r(A^I \leq B^I) = \frac{1}{2}$, only when $A^L + A^R = B^L + B^R$.
6. (Transitivity) For three intervals A^I, B^I and C^I, if $p_r(A^I \leq B^I) \geq q$ and $p_r(B^I \leq C^I) \geq q$, then $p_r(A^I \leq C^I) \geq q$, where $q \in [-\infty, \infty]$.

Figures 3 and 4 are presented to illustrate the variation pattern of $p_r(A^I \leq B^I)$. In Fig. 3, A^c, A^w and B^w are fixed, and the midpoint B^c can be only changed. Moving interval B^I along the horizontal axis, the values of $p_r(A^I \leq B^I)$ (dash line) take on

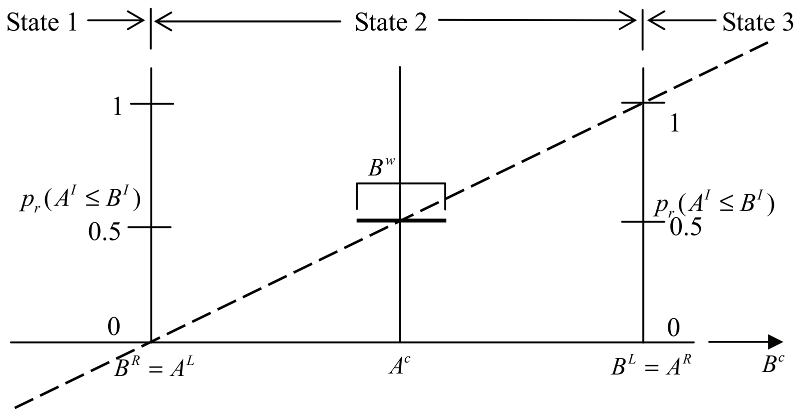

Figure 3: Variation pattern of reliability-based possibility degree of interval by only changing B^c

a consistent linear variation state, which is different from the possibility degree of interval as shown in Fig. 1. With increasing of B^c, the possibility or reliability of $A^I \leq B^I$ apparently becomes larger, and correspondingly the value of $p_r(A^I \leq B^I)$ increases monotonously, which is well in accordance with the reliability variation. In Fig. 4, A^c, A^w and B^c are all fixed, and the radius B^w can be only changed. The symbol p_{r0} is used to denote the value of $p_r(A^I \leq B^I)$ when $B^w = 0$. Increasing the value of B^w from 0 to ∞, two kinds of nonlinear variation patterns can be obtained for RPDI (dash line) according to different values of p_{r0}. For $p_{r0} > 0.5$ and $p_{r0} < 0.5$, the variation curves of $p_r(A^I \leq B^I)$ are monotonously decreasing and increasing respectively, and they both infinitely approach 0.5. For $p_{r0} = 0.5$, the value of $p_r(A^I \leq B^I)$ will be kept constant.

As shown in Fig. 3, RPDI can work well not only for overlapped intervals but also for completely separated intervals. For two overlapped intervals, the fine comparing function of the *possibility degree of interval* is inherited, and the same results can be obtained regardless of using RPDI or the above possibility degree of interval. For two separated intervals, the value of RPDI no longer falls into a fixed value 1 or 0 while related to the relative position of intervals, and furthermore its variation trend is in accordance with the variation of the reliability very well. Based on

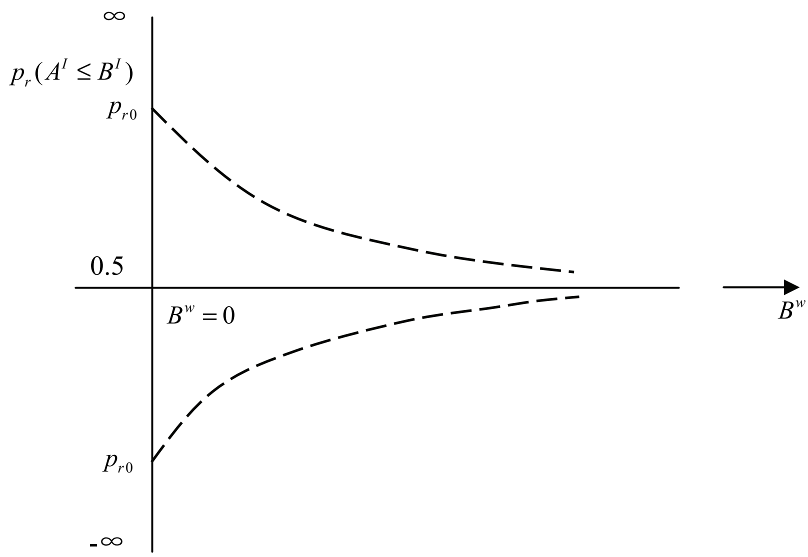

Figure 4: Variation pattern of reliability-based possibility degree of interval by only changing B^w

this property, RPDI can be used as an effective mathematical tool for engineering reliability analysis. Now, we adopt RPDI also to analyze the same example defined by Eq. (7). We can get $p_{r0} = 1.75$ and $p_{r0} = 2.5$ for cases 1 and 2, $p_{r0} = 0.625$ and $p_{r0} = 0.875$ for cases 3 and 4, $p_{r0} = -1.0$ and $p_{r0} = -0.25$ for cases 5 and 6, respectively. As aforementioned that the reliability of cases 2, 4 and 6 is better than cases 1, 3 and 5, respectively, and therefore it can be found that the RPDI can reflect these reliability information very well through its values in which a greater value of RPDI indicates a better reliability.

On the other hand, as shown in Fig. 3, the RPDI has a consistent and smoothed variation state, and the inflection points as shown in Fig. 1 do not exist any more. Therefore, RPDI generally will not lead to the problem of non-differentiability, and hence the numerical computation can be realized more conveniently and easily than the other V-ICRs including the above possibility degree of interval. As an example, we use RPDI to also analyze Eq. (8) and the following result can be obtained:

$$p_r([x+1,\ x+2] \le [8,10]) = \frac{9-x}{3},\ -\infty < x < \infty \tag{12}$$

It can be found that the obtained function is continuous and differentiable with respect to the variable x on the whole real number field, while there exist two non-differentiable points using the possibility degree of interval as shown in Eq. (9).

Through the above analysis, it can be found that the major defects of the current V-ICRs have been eliminated in our new developed relation. Furthermore, the RPDI has an easier mathematical expression which is completely explicit. Comparing with the current V-ICRs, the suggested RPDI seems more suitable for reliability analysis and numerical computation, and can be applied to practical engineering problems more conveniently. On the other hand, the RPDI should not be regarded as only a simple modification of the possibility degree of interval, though they have a similar mathematical expression a certain extent. Actually, it is a significant extension in concept for the V-ICR, through which the value range of V-ICRs are extended from $[0,1]$ to $[-\infty,\infty]$ and the precise comparison is made possible for any pairs of intervals on the real line. More importantly, based on this extension the V-ICR can be used as a new kind of reliability index for engineering reliability analysis.

5 Interval number programming based on the RPDI

Using intervals to quantify the uncertainty of the imprecise parameters in a programming problem, an *interval number programming* problem can be then formulated. As a class of important uncertain optimization methods, the interval number programming has been obtaining more and more attentions, and some prominent works have been published in this field, including linear interval number programming (Ishibuchi and Tanka, 1990; Chanas and Kuchta, 1996; Sengupta and Pal, 2001; Xu and Da, 2003; Tong, 1994; Inuiguchi and Kume, 1992; Inuiguchi and Sakawa, 1995; Mausser and Laguna, 1999; Chinneck and Ramadan, 2000; Averbakh and Lebedev, 2005; Oliveira and Antunes, 2007), nonlinear interval number programming (Ma, 2002; Li and Azarm, 2008; Liu, 2008; Liu and Wang, 2007; Jiang et al., 2008; Jiang and Han, 2007; Jiang et al., 2007), optimality condition for interval number programming (Wu, 2007, 2008, 2009), and etc. In the following text, the proposed RPDI will be employed to deal with the interval constraint or interval objective function, and whereby several transformation models are developed for linear and nonlinear interval number programming, through which the uncertain optimization problems can be transformed into deterministic optimization problems.

5.1 Linear interval number programming

A general linear interval number programming problem can be formulated:

$$\min_{\mathbf{X}} f(\mathbf{X},\mathbf{c}) = \sum_{i=1}^{n} c_i^I X_i$$

subject to

$$g_j(\mathbf{X},\mathbf{a}) = \sum_{i=1}^{n} a_{ij}^I X_i \le b_j^I,\ j = 1,...,l$$

$$X_i \ge 0,\ i = 1,2,...,n \tag{13}$$

where $\mathbf{X}$ is an$n$-dimensional decision vector. f and g are objective function and constraint, respectively, and l is the number of the constraints. $\mathbf{c}$ and $\mathbf{a}$ are n-dimensional coefficient vector and $n \times l$ coefficient matrix existing in the interval objective function and constraints, and their uncertainty is quantified by an interval vector $\mathbf{c}^I$ and an interval matrix $\mathbf{a}^I$, respectively. b_i^I denotes the allowable interval of the ith constraint. For each specific $\mathbf{X}$, the possible values of the objective function or any constraint will form an interval instead of a real number, which is different from the traditional deterministic optimization problems.

In the stochastic optimization (Liu et al., 2003), we often make an uncertain constraint satisfied with a confidence level and whereby transform the uncertain constraint into a deterministic one. Similarly, we can use the proposed RPDI to deal with the inequality constraints in Eq. (13), and make them satisfied with certain levels so as to achieve deterministic constraints:

$$p_r\left(g_j^I(\mathbf{X}) \le b_j^I\right) = \frac{b_j^R - g_j^L(\mathbf{X})}{2g_j^w(\mathbf{X}) + 2b_j^w} \ge \lambda_j,\ j = 1,2,...,l \tag{14}$$

where $g_j^I(\mathbf{X})$ represents an interval of the jth constraint at a specific $\mathbf{X}$ caused by the interval coefficients. $\lambda_j \in [-\infty,\infty]$ is a predetermined RPDI level for the jth constraint, and it can be adjusted to control the feasible field of $\mathbf{X}$. A larger λ means a smaller feasible field of Eq. (14), but a greater reliability of the uncertain constraint. λ can be allocated different values for different constraints according to their respective reliability requirements. Subsequently, we use two different approaches to deal with the uncertain objective function, and whereby construct two kinds of transformation models for the interval number programming.

5.1.1 The first transformation model

In reference (Ishibuchi and Tanaka, 1990), a preference-based interval comparison relation (P-ICR) expressed as $\leq_{mw}$ was formulated for minimization problems:

$$A^I \leq_{cw} B^I \text{ if } A^c \geq B^c \text{ and } A^w \geq B^w$$

$$A^I <_{cw} B^I \text{ if } A^I \leq_{cw} B^I \text{ and } A^I \neq B^I \tag{15}$$

Based on $\leq_{cw}$, an interval with both of smaller midpoint and radius is preferred. Using $\leq_{cw}$ to deal with the interval objective function, thus we naturally hope to find an optimal decision vector to obtain an objective interval with not only smallest midpoint but also smallest radius, and therefore the interval objective function in Eq. (13) can be transformed into a deterministic two-objective optimization problem:

$$\min_{\mathbf{X}} [f^c(\mathbf{X}), f^w(\mathbf{X})] \tag{16}$$

$$f^c(\mathbf{X}) = \frac{1}{2}(f^L(\mathbf{X}) + f^R(\mathbf{X}))$$

$$f^w(\mathbf{X}) = \frac{1}{2}(f^R(\mathbf{X}) - f^L(\mathbf{X}))$$

where $f^c(\mathbf{X})$ and $f^w(\mathbf{X})$ denote midpoint and radius of the objective function interval at a specific $\mathbf{X}$.

Thus the interval number programming problem Eq. (13) can be transformed into a following deterministic programming problem, which forms our first kind of transformation model:

$$\min_{\mathbf{X}} [f^c(\mathbf{X}), f^w(\mathbf{X})]$$

subject to

$$p_r\left(g_j^I(\mathbf{X}) \leq b_j^I\right) = \frac{b_j^R - g_j^L(\mathbf{X})}{2g_j^w(\mathbf{X}) + 2b_j^w} \geq \lambda_j,\ j = 1, 2, ..., l$$

$$X_i \geq 0,\ i = 1, 2, ..., n \tag{17}$$

For linear interval number programming, intervals of the uncertain objective function and constraints at any specific $X_i \geq 0,\ i = 1,2,...,n$ can be obtained explicitly:

$$f^L(\mathbf{X}) = \sum_{i=1}^{n} c_i^L X_i,\ f^R(\mathbf{X}) = \sum_{i=1}^{n} c_i^R X_i$$

$$g_j^L(\mathbf{X}) = \sum_{i=1}^{n} a_{ij}^L X_i,\ g_j^R(\mathbf{X}) = \sum_{i=1}^{n} a_{ij}^R X_i,\ j = 1,2,...,l \tag{18}$$

Substituting Eq. (18) into Eq. (17) leads to:

$$\min_{\mathbf{X}} \left[\frac{\sum\limits_{i=1}^{n} c_i^L X_i + \sum\limits_{i=1}^{n} c_i^R X_i}{2},\ \frac{\sum\limits_{i=1}^{n} c_i^R X_i - \sum\limits_{i=1}^{n} c_i^L X_i}{2} \right]$$

subject to

$$\sum_{i=1}^{n} \left[\lambda_j \left(a_{ij}^R - a_{ij}^L \right) + a_{ij}^L \right] X_i \leq b_j^R - \lambda_j \left(b_j^R - b_j^L \right),\ j = 1,2,...,l$$

$$X_i \geq 0,\ i = 1,2,...,n \tag{19}$$

Using the linear combination method (Hu, 1990), a deterministic single-objective optimization problem can be finally obtained in terms of a desirability function f_d:

$$\min_{\mathbf{X}} f_d(\mathbf{X}) = \sum_{i=1}^{n} \left[\left(\frac{1}{2} - \beta \right) c_i^L + \frac{1}{2} c_i^R \right] X_i$$

subject to

$$\sum_{i=1}^{n} \left[\lambda_j \left(a_{ij}^R - a_{ij}^L \right) + a_{ij}^L \right] X_i \leq b_j^R - \lambda_j \left(b_j^R - b_j^L \right),\ j = 1,2,...,l$$

$$X_i \geq 0,\ i = 1,2,...,n \tag{20}$$

where $0 \leq \beta \leq 1$ is a weighting factor of the two objective functions. Obviously, Eq. (20) is a traditional linear programming problem, and it can be easily solved by the simplex method (Nocedal and Wright, 1999).

5.1.2 The second transformation model

For practical engineering problems, sometimes we need to maximize the possibility or reliability that the objective function satisfies a certain requirement. Then the second transformation model can be constructed for the above interval number programming, in which the RPDI is expected to be maximized for the interval objective function:

$$\max_{\mathbf{X}} p_r\left(f^I(\mathbf{X}) \leq e^I\right) = \frac{e^R - f^L(\mathbf{X})}{2f^w(\mathbf{X}) + 2e^w}$$

subject to

$$p_r\left(g_j^I(\mathbf{X}) \leq b_j^I\right) = \frac{b_j^R - g_j^L(\mathbf{X})}{2g_j^w(\mathbf{X}) + 2b_j^w} \geq \lambda_j,\ j = 1,2,...,l$$

$$X_i \geq 0,\ i = 1,2,...,n \tag{21}$$

where e^I is a predetermined performance interval which needs to be satisfied by the interval objective function as far as possible.

Also based on Eq. (18), Eq. (21) can be changed into the following explicit expression:

$$\max_{\mathbf{X}} \frac{e^R - \sum_{i=1}^{n} c_i^L X_i}{2e^w + \sum_{i=1}^{n}\left(c_i^R - c_i^L\right) X_i}$$

subject to

$$\sum_{i=1}^{n}\left[\lambda_j\left(a_{ij}^R - a_{ij}^L\right) + a_{ij}^L\right] X_i \leq b_j^R - \lambda_j\left(b_j^R - b_j^L\right),\ j = 1,2,...,l$$

$$X_i \geq 0,\ i = 1,2,...,n \tag{22}$$

Obviously, Eq. (22) is a traditional nonlinear programming problem with linear constraints, and it can be efficiently solved by many well-established algorithms such as feasible direction method, sequential linear programming (SLP) method and etc (Nocedal and Wright, 1999).

5.2 Nonlinear interval number programming

A general nonlinear interval number programming problem can be formulated:

$$\min_{\mathbf{X}} f(\mathbf{X},\mathbf{U})$$

subject to

$$g_j(\mathbf{X},\mathbf{U}) \leq b_j^I,\ j=1,...,l$$

$$\mathbf{U} \in \mathbf{U}^I = \left[\mathbf{U}^L,\mathbf{U}^R\right],\ U_i \in U_i^I = \left[U_i^L,U_i^R\right],\ i=1,2,...,q$$

$$X_i \geq 0,\ i=1,2,...,n \tag{23}$$

where the q-dimensional interval vector $\mathbf{U}^I$ collects all of the uncertain parameters in the objective function and constraints. Here, the objective function f and constraints g are nonlinear and continuous functions with respect to $\mathbf{X}$ or $\mathbf{U}$.

Like the linear interval number programming, we can create two similar transformation models for the above problem as Eqs. (17) and (21). The only difference is that for nonlinear interval number programming the intervals of the objective function and constraints at each specific $\mathbf{X}$ can not obtained explicitly by Eq. (18) but the following optimization processes:

$$f^L(\mathbf{X}) = \min_{U\in\Gamma} f(\mathbf{X},\mathbf{U}),\ f^R(\mathbf{X}) = \max_{U\in\Gamma} f(\mathbf{X},\mathbf{U})$$

$$g_j^L(\mathbf{X}) = \min_{U\in\Gamma} g_j(\mathbf{X},\mathbf{U}),\ g_j^R(\mathbf{X}) = \max_{U\in\Gamma} g_j(\mathbf{X},\mathbf{U}),\ j=1,2,...,l$$

$$\mathbf{U} \in \Gamma = \left\{\mathbf{U}\,\middle|\,\mathbf{U}^L \leq \mathbf{U} \leq \mathbf{U}^R\right\} \tag{24}$$

Thus for nonlinear interval number programming, the finally obtained deterministic optimization problems like Eqs. (17) and (21) can not be explicitly expressed, which is different from the linear interval number programming. Furthermore, the two-layer nesting optimization is generally involved when solving these obtained deterministic optimization problems, in which the outer layer optimization is used to optimize the design vector to achieve a minimal desirability function (the first transformation model) or a maximal RPDI value (the second transformation model), and the inner layer optimization is used to compute the bounds of the uncertain objective function and constraints. In this paper, a GA-based nesting optimization method (Jiang et al., 2008) is employed to solve the deterministic optimization problems, in which the intergeneration projection genetic algorithm (IP-GA) (Liu and Han, 2003) with fine global convergence performance is adopted as optimization solver for both of the outer layer and inner layer optimization.

6 Numerical examples and discussion

6.1 Numerical example for linear interval number programming

The following linear interval number programming problem is investigated:

$$\min_{\mathbf{X}} [-3.0,-2.0]X_1 + [-2.0,-1.0]X_2 + [-2.0,-1.0]X_3$$

subject to

$$[0.5,1.5]X_1 + [0.5,1.5]X_2 + [1.5,3.0]X_3 \leq [11.0,13.0]$$

$$[0.5,2.0]X_1 + [1.0,2.0]X_2 + [-2.0,0.0]X_3 \leq [10.0,12.0]$$

$$X_1 \geq 1.0,\ X_2 \geq 1.0,\ X_3 \geq 1.0 \tag{25}$$

Firstly, the first transformation model is adopted to deal with the above problem, and the weighting factor β is set to 0.5 which means that the midpoint and radius of the interval objective function are given a same preference. The RPDI levels λ_1 and λ_2 are given a same value for the two interval constraints, and the computation results under different RPDI levels are listed in Table 1. It can be found that different RPDI levels correspond to different optimization results. With increasing of the RPDI level from 0.0 to 1.8, the desirability function f_d increases from -23.0 to -2.3, namely becomes worse. It is because that a larger RPDI level produces a smaller feasible field of the transformed deterministic constraints and whereby leads to a worse result for the objective function. However, when the RPDI level reaches to an excessively large value 2.0, the feasible field starts to become empty and the optimum does not exist. On the other hand, the obtained optimal design makes the

Table 1: Computation results under different RPDI levels for the first transformation model (numerical example 1)

λ_1, λ_2	Optimal design vector	Interval of the objective function	Interval of constraint 1	Interval of constraint 2	f_d	RPDI of the two constraints
0.0, 0.0	(22.0, 1.0, 1.0)	[-70.0, -46.0]	[13.0, 37.5]	[10.0, 46.0]	-23.0	0.00, 0.05
0.5, 0.5	(8.5, 1.0, 1.1)	[-29.7, -19.1]	[6.4, 17.6]	[3.0, 19.0]	-9.5	0.50, 0.50
1.0, 1.0	(4.0, 1.0, 1.2)	[-16.3, -10.2]	[4.3, 11.0]	[0.7, 10.0]	-5.1	1.00, 1.00
1.5, 1.5	(2.0, 1.0, 1.1)	[-10.1, -6.0]	[3.1, 7.7]	[-0.2, 5.9]	-3.0	1.50, 1.50
1.8, 1.8	(1.3, 1.0, 1.1)	[-7.8, -4.5]	[2.6, 6.4]	[-0.4, 4.5]	-2.3	1.80, 1.80
2.0, 2.0	Infeasible	—	—	—	—	—

two uncertain constraints have better reliability with increasing of the RPDI levels. The variation pattern of relative positions between the allowable interval and the constraint interval under different RPDI levels is illustrated in Figure 5 for the first constraint. For $\lambda_1 = 0$ the interval of the first constraint at the optimal design vector is completely on the right of the allowable interval, and it moves gradually along the negative direction of the coordinate axis with increasing of the RPDI levels. For $\lambda_1 = 1.8$, a leftmost interval apart from the allowable interval is gotten, which indicates a best reliability of the first constraint.

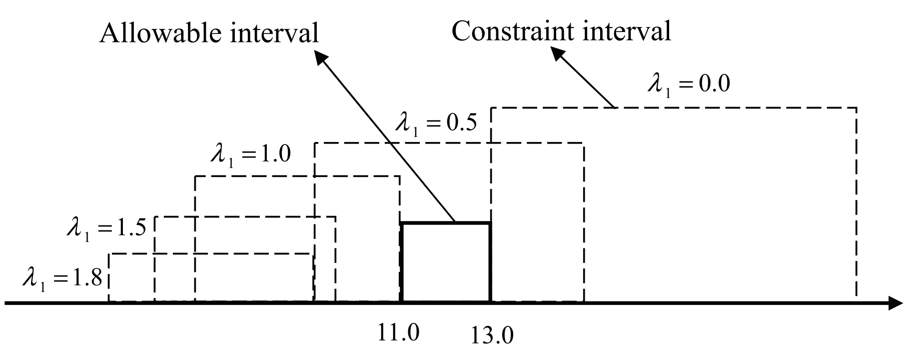

Figure 5: Relative positions between the allowable interval and the constraint 1 interval under different RPDI levels

Table 2: Computation results under different RPDI levels for the second transformation model (numerical example 1)

λ_1, λ_2	Optimal design vector	Interval of the objective function	Interval of constraint 1	Interval of constraint 2	RPDI of the objective function	RPDI of the two constraints
-0.2, -0.2	(39.7, 1.0, 1.0)	[-123.0, -81.3]	[21.8, 64.0]	[18.8, 81.3]	2.93	-0.20, -0.11
0.2, 0.2	(14.4, 1.0, 1.0)	[-47.3, -30.9]	[9.2, 26.1]	[6.2, 30.9]	2.84	0.20, 0.22
0.7, 0.7	(6.1, 1.0, 1.0)	[-22.4, -14.3]	[5.1, 13.7]	[2.1, 14.3]	2.70	0.75, 0.70
1.2, 1.2	(3.0, 1.0, 1.0)	[-13.1, -8.1]	[3.5, 9.1]	[0.5, 8.1]	2.57	1.26, 1.20
1.7, 1.7	(1.5, 1.0, 1.0)	[-8.4, -5.0]	[2.7, 6.7]	[-0.3, 5.0]	2.45	1.72, 1.70

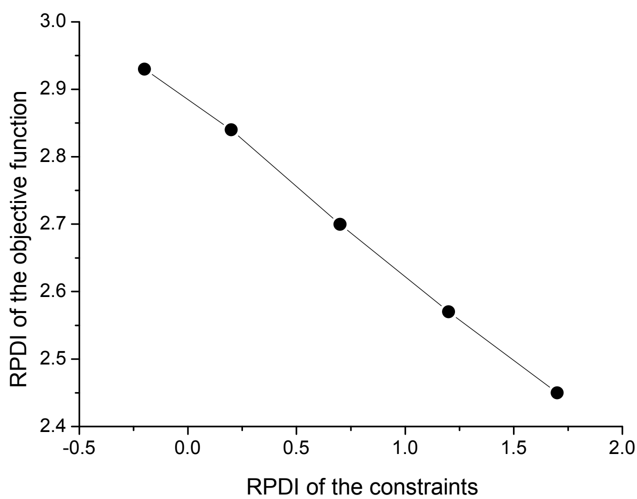

Figure 6: Relation between the RPDI of the constraints and the optimal RPDI of the objective function

The second transformation model is also adopted to deal with the above problem, and the performance interval e^I is set to $[3.0, 5.0]$. The SLP method (Nocedal and Wright, 1999) is adopted to solve the obtained deterministic optimization problem. The RPDI levels are also given different values for the two constraints, and the computation results are listed in Table 2. It can be found that with increasing of the RPDI level the optimal RPDI of the objective function and the RPDI of the constraints behave two opposite variation trends, which can also be shown in Fig. 6. For $\lambda_1 = -0.2$, we have a maximal RPDI value 2.93 for the objective function but minimal RPDI values -0.20 and -0.11 for the two interval constraints, while for $\lambda_1 = 1.7$ we have a minimal RPDI value 2.45 for the objective function but maximal RPDI values 1.72 and 1.70 for the constraints. Thus to achieve a greater possibility or reliability of the objective function satisfying the performance interval, relaxing the reliability requirement of the constraints a certain extent is generally needed. Without loss of generality, the case that the RPDI level takes a negative value -0.2 is investigated in this example. Actually, for a practical engineering problem allocating a negative RPDI level to the interval constraint is generally meaningless, as it is likely to bring about a completely unreliable design.

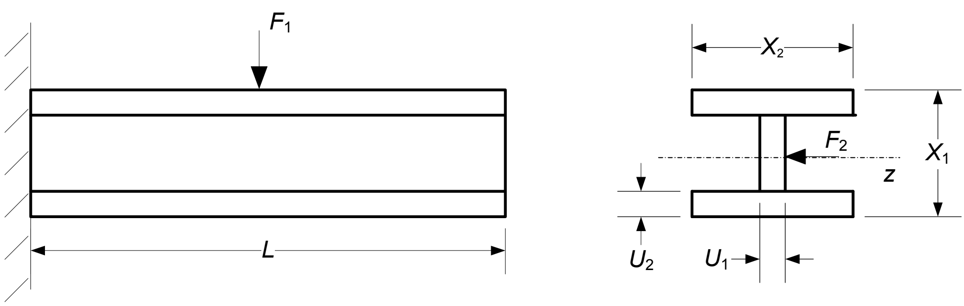

Figure 7: A beam design problem

6.2 Numerical example for nonlinear interval number programming

A practical beam design problem as shown in Fig. 7 is investigated, which is modified from the numerical example in reference (Wang, 2003). The cross-sectional dimensions X_1 and X_2 are needed to be optimized to obtain a minimum vertical deflection of the beam. The other two cross-sectional dimensions U_1 and U_2 are uncertain parameters, and their variation intervals are $[0.9\text{cm}, 1.1\text{cm}]$ and $[1.8\text{cm}, 2.2\text{cm}]$, respectively. Meanwhile, two constraints are applied, namely the cross-sectional area and maximal stress of the beam should not be more than $\left[270\text{cm}^2, 330\text{cm}^2\right]$ and $\left[10\text{kN/cm}^2, 13\text{kN/cm}^2\right]$, respectively. The Young's Modulus E, bending forces F_1 and F_2, length L of the beam are $2 \times 10^4\text{kN/cm}^2$, 600kN, 50kN and 200cm, respectively. Then, a complex nonlinear interval number programming problem can be formulated as follows:

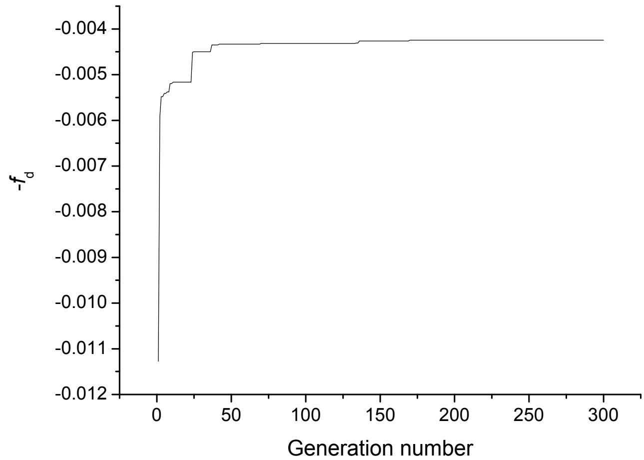

Figure 8: Convergence curve of the outer layer IP-GA for $\lambda_1 = 0.8$ and $\lambda_2 = 1.0$

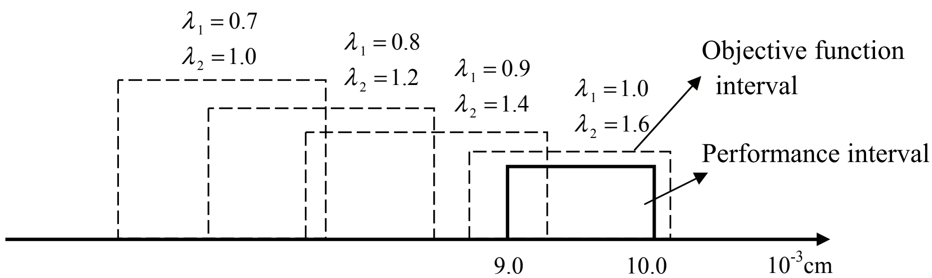

Figure 9: Relative positions between the performance interval and the optimal objective function interval under different RPDI levels

$$\min_{\mathbf{X}} f(\mathbf{X},\mathbf{U}) = \frac{F_1 L^3}{48EI_z} = \frac{5000}{\frac{1}{12}U_1\left(X_1-2U_2\right)^3+\frac{1}{6}X_2U_2^3+2X_2U_2\left(\frac{X_1-U_2}{2}\right)^2}$$

subject to

$$g_1\left(\mathbf{X},\mathbf{U}\right) = 2X_2U_2+U_1\left(X_1-2U_2\right) \leq \left[270\text{cm}^2, 330\text{cm}^2\right]$$

$$g_2(\mathbf{X},\mathbf{U}) = \frac{180000X_1}{U_1(X_1-2U_2)^3+2X_2U_2\left[4U_2^2+3X_1(X_1-2U_2)\right]} + \frac{15000X_2}{(X_1-2a_2)U_1^3+2U_2X_2^3}$$
$$\leq \left[10\text{kN/cm}^2, 13\text{kN/cm}^2\right]$$

$$a_1 \in [0.9\text{cm}, 1.1\text{cm}],\ a_2 \in [1.8\text{cm}, 2.2\text{cm}]$$

$$10.0\text{cm} \leq X_1 \leq 120.0\text{cm},\ 10.0\text{cm} \leq X_2 \leq 120.0\text{cm} \tag{26}$$

where the objective function f represents the vertical deflection of the beam.

Table 3: Computation results under different RPDI levels for the first transformation model (numerical example 2)

λ_1, λ_2	Optimal design vector (cm)	Interval of the objective function (10^{-3}cm)	Interval of constraint 1 (cm^2)	Interval of constraint 2 (kN/cm^2)	f_d (10^{-3})	RPDI of the two constraints
0.8, 1.0	(119.9, 37.4)	[7.0, 8.5]	[239.4, 291.7]	[5.0, 6.0]	4.25	0.81, 1.97
0.9, 1.2	(120.0, 29.7)	[8.4, 1.0]	[211.6, 257.7]	[6.9, 8.4]	5.09	1.12, 1.36
1.1, 1.5	(85.6, 38.3)	[14.6, 17.7]	[211.6, 257.7]	[6.1, 7.4]	8.83	1.12, 1.61

The first transformation model is firstly adopted to solve the above problem, and the weighting factor β is also set to 0.5. When using the GA-based nesting optimization method (Jiang et al., 2008) to solve the obtained deterministic optimization problem, the maximum generations for the inner layer IP-GA and the outer layer IP-GA are specified as 100 and 300, respectively, and the population size and probability of crossover for the IP-GA are set to 5 and 0.5, respectively. Considering that the stress constraint is more important than the area constraint as its violation may lead to failure of the beam, it is given a relatively larger RPDI level. The computation results under different combinations of λ_1 and λ_2 are listed in Table 3. It can be found that with increasing of λ_1 and λ_2 the desirability function f_d also becomes worse, while reliability of the constraints becomes better. For $\lambda_1 = 0.8$ and $\lambda_2 = 1.0$, the obtained interval of the area constraint is $\left[239.4\text{cm}^2, 291.7\text{cm}^2\right]$, which is partially overlapped with the corresponding allowable interval $\left[270\text{cm}^2, 330.0\text{cm}^2\right]$. However, for the other two cases, the obtained intervals of the area constraint and stress constraint are both completely separated with respective allowable intervals, and hence the RPDI values of the constraints at the optimal design are larger than 1.0 which indicates a better reliability. The convergence curve of the outer layer IP-GA for the case of $\lambda_1 = 0.8$ and $\lambda_2 = 1.0$

Table 4: Computation results under different RPDI levels for the second transformation model (numerical example 2)

λ_1, λ_2	Optimal design vector (cm)	Interval of the objective function (10^{-3}cm)	Interval of constraint 1 (cm^2)	Interval of constraint 2 (kN/cm^2)	RPDI of the objective function	RPDI of the two constraints
0.7, 1.0	(119.8, 40.3)	[6.6, 8.0]	[249.7, 304.3]	[4.5, 5.4]	1.41	0.70, 2.15
0.8, 1.2	(120.0, 37.4)	[7.0, 8.5]	[239.5, 291.8]	[5.0, 6.0]	1.21	0.81, 1.97
0.9, 1.4	(120.0, 34.0)	[7.6, 9.2]	[227.1, 276.7]	[5.7, 6.9]	0.94	0.94, 1.73
1.0, 1.6	(111.4, 34.6)	[8.8, 10.1]	[221.7, 270.0]	[5.8, 7.0]	0.41	1.00, 1.70

is also provided in Fig. 8. It can be found that after a certain amount of generations the outer layer optimization converges at a relatively stationary value.

Now, the second transformation model is also adopted to deal with the above beam design problem, in which the performance interval is set to $\left[9.0\times10^{-3}\text{cm}, 10.0\times10^{-3}\text{cm}\right]$ for the objective function. The GA-based nesting optimization method is also adopted to solve the obtained deterministic optimization problem, and all the concerned parameters are kept same. The computational results under different combinations of λ_1 and λ_2 are listed in Table 4. For the optimal RPDI of the objective function and the RPDI of the constraints, two opposite variation trends are also observed. By increasing λ_1 and λ_2 from 0.7, 1.0 to 1.0, 1.6, the RPDI of the objective function declines from 1.41 to 0.41, while the RPDI of the two constraints rises from 0.70 and 2.15 to 1.00 and 1.70. The relative position pattern of the performance interval and the optimal objective function interval under different RPDI levels is shown in Fig. 9. For the case of $\lambda_1 = 1.0$ and $\lambda_2 = 1.6$, the optimal objective function interval is fully overlapped with the performance interval, and the corresponding RPDI value is relatively small. With decreasing of λ_1 and λ_2, the obtained optimal objective function interval moves towards the left side of the performance interval on the whole, and for $\lambda_1 = 0.9$ and $\lambda_2 = 1.4$ it becomes only partially overlapped with the performance interval. And for the other two cases with lower RPDI levels, the optimal objective function intervals are finally completely separately with the performance interval, which indicates a larger possibility or reliability of the uncertain objective function satisfying the performance interval of the vertical deflection.

7 Conclusion

In this paper, we propose a new kind of V-ICR termed as reliability-based possibility degree of interval, which can overcome the major limitations of the current V-ICRs. The RPDI can work well not only for overlapped intervals but also for completely separated intervals, and hence the precise comparison is made possible for any pairs of intervals on the real line. In addition, the RPDI has an easier and completely explicit mathematical expression, and whereby can be more conveniently applied to practical engineering problems. It should be noticed that the RPDI can not be regarded as only a simple modification of the traditional V-ICRs, on the contrary, it is a significant extension in concept for the V-ICR, and based on this extension we can use the V-ICR to carry out engineering reliability analysis. The similar extension can be also accomplished in the other probability-based V-ICRs, and corresponding new reliability-based relations can be constructed.

The RPDI is also applied to the interval number programming, and two kinds of transformation models are developed for both of the linear and nonlinear interval number programming problems, based on which two deterministic optimization problems can be finally achieved. These two transformation models both use the RPDI to deal with the uncertain constraints while different approaches for the uncertain objective function. For linear interval number programming, the obtained deterministic optimization problems can be explicitly expressed and efficiently solved, while for nonlinear interval number programming they are implicit two-layer nesting optimization problems which are more difficult to solve. In the two numerical examples, both linear and nonlinear interval number programming problems are investigated, and for each one both of the two transformations models are used. The computation results indicate that with increasing of the RPDL levels the reliability of the uncertain constraints becomes better while the objective function becomes worse. Thus for practical engineering problems, appropriate values should be allocated for the RPDL levels based on an overall consideration of the manufacturing cost, design requirements, working reliability and etc.

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