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Dispersion of Axisymmetric Longtudinal Waves in a Pre-Strained Imperfectly Bonded Bi-Layered Hollow Cylinder

S. D. Akbarov[1,2] **and C. Ipek** [3]

Abstract: This paper studies the dispersion of the axisymmetric longitudinal wave propagation in the pre-strained hollow cylinder consisting of two-layers under the shear-spring type imperfectness of the contact conditions between these layers. The investigations are made within the framework of the piecewise homogeneous body model by utilizing the 3D linearized theory of elastic waves in elastic bodies with initial stresses. It is assumed that the layers of the cylinder are made from compressible hyper-elastic materials and their elasticity relations are given through the harmonic potential. The degree of the mentioned imperfectness is estimated by the shear-spring parameter. Numerical results on the influence of this parameter on the behavior of the dispersion curves related to the fundamental mode are presented and discussed. It is established that the considered type imperfectness of the contact conditions causes two branches of the dispersion curve related to the fundamental mode to appear: the first disappears, but the second approaches the dispersion curve obtained for the perfect interface case by decreasing the shear-spring parameter.

Keywords: Bi-layered hollow cylinder, shear-spring type imperfectness, initial strains, axisymmetric waves, wave dispersion

1 Introduction

Initial strains (or stresses) in construction elements are one of the reference details or factors which must be taken into account not only because of their static behavior but particularly for their dynamic behavior. It is known that these initial strains

[1] Yildiz Technical University, Faculty of Mechanical Engineering, Department of Mechanical Engineering, Yildiz Campus, 34349, Besiktas, Istanbul-TURKEY. *E-mail address:* akbarov@yildiz.edu.tr (S. D. Akbarov)

[2] Institute of Mathematics and Mechanics of National Academy of Sciences of Azerbaijan, 37041, Baku, Azerbaijan

[3] Istanbul Technical University, Faculty of Civil Engineering, Department of Civil Engineering, Maslak, Istanbul-TURKEY. *E-mail address:* cengipek@yahoo.com

occur in structural elements during their manufacture and assembly, in the Earth's crust under the action of geostatic forces, in composite materials, etc. Moreover, construction elements are loaded by external forces, as well as by additional forces acting on the external forces, during the construction process. When it is necessary to identify the mechanical problems caused by these additional forces, then the stresses caused by the working load can be taken as the initial stresses.

Thus, the scope of the problem regarding the initially stressed body is significantly wide and it is of utmost importance to study it in both the practical as well as theoretical sense.

Wave propagation in pre-strained bodies has been studied by many researchers, the systematic analyses of which are given in the monographs by Biot (1965), Guz (2004) and Eringen and Suhubi (1975). A review of these investigations is given in papers by Guz (2002, 2005), and Akbarov (2007). A considerable part of these studies relates to wave propagation in pre-strained cylinders and plates.

Here we consider a brief review of the investigations related to time-harmonic wave dispersion in the pre-strained homogeneous and layered cylinders with circular cross sections which are directly relevant to the present paper. Note that the pioneer work in this field was made by Green (1961) in which the torsional wave propagation in the pre-stretched homogeneous cylinder was studied. The paper by Demiray and Suhubi (1970) analyzed the axisymmetric wave propagation in an initially twisted circular cylinder. It was established that the initial twisting of the circular cylinder causes the coupled wave propagation field between the axisymmetric torsional and longitudinal waves to occur. In other words, it was established that in the initially twisted circular cylinder the axisymmetric torsional and longitudinal waves cannot be propagated separately. However, in the paper by Demiray and Suhubi (1970), as an example of the numerical results, only the approximate analytical expression for the perturbation of the torsional oscillation frequency caused by the initial twisting is given.

In a paper by Belward (1976), the wave propagation in a pre-strained cylinder made from an incompressible material was studied. Initial strains in the cylinder were determined within the framework of the non-linear theory of elasticity. The study of the longitudinal wave propagation in the homogeneous cylinder was also a subject of papers by Guz et al. (1975) and Kushnir (1979). The results of these later papers were also included in a monograph by Guz (2004).

Note that in the foregoing papers the subject of the study was the homogeneous circular cylinder. Before the beginning of the 21^{st} century there was no investigation of the wave propagation problems in a pre-stressed bi-material compounded cylinder. The first attempt in this field was made by Akbarov and Guz (2004) in which it was assumed that the initial stretching is small and the initial stress state in the compound cylinder is calculated within the scope of the first version of the small initial deformation theory, the meaning of which was described in the monographs by Guz (1999, 2004).

Ozturk and Akbarov (2008, 2009a, 2009b) studied torsional wave dispersion in the initially stretched bi-material compound hollow cylinder, in the pre-strained cylinder embedded in an infinite pre-strained elastic medium and in the initially stretched bi-material solid compound cylinder. In these papers it was assumed that the initial strains are small and the initial stress-strain state in the cylinders was determined within the scope of the classical linear theory of elasticity. Moreover, the elasticity relations of the materials of the constituents were described by the Murnaghan potential.

A paper by Akbarov and Guliev (2009) extended the work by Akbarov and Guz (2004) to the case where the initial strains are finite and the mechanical relations of their materials are assumed to be compressible and are given by the harmonic-type potential. Within the same assumptions Akbarov and Guliev (2010) studied the influence of the finite initial strains on the axisymmetric wave dispersion in a circular cylinder embedded in a compressible elastic medium. In paper by Akbarov et al. (2010) the problem considered in papers by Akbarov and Guliev (2009, 2010) was studied for the case where the materials of the components of the system are incompressible and the stress-strain relations for them are given through the Treloar potential. Numerical results regarding the influence of the initial strains in the cylinder and embedded body on the wave dispersion are presented and discussed. It should be noted that in the investigations reviewed above it was assumed that the initial strains are caused by the uni-axial stretching or compression along the wave propagation direction, i.e. along the cylinders. The dispersion of the axisymmetric longitudinal wave in the initially twisted compound cylinder was a subject of a paper by Akbarov et al. (2011). It was assumed that in the initial state the cylinders are twisted and each of them has a constant twist per unit length and this initial stress-strain state is determined within the scope of the classical linear theory of elasticity. The materials of the constituents are isotropic and homogeneous.

In all the foregoing papers, it is assumed that the contact condition on the interface between the inner and outer cylinders is a perfect one; i.e., it is assumed that the forces and displacements are continuous functions across the interface surface. However, in many cases (for an example, in the case where the reinforced cables are modeled as bi-material compounded cylinders), it is unrealistic to assume a perfectly bounded interface. Consequently, in order to apply the results of the theoretical investigations to the indicated cases, it is necessary to take the imperfectness of the contact conditions into account in the study of the wave propagation in the bi-material compounded circular cylinders. Note that the study of the torsional wave propagation in the bi-material compounded cylinder (without initial stresses) with an imperfect interface is studied in the paper by Berger et al. (2000) in which the imperfection of the contact condition is presented according to the model proposed by Jones and Whitter (1967). Kepceler (2010) has carried out investigations of a similar type for the initially stressed bi-material compound cylinder. Note that in this paper, the investigations carried out by Ozturk and Akbarov (2008, 2009b) are developed for the case where the contact condition on the interface surface is imperfect and the imperfectness of the contact condition is formulated according to the model used by Berger et al. (2000). Moreover, in the present paper, as in papers by Ozturk and Akbarov (2008, 2009b), the mathematical formulations for the corresponding eigen-value problems are made within the scope of the piecewise homogenous body model with the use of the equations and relations of the Three-dimensional Linearized Theory of Elastic Waves in Initially Stressed Bodies (TLTEWISB). It is also assumed that the elasticity relations of the cylinders' materials are given through the Murnaghan potential. Note that the obtaining of the equations and relations of TLTEWISB for the case under consideration will be detailed in section 2.

To the best of the authors' knowledge, up to publication of the paper by Akbarov and Ipek (2010) there has not been any investigation related to the study of the influence of the imperfectness of the contact conditions on the axisymmetric longitudinal wave propagation either in the compound cylinder with initial strains or in the compound cylinder without initial strains. The first attempt in this field was made in the mentioned paper by the authors, in which the influence of the shear-spring type of the imperfectness of the contact conditions on the dispersion on the axisymmetric longitudinal wave propagation in a pre-strained bi-material solid cylinder was studied. In the present paper, the investigations started in the paper by Akbatov and Ipek (2010) are extended for the pre-strained bi-material hollow cylinder. As in the paper by Akbatov and Ipek (2010), it is assumed that the materials of the constituents are hyper-elastic compressible ones and their elasticity relations are described by the harmonic potential. The corresponding numerical results are presented and discussed.

2 Formulation of the problem

We consider the sandwich hollow circular cylinder shown in Fig. 1 and assume that in the natural state the radius of the external circle of the inner hollow cylinder is R and the thickness of the inner and outer cylinders are $h^{(2)}$ and $h^{(1)}$, respectively. In the natural state we determine the position of the points of the cylinders by the Lagrangian coordinates in the cylindrical system of coordinates $Or\theta z$. The values related to the inner and external hollow cylinders will be denoted by the upper indices (2) and (1), respectively. Furthermore, we denote the values related to the initial state by an additional upper index 0.

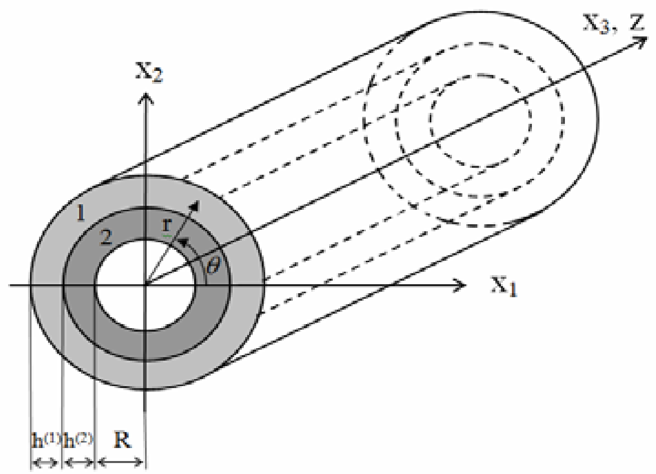

Figure 1: The geometry of the considered cylinder.

It is assumed that the cylinders have infinite length in the direction of the Oz axis and the initial strain-stress state in each component of the considered body is axisymmetric with respect to this axis and homogeneous. Moreover, it is assumed that the mentioned initial strain-stress state in the inner and external hollow cylinders are determined through the following displacement fields:

$$u_r^{(k),0} = (\lambda_1^{(k)} - 1)r, \quad u_\theta^{(k),0} = 0, \quad u_z^{(k),0} = (\lambda_3^{(k)} - 1)z, \quad \lambda_1^{(k)} \neq \lambda_3^{(k)}, \quad k = 1,2 \quad (1)$$

where $u_r^{(k),0}$ ($u_z^{(k),0}$) is the displacement along the radial direction (along the direction of the Oz axis) and $\lambda_1^{(k)}$ ($\lambda_3^{(k)}$) is the elongation parameters.

Such an initial stress field may be present with stretching or compression of the considered body along the Oz axis. The stretching or compression may be conducted for the inner and the external hollow cylinders separately before they are compounded. However, the results which will be discussed below can also be related to the case where the inner and external hollow cylinders are stretched or compressed together after the compounding. In this case, as a result of the difference of the radial and circumferential deformations of the inner and external cylinders' materials (similar to the deformations in the classical linear theory of elasticity which are determined through the Poisson's coefficient), the inhomogeneous initial stresses acting on the areas which are parallel to the Oz axis may arise. Nevertheless, according to the well known mechanical consideration, the aforementioned inhomogeneous initial stresses can be neglected under corresponding investigations, in the cases where these stresses are less significant than those acting on the areas which are perpendicular to the Oz axis. Otherwise, it is necessary to take the mentioned inhomogeneous stresses into account when considering the corresponding problems which may be a subject of other investigations.

For the initial state of the cylinders, we associate the Lagrangian cylindrical system of coordinates $O'r'\theta'z'$ and introduce the following notation:

$$r' = \lambda_1^{(k)} r, \quad z' = \lambda_3^{(k)} z, \quad R' = \lambda_1^{(2)} R, \tag{2}$$

$k = 2$ for $R \le r \le R + h^{(2)}$, $k = 1$ for $R + h^{(2)} < r \le R + h^{(1)} + h^{(2)}$.

The values related to the system of coordinates associated with the initial state below, i.e. with $O'r'\theta'z'$, will be denoted by an upper prime.

Within this framework, let us investigate the axisymmetric wave propagation along the $O'z'$ axis in the considered body. We make this investigation by the use of coordinates r' and z' in the framework of the TLTEWISB. We will follow the style and notation used in the paper by Akbarov and Guliev (2008). Thus, we write the basic relations of the TLTEWISB for the compressible body under an axisymmetrical state. These relations are satisfied within each constituent of the considered body because we use the piecewise homogeneous body model.

The equations of motion are:

$$\frac{\partial}{\partial r'} Q'^{(k)}_{r'r'} + \frac{\partial}{\partial z'} Q'^{(k)}_{r'z'} + \frac{1}{r'}\left(Q'^{(k)}_{r'r'} - Q'^{(k)}_{\theta'\theta'}\right) = \rho'^{(k)} \frac{\partial^2}{\partial t^2} u'^{(k)}_{r'},$$

$$\frac{\partial}{\partial r'} Q'^{(k)}_{r'z'} + \frac{\partial}{\partial z'} Q'^{(k)}_{z'z'} + \frac{1}{r'} Q'^{(k)}_{r'z'} = \rho'^{(k)} \frac{\partial^2}{\partial t^2} u'^{(k)}_{z'}. \tag{3}$$

The mechanical relations are:

$$Q'^{(k)}_{r'r'} = \omega'^{(k)}_{1111} \frac{\partial u'^{(k)}_{r'}}{\partial r'} + \omega'^{(k)}_{1122} \frac{u'^{(k)}_{r'}}{r'} + \omega'^{(k)}_{1133} \frac{\partial u'^{(k)}_{z'}}{\partial z'},$$

$$Q'^{(k)}_{\theta'\theta'} = \omega'^{(k)}_{2211} \frac{\partial u'^{(k)}_{r'}}{\partial r'} + \omega'^{(k)}_{2222} \frac{u'^{(k)}_{r'}}{r'} + \omega'^{(k)}_{2233} \frac{\partial u'^{(k)}_{z'}}{\partial z'},$$

$$Q'^{(k)}_{z'z'} = \omega'^{(k)}_{3311} \frac{\partial u'^{(k)}_{r'}}{\partial r'} + \omega'^{(k)}_{3322} \frac{u'^{(k)}_{r'}}{r'} + \omega'^{(k)}_{3333} \frac{\partial u'^{(k)}_{z'}}{\partial z'},$$

$$Q'^{(k)}_{r'z'} = \omega'^{(k)}_{1313} \frac{\partial u'^{(k)}_{r'}}{\partial z'} + \omega'^{(k)}_{1331} \frac{\partial u'^{(k)}_{z'}}{\partial r'}, \quad Q'^{(k)}_{z'r'} = \omega'^{(k)}_{3113} \frac{\partial u'^{(k)}_{r'}}{\partial z'} + \omega'^{(k)}_{3131} \frac{\partial u'^{(k)}_{z'}}{\partial r'}. \tag{4}$$

In (3) and (4) through $Q'^{(k)}_{r'r'}, \ldots, Q'^{(k)}_{z'r'}$ the perturbation of the components of the Kirchoff stress tensor are denoted. The notation $u'^{(k)}_{r'}$, $u'^{(k)}_{z'}$ shows the perturbations of the components of the displacement vector. The constants $\omega'^{(k)}_{1111}, \ldots, \omega'^{(k)}_{3333}$ in (3) and (4) are determined by the mechanical constants of the inner and outer cylinders' materials and through the initial stress state. $\rho'^{(k)}$ is the density of the k-th material.

For an explanation of the foregoing equations and relations, according to monographs by Eringen and Suhubi (1975) and Guz (2004), let us consider briefly some basic relationships of the large (finite) elastic deformation theory for hyper-elastic bodies and their linearization which are used in the present investigation.

2.1 Some related relations of the non-linear theory of elasticity for hyper-elastic bodies

Consider the definition of the stress and strain tensors in the large elastic deformation theory under an axisymmetric case. For this purpose we use the Lagrange coordinates r, θ and z in the cylindrical system of coordinates $Or\theta z$. In this case, the physical components of Green's strain tensor $\tilde{\boldsymbol{\varepsilon}}$ in the $Or\theta z$ coordinate system are determined by the physical components u_r and u_z of the displacement vector $\mathbf{u}$ through the following relations:

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} + \frac{1}{2}\left\{\left(\frac{\partial u_r}{\partial r}\right)^2 + \left(\frac{\partial u_z}{\partial r}\right)^2\right\},$$

$$\varepsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) + \frac{1}{2}\left\{\frac{\partial u_r}{\partial r}\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\frac{\partial u_z}{\partial z}\right\}$$

$$\varepsilon_{\theta\theta} = \frac{u_r}{r} + \left(\frac{u_r}{r}\right)^2, \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z} + \frac{1}{2}\left\{\left(\frac{\partial u_r}{\partial z}\right)^2 + \left(\frac{\partial u_z}{\partial z}\right)^2\right\}. \tag{5}$$

Consider the determination of the Kirchhoff stress tensor. The use of various types of stress tensors in the large (finite) elastic deformation theory is connected with the reference of the components of these tensors to the unit area of the relevant surface elements in the deformed or un-deformed state. This is because, in contrast to the linear theory of elasticity, in the finite elastic deformation theory the difference between the areas of the surface elements taken before and after deformation must be accounted for in the derivation of the equation of motion and under satisfaction of the boundary conditions. According to the aim of the present investigation, we consider here two types of stress tensors denoted by $\tilde{\mathbf{q}}$ and $\tilde{\mathbf{s}}$ the components of which refer to the unit area of the relevant surface elements in the un-deformed state, but act on the surface elements in the deformed state. The physical components $s_{(ij)}$of the stress tensor $\tilde{\mathbf{s}}$ are determined through strain energy potential $\Phi = \Phi(\varepsilon_{rr}, \varepsilon_{\theta\theta}, ..., \varepsilon_{zz})$ by the use of the following expression:

$$s_{(ij)} = \frac{1}{2}\left(\frac{\partial}{\partial \varepsilon_{(ij)}} + \frac{\partial}{\partial \varepsilon_{(ji)}}\right)\Phi, \tag{6}$$

where $(ij) = rr,\ \theta\theta,\ zz,\ rz.$

The physical components of the stress tensor $\tilde{\mathbf{q}}$ are determined by the following expressions:

$$q_{rr} = s_{rr}\left(1 + \frac{\partial u_r}{\partial r}\right) + s_{rz}\frac{\partial u_r}{\partial z}, \quad q_{rz} = s_{rr}\frac{\partial u_z}{\partial r} + s_{rz}\left(1 + \frac{\partial u_z}{\partial z}\right),$$

$$q_{\theta\theta} = s_{\theta\theta}\left(1 + \frac{u_r}{r}\right), \quad q_{zr} = s_{zr}\left(1 + \frac{\partial u_r}{\partial r}\right) + s_{zz}\frac{\partial u_r}{\partial z},$$

$$q_{zz} = s_{zr}\frac{\partial u_z}{\partial r} + s_{zz}\left(1 + \frac{\partial u_z}{\partial z}\right), \tag{7}$$

The stress tensor $\tilde{\mathbf{q}}$ with components determined by expression (7) is called the Kirchhoff stress tensor, but the stress tensor $\tilde{\mathbf{s}}$ is called the Lagrange stress tensor. According to the expressions (6) and (7), the stress tensor $\tilde{\mathbf{s}}$ is symmetric, but the stress tensor $\tilde{\mathbf{q}}$ is non-symmetric. In this case the equation of motion is written as follows:

$$\frac{\partial q_{rr}}{\partial r} + \frac{\partial q_{zr}}{\partial z} + \frac{1}{r}(q_{rr} - q_{\theta\theta}) = \rho\frac{\partial^2 u_r}{\partial t^2}, \quad \frac{\partial q_{rz}}{\partial r} + \frac{1}{r}q_{rz} + \frac{\partial q_{zz}}{\partial z} = \rho\frac{\partial^2 u_z}{\partial t^2}, \tag{8}$$

Under determination of the stress-strain relations, it is necessary to give the explicit expression for the strain energy potential Φ in expression (6). In the present paper, we will use the following expression for the potential Φ which was proposed in a paper by John (1960) and was called the harmonic potential:

$$\Phi = \frac{1}{2}\lambda e_1^2 + \mu e_2, \tag{9}$$

where

$$e_1 = \sqrt{1 + 2\varepsilon_1} + \sqrt{1 + 2\varepsilon_2} + \sqrt{1 + 2\varepsilon_3} - 3,$$

$$e_2 = \left(\sqrt{1+2\varepsilon_1}-1\right)^2 + \left(\sqrt{1+2\varepsilon_2}-1\right)^2 + \left(\sqrt{1+2\varepsilon_3}-1\right)^2. \quad (10)$$

In relations (9) and (10), λ and μ are material constants and $\varepsilon_i (i = 1,2,3)$ are the principal values of Green's strain tensor.

Thus, with this we restrict ourselves to consideration of the definition of the stress and strain tensor, the determination of the relations between them, as well as the equation of motion in the finite elastic deformation theory.

2.2 Determination of the initial strains and stresses

Substituting the expression (1) into the relation (5) and supplying it with the corresponding upper indices we obtain the following initial strains:

$$\varepsilon_{rr}^{(k),0} = \varepsilon_{\theta\theta}^{(k),0} = \frac{1}{2}\left(\left(\lambda_1^{(k)}\right)^2 - 1\right), \quad \varepsilon_{zz}^{(k),0} = \frac{1}{2}\left(\left(\lambda_3^{(k)}\right)^2 - 1\right), \quad \varepsilon_{rz}^{(k),0} = 0. \quad (11)$$

It follows from (11) that in the considered initial state, the principal values of Green's strain tensor $\varepsilon_1^{(k),0}$, $\varepsilon_2^{(k),0}$ and $\varepsilon_3^{(k),0}$ coincide with $\varepsilon_{rr}^{(k),0}$, $\varepsilon_{\theta\theta}^{(k),0}$ and $\varepsilon_{zz}^{(k),0}$, respectively. Consequently, substituting the expression (11) into the relations (9) and (10) we obtain the following expression for the strain energy potential in the initial state:

$$\Phi^{(k),0} = \frac{1}{2}\lambda^{(k)}\left(2\lambda_1^{(k)} + \lambda_3^{(k)} - 3\right)^2 + \mu^{(k)}\left(2\left(\lambda_1^{(k)} - 1\right)^2 + \left(\lambda_3^{(k)} - 1\right)^2\right). \quad (12)$$

According to the expression (11), the following relations can be written:

$$\frac{\partial}{\partial \varepsilon_{rr}^{(k),0}} = \frac{\partial}{\partial \varepsilon_{\theta\theta}^{(k),0}} = \frac{1}{\lambda_1^{(k)}}\frac{\partial}{\partial \lambda_1^{(k)}}, \quad \frac{\partial}{\partial \varepsilon_{zz}^{(k),0}} = \frac{1}{\lambda_3^{(k)}}\frac{\partial}{\partial \lambda_3^{(k)}}. \quad (13)$$

Using (12) and (13) we obtain the following expressions for the stresses in the initial state:

$$s_{zz}^{(k),0} = \left[\lambda^{(k)}\left(2\lambda_1^{(k)} + \lambda_3^{(k)} - 3\right) + 2\mu^{(k)}\left(\lambda_3^{(k)} - 1\right)\right]\left(\lambda_3^{(k)}\right)^{-1},$$

$$s_{r\theta}^{(k),0} = s_{rz}^{(k),0} = s_{z\theta}^{(k),0} = 0,$$

$$s_{rr}^{(k),0} = s_{\theta\theta}^{(k),0} = \left[\lambda^{(k)}\left(2\lambda_1^{(k)} + \lambda_3^{(k)} - 3\right) + 2\mu^{(k)}\left(\lambda_1^{(k)} - 1\right)\right]\left(\lambda_1^{(k)}\right)^{-1}. \quad (14)$$

According to the problem statement, we can write:

$$s_{rr}^{(k),0} = s_{\theta\theta}^{(k),0} = \left[\lambda^{(k)}\left(2\lambda_1^{(k)} + \lambda_3^{(k)} - 3\right) + 2\mu^{(k)}\left(\lambda_1^{(k)} - 1\right)\right]\left(\lambda_1^{(k)}\right)^{-1} = 0,$$

and from which we obtain:

$$\lambda_1^{(k)} = \left[2 - \frac{\lambda^{(k)}}{\mu^{(k)}}\left(\lambda_3^{(k)} - 3\right)\right]\left[2\left(\frac{\lambda^{(k)}}{\mu^{(k)}} + 1\right)\right]^{-1}. \tag{15}$$

Also, we obtain from (14), (1) and (7) the following expressions for the Kirchhoff stress tensor in the initial state:

$$q_{zz}^{(k),0} = \lambda_3^{(k)} s_{zz}^{(k),0}, \quad q_{rr}^{(k),0} = \lambda_1^{(k)} s_{rr}^{(k),0} = 0, \quad q_{\theta\theta}^{(k),0} = \lambda_1^{(k)} s_{\theta\theta}^{(k),0} = 0,$$

$$q_{\theta r}^{(k),0} = q_{r\theta}^{(k),0} = q_{rz}^{(k),0} = q_{zr}^{(k),0} = q_{z\theta}^{(k),0} = q_{\theta z}^{(k),0} = 0. \tag{16}$$

It follows from the relations (1) and (16) that the equation (8) satisfies automatically the initial strain-stress state.

2.3 Determination of the relations related to the perturbation state

Now we assume that the considered three-layered hollow cylinder with the foregoing initial strain-stress state has additional small perturbations determined by a displacement vector with components $u_r^{(k)} = u_r^{(k)}(r,z,t)$ and $u_z^{(k)} = u_z^{(k)}(r,z,t)$. Taking into account the smallness of the displacement perturbation we linearize the relationships (5) – (10) for the perturbed state in the vicinity of the appropriate values for the initial state and then subtract from them the relationships for the initial state. As a result, we obtain the equations of the TLTEWISB. As an example, in the case under consideration, as a result of the mentioned linearization we obtain the following expressions for perturbations of the components of Green's strain tensor:

$$\varepsilon_{rr}^{(k)} = \lambda_1^{(k)} \frac{\partial u_r^{(k)}}{\partial r}, \quad \varepsilon_{\theta\theta}^{(k)} = \lambda_1^{(k)} \frac{u_r^{(k)}}{r}, \quad \varepsilon_{zz}^{(k)} = \lambda_3^{(k)} \frac{\partial u_z^{(k)}}{\partial z},$$

$$\varepsilon_{rz}^{(k)} = \frac{1}{2}\left(\lambda_1^{(k)} \frac{\partial u_r^{(k)}}{\partial z} + \lambda_3^{(k)} \frac{\partial u_z^{(k)}}{\partial r}\right). \tag{17}$$

The perturbation of the components of the stress tensor $\tilde{\mathbf{s}}^{(k)}$ (denote them by capital letter $S_{(ij)}^{(k)}$) are determined from the linearization of the relation (6). We do not consider here the details of this linearization procedure, but note that as a result of this linearization the following expressions for $S_{(ij)}^{(k)}$ are obtained:

$$S_{rr}^{(k)} = \left(\lambda^{(k)} + 2\mu^{(k)}\right) \frac{\partial u_r^{(k)}}{\partial r} + \lambda^{(k)} \frac{u_r^{(k)}}{r} + \lambda^{(k)} \frac{\partial u_z^{(k)}}{\partial z},$$

$$S_{\theta\theta}^{(k)} = \lambda^{(k)} \frac{\partial u_r^{(k)}}{\partial r} + \left(\lambda^{(k)} + 2\mu^{(k)}\right) \frac{u_r^{(k)}}{r} + \lambda^{(k)} \frac{\partial u_z^{(k)}}{\partial z},$$

$$S_{zz}^{(k)} = \lambda^{(k)} \frac{\partial u_r^{(k)}}{\partial r} + \lambda^{(k)} \frac{u_r^{(k)}}{r} + \left(\lambda^{(k)} + \frac{\lambda_1^{(k)}}{\lambda_3^{(k)}} 2\mu^{(k)}\right) \frac{\partial u_z^{(k)}}{\partial z},$$

$$S_{rz}^{(k)} = \frac{2\lambda_1^{(k)}}{\lambda_1^{(k)} + \lambda_3^{(k)}} \mu^{(k)} \frac{\partial u_r^{(k)}}{\partial z} + \frac{2\lambda_3^{(k)}}{\lambda_1^{(k)} + \lambda_3^{(k)}} \mu^{(k)} \frac{\partial u_z^{(k)}}{\partial r}. \tag{18}$$

Taking the relations (18) into account we obtain from (7) the following expression for perturbation of the components of the Kirchhoff stress tensor $\tilde{\mathbf{q}}^{(k)}$ (denote them by capital letter $Q_{(ij)}^{(k)}$) which differ from zero.

$$Q_{zz}^{(k)} = \lambda_3^{(k)} S_{zz}^{(k)} + s_{zz}^{(k),0} \frac{\partial u_r^{(k)}}{\partial r}, \quad Q_{rr}^{(k),0} = \lambda_1^{(k)} S_{rr}^{(k)}, \quad Q_{\theta\theta}^{(k)} = \lambda_1^{(k)} S_{\theta\theta}^{(k)},$$

$$Q_{rz}^{(k)} = \lambda_1^{(k)} S_{rz}^{(k)}, \quad Q_{zr}^{(k)} = \lambda_3^{(k)} S_{rz}^{(k)} + s_{zz}^{(k),0} \frac{\partial u_r^{(k)}}{\partial z}. \tag{19}$$

Thus, substituting $\left(q_{(ij)}^{(k),0} + Q_{(il)}^{(k)}\right)$, $\left(u_r^{(k),0} + u_r^{(k)}\right)$ and $\left(u_z^{(k),0} + u_z^{(k)}\right)$ for $q_{(ij)}$, u_r and u_z respectively in equation (8) we obtain:

$$\frac{\partial Q_{rr}^{(k)}}{\partial r} + \frac{\partial Q_{zr}^{(k)}}{\partial z} + \frac{1}{r}\left(Q_{rr}^{(k)} - Q_{\theta\theta}^{(k)}\right) = \rho^{(k)} \frac{\partial^2 u_r^{(k)}}{\partial t^2},$$

$$\frac{\partial Q_{rz}^{(k)}}{\partial r} + \frac{1}{r} Q_{rz}^{(k)} + \frac{\partial Q_{zz}^{(k)}}{\partial z} = \rho^{(k)} \frac{\partial^2 u_z^{(k)}}{\partial t^2}. \tag{20}$$

Multiplying the equation (20) with $\left(\left(\lambda_1^{(k)}\right)^2 \lambda_3^{(k)}\right)^{-1}$ and using the notation:

$$\rho'^{(k)} = \rho^{(k)} \left(\left(\lambda_1^{(k)}\right)^2 \lambda_3^{(k)}\right)^{-1}, \quad Q'^{(k)}_{r'r'} = Q_{rr}^{(k)} \left(\lambda_1^{(k)} \lambda_3^{(k)}\right)^{-1},$$

$$Q'^{(k)}_{z'r'} = Q_{zr}^{(k)} \left(\lambda_1^{(k)}\right)^{-2}, \quad r' = \lambda_1^{(k)} r Q'^{(k)}_{\theta'\theta'} = Q_{\theta\theta}^{(k)} \left(\lambda_1^{(k)} \lambda_3^{(k)}\right)^{-1},$$

$$Q'^{(k)}_{z'z'} = Q_{zz}^{(k)} \left(\lambda_1^{(k)}\right)^{-2}, \quad u'^{(k)}_{r'} = u_r^{(k)}, \quad u'^{(k)}_{z'} = u_z^{(k)}, \quad z' = \lambda_3^{(k)} z. \tag{21}$$

we obtain the equation (3) from equation (20). Moreover, from the relationships (18), (19) and (21) we derive the following expressions for the components $\omega'^{(k)}_{1111}$, $\omega'^{(k)}_{3333}$, $\omega'^{(k)}_{1122}$, $\omega'^{(k)}_{1133}$, $\omega'^{(k)}_{1221}$, $\omega'^{(k)}_{3113}$ and $\omega'^{(k)}_{1313}$ which enter the relation (4):

$$\omega'^{(k)}_{1111} = \left(\lambda_3^{(k)}\right)^{-1}\left(\lambda^{(k)} + 2\mu^{(k)}\right), \quad \omega'^{(k)}_{3333} = \left(\frac{\lambda_3^{(k)}}{\lambda_1^{(2)}}\right)^2\left(\lambda^{(k)} + 2\mu^{(k)}\right),$$

$$\omega'^{(k)}_{1122} = \left(\lambda_3^{(k)}\right)^{-1}\lambda^{(k)}, \quad \omega'^{(k)}_{1133} = \left(\lambda_1^{(k)}\right)^{-1}\lambda^{(k)},$$

$$\omega'^{(k)}_{1221} = \left(\lambda_3^{(k)}\right)^{-1}\mu^{(k)}, \quad \omega'^{(k)}_{1313} = 2\mu^{(k)}\left(\lambda_1^{(k)} + \lambda_3^{(k)}\right)^{-1},$$

$$\omega'^{(k)}_{3113} = 2\mu^{(k)}\left(\lambda_1^{(k)}\right)^{-2}\left(\lambda_3^{(k)}\right)^2\left(\lambda_1^{(k)} + \lambda_3^{(k)}\right)^{-1}. \tag{22}$$

Although the foregoing operations are made for the harmonic potential (9) in the cylindrical system of coordinates $Or\theta z$, but these operations, as in Refs. Ogden and Roxburgh (1993), Roxburgh and Ogden (1994) and Guz (2004), can be also made for the general form of the potential Φ in the mentioned cylindrical system of coordinates. For this purpose the various type invariants of the strain tensor, through which this potential is expressed, must be rewritten by physical components of this tensor. After this procedure, the expression (6) can be employed for the operations similar above ones for each possible type elastic potential.

Thus, the propagation of the longitudinal axisymmetric wave in the considered systems will be investigated by the use of Eqs. (3), (4) and (22). These equations must be supplied with the corresponding boundary and contact conditions. First, we consider the boundary conditions which can be written as follows:

$$Q'^{(1)}_{r'r'}\Big|_{r'=R'+h'^{(1)}} = 0, \quad Q'^{(1)}_{r'z'}\Big|_{r'=R'+h'^{(1)}} = 0,$$

$$Q'^{(2)}_{r'r'}\Big|_{r'=R'-h'^{(2)}} = 0, \quad Q'^{(2)}_{r'z'}\Big|_{r'=R'-h'^{(2)}} = 0. \tag{23}$$

Now we consider the formulation of the imperfect contact conditions on the interface surface between the inner and outer cylinders. It should be noted that, in general, the imperfectness of the contact conditions is identified by discontinuities of the displacements and forces across the mentioned interface. A review of the mathematical modeling of the various types of incomplete contact conditions for elastodynamics problems has been detailed in a paper by Martin (1992). It follows from this paper that for most models the discontinuity of the displacement $\mathbf{u}^+$ and force $\mathbf{f}^+$ vectors on one side of the interface are assumed to be linearly related to the displacement $\mathbf{u}^-$ and force $\mathbf{f}^-$ vectors on the other side of the interface. This statement, as in the paper by Rokhlin and Wang (1991), can be presented as follows:

$$[\mathbf{f}] = \mathbf{C}\mathbf{u}^- + \mathbf{D}\mathbf{f}^-, \quad [\mathbf{u}] = \mathbf{G}\mathbf{u}^- + \mathbf{F}\mathbf{f}^-, \tag{24}$$

where $\mathbf{C}$, $\mathbf{D}$, $\mathbf{G}$ and $\mathbf{F}$ are three-dimensional (3×3) matrices and the square brackets indicate a jump in the corresponding quantity across the interface. Consequently, if the interface is at $r' = R'$, then:

$$[\mathbf{u}] = \mathbf{u}|_{r'=R'+0} - \mathbf{u}|_{r'=R'-0}, \quad [\mathbf{f}] = \mathbf{f}|_{r'=R'+0} - \mathbf{f}|_{r'=R'-0}. \tag{25}$$

It follows from (24) that we can write incomplete contact conditions for various particular cases by selection of the matrices $\mathbf{C}$, $\mathbf{D}$, $\mathbf{G}$ and $\mathbf{F}$. One such selection was made in the paper by Jones and Whitter (1967), according to which, it was assumed that $\mathbf{C} = \mathbf{D} = \mathbf{G} = \mathbf{0}$. In this case the following can be obtained from (25):

$$[\mathbf{f}] = \mathbf{0}, \quad [\mathbf{u}] = \mathbf{F}\mathbf{f}^-, \tag{26}$$

where $\mathbf{F}$ is a constant diagonal matrix. The model (26) simplifies significantly the solution procedure of the corresponding problems and is adequate in many real cases. Therefore, this model (i.e. the model (26)) is called a shear-spring type resistance model and has been used in many investigations carried out within the framework of classical elastodynamics by Jones and Wittier (1967), Berger, Martin and McCaffery (2000), Kepceler (2010) and others. According to this statement, we also use the model (26) for the mathematical formulation of the imperfectness of the contact conditions. For the problems under consideration these conditions can be written as follows:

$$Q'^{(1)}_{r'r'}\Big|_{r'=R'} = Q'^{(2)}_{r'r'}\Big|_{r'=R'}, \; Q'^{(1)}_{r'z'}\Big|_{r'=R'} = Q'^{(2)}_{r'z'}\Big|_{r'=R'}, \quad u'^{(1)}_{r'}\Big|_{r'=R'} = u'^{(2)}_{r'}\Big|_{r'=R'},$$

$$u'^{(1)}_{z'}\Big|_{r'=R'} - u'^{(2)}_{z'}\Big|_{r'=R'} = F\frac{R}{\mu^{(2)}} Q'^{(2)}_{r'z'}, \tag{27}$$

where F is the non-dimensional shear-spring parameter. The case where $F = 0$ corresponds to the perfect contact condition, but the case where $F = \infty$ corresponds to the fully slipping imperfectness of the contact condition.

This completes the formulation of the problem. It should be noted that in the case where $\lambda_3^{(k)} = \lambda_1^{(k)} = 1.0$, $(k = 1, 2)$ the above described formulation transforms to the corresponding one of the classical linear theory of elastodynamics for the compressible body.

3 Solution procedure and obtaining the dispersion equation

Substituting (4) in (3) we obtain the following equation of motion in displacement terms:

$$\omega'^{(k)}_{1111}\frac{\partial^2 u'^{(k)}_{r'}}{\partial r'^2}+\omega'^{(k)}_{1122}\frac{\partial}{\partial r'}\left(\frac{u'^{(k)}_{r'}}{r'}\right)+\left(\omega'^{(k)}_{1133}+\omega'^{(k)}_{1331}\right)\frac{\partial^2 u'^{(k)}_{r'}}{\partial r'\partial z'}+$$

$$+\omega'^{(k)}_{1313}\frac{\partial^2 u'^{(k)}_{r'}}{\partial z'^2}+\frac{1}{r'}\left(\omega'^{(k)}_{1111}-\omega'^{(k)}_{2211}\right)\times$$

$$\times\frac{\partial u'^{(k)}_{r'}}{\partial r'}+\left(\omega'^{(k)}_{1122}-\omega'^{(k)}_{2222}\right)\frac{u'^{(k)}_{r'}}{r'^2}+\left(\omega'^{(k)}_{1133}-\omega'^{(k)}_{2233}\right)\frac{1}{r'}\frac{\partial u'^{(k)}_{3}}{\partial z'}=\rho'^{(k)}\frac{\partial^2 u'^{(k)}_{r'}}{\partial t^2},$$

$$\omega'^{(k)}_{3133}\frac{\partial^2 u'^{(k)}_{r'}}{\partial r'\partial z'}+\omega'^{(k)}_{3131}\frac{\partial^2 u'^{(k)}_{3}}{\partial r'^2}+\frac{1}{r'}\omega'^{(k)}_{3113}\frac{\partial u'^{(k)}_{r'}}{\partial z'}+\frac{1}{r'}\omega'^{(k)}_{3131}\frac{\partial u'^{(k)}_{3}}{\partial r'}+\omega'^{(k)}_{3311}\frac{\partial^2 u'^{(k)}_{r'}}{\partial r'\partial z'}+$$

$$\omega'^{(k)}_{3322}\frac{1}{r'}\frac{\partial u'^{(k)}_{r'}}{\partial z'}+\omega'^{(k)}_{3333}\frac{\partial^2 u'^{(k)}_{3}}{\partial z'^2}=\rho'^{(k)}\frac{\partial^2 u'^{(k)}_{3}}{\partial t^2}. \tag{28}$$

According to the monograph by Guz (2004), we use the following representation for the displacement:

$$u'^{(k)}_{r'}=-\frac{\partial^2}{\partial r'\partial z'}\mathrm{X}^{(k)},$$

$$u'^{(k)}_{3}=\frac{1}{\omega'^{(k)}_{1133}+\omega'^{(k)}_{1313}}\left(\omega'^{(k)}_{1111}\Delta'_1+\omega'^{(k)}_{3113}\frac{\partial^2}{\partial z'^2}-\rho'^{(k)}\frac{\partial^2}{\partial t^2}\right)\mathrm{X}^{(k)}, \tag{29}$$

where $\mathrm{X}^{(k)}$ satisfies the following equation:

$$\left[\left(\Delta'_1+\left(\xi'^{(k)}_{2}\right)^2\frac{\partial^2}{\partial z'^2}\right)\left(\Delta'_1+\left(\xi'^{(k)}_{3}\right)^2\frac{\partial^2}{\partial z'^2}\right)\right.$$

$$-\rho'^{(k)}\left(\frac{\omega'^{(k)}_{1111}+\omega'^{(k)}_{1331}}{\omega'^{(k)}_{1111}\omega'^{(k)}_{1331}}\Delta'_1+\frac{\omega'^{(k)}_{3333}+\omega'^{(k)}_{3113}}{\omega'^{(k)}_{1111}\omega'^{(k)}_{1331}}\frac{\partial^2}{\partial z'^2}\right)\frac{\partial^2}{\partial t^2}+$$

$$\left.+\frac{\rho'^{(k)}}{\omega'^{(k)}_{1111}\omega'^{(k)}_{1331}}\frac{\partial^4}{\partial t^4}\right]\mathrm{X}^{(k)}=0. \tag{30}$$

In (29) and (30) the following notation is used:

$$\Delta'_1=\frac{d^2}{dr'^2}+\frac{1}{r'}\frac{d}{dr'},\quad \left(\xi'^{(k)}_{2,3}\right)^2=d^{(k)}\pm\left[\left(d^{(k)}\right)^2-\omega'^{(k)}_{3333}\omega'^{(k)}_{3113}\left(\omega'^{(k)}_{1111}\omega'^{(k)}_{1331}\right)^{-1}\right]^{\frac{1}{2}},$$

$$d^{(k)} = \left(2\omega'^{(k)}_{1111}\omega'^{(k)}_{1331}\right)^{-1}\left[\omega'^{(k)}_{1111}\omega'^{(k)}_{3333} + \omega'^{(k)}_{1331}\omega'^{(k)}_{3113} - \left(\omega'^{(k)}_{1133} + \omega'^{(k)}_{1313}\right)\right]. \tag{31}$$

We represent the function $\mathrm{X}^{(m)} = \mathrm{X}^{(m)}\left(r', y'_3, t\right)$ as

$$\mathrm{X}^{(m)} = \mathrm{X}^{(m)}_1\left(r'\right)\cos\left(kz' - \omega t\right), \quad m = 1, 2. \tag{32}$$

Substituting (32) in (30) and doing some manipulations we obtain the following equation for $\mathrm{X}^{(m)}_1\left(r'\right)$:

$$\left(\Delta'_1 + \left(\zeta'^{(m)}_2\right)^2\right)\left(\Delta'_1 + \left(\zeta'^{(m)}_3\right)^2\right)\mathrm{X}^{(m)}_1(r') = 0. \tag{33}$$

The constants $\zeta'^{(k)}_{2,3}$ are determined from the following equation:

$$\omega'^{(m)}_{1111}\omega'^{(m)}_{1331}\left(\zeta'^{(m)}\right)^4 -$$
$$k^2\left(\zeta'^{(m)}\right)^2\left[\omega'^{(m)}_{1111}\left(\rho^{(m)}c^2 - \omega'^{(m)}_{3333}\right) + \omega'^{(m)}_{1331}\left(\rho^{(m)}c^2 - \omega'^{(m)}_{3113}\right) +\right.$$
$$\left. + \left(\omega'^{(m)}_{1133} + \omega'^{(m)}_{1313}\right)^2\right] + k^4\left(\rho^{(m)}c^2 - \omega'^{(m)}_{3333}\right)\left(\rho^{(m)}c^2 - \omega'^{(m)}_{3113}\right) = 0, \tag{34}$$

where $c = \omega/k$, i.e. c is the phase velocity of the propagating wave. We determine the following expression for $\mathrm{X}^{(m)}_1\left(r'\right)$ from equations (33) and (34):

$$\mathrm{X}^{(1)}_1\left(r'\right) =$$
$$A^{(1)}_2 E^{(1)}_0\left(kr'\zeta'^{(1)}_2\right) + A^{(1)}_3 E^{(1)}_0\left(kr'\zeta'^{(1)}_3\right) + B^{(1)}_2 G^{(1)}_0\left(kr'\zeta'^{(1)}_2\right) + B^{(1)}_3 G^{(1)}_0\left(kr'\zeta'^{(1)}_3\right),$$

$$\mathrm{X}^{(2)}_1\left(r'\right) =$$
$$A^{(2)}_2 E^{(2)}_0\left(kr'\zeta'^{(2)}_2\right) + A^{(2)}_3 E^{(2)}_0\left(kr'\zeta'^{(2)}_3\right) + B^{(2)}_2 G^{(2)}_0\left(kr'\zeta'^{(2)}_2\right) + B^{(2)}_3 G^{(2)}_0\left(kr'\zeta'^{(2)}_3\right), \tag{35}$$

where

$$E^{(k)}_0\left(kr'\zeta'^{(k)}_m\right) = \begin{cases} J_0\left(kr'\zeta'^{(k)}_m\right) & if \quad \left(\zeta^{(k)}_m\right)^2 > 0, \\ I_0\left(kr'\left|\zeta'^{(k)}_m\right|\right) & if \quad \left(\zeta^{(k)}_m\right)^2 < 0, \end{cases}.$$

$$G_0^{(k)}\left(kr'\zeta'^{(k)}_m\right) = \begin{cases} Y_0\left(kr'\zeta'^{(k)}_m\right) & if \quad \left(\zeta_m^{(k)}\right)^2 > 0, \\ K_0\left(kr'\left|\zeta'^{(k)}_m\right|\right) & if \quad \left(\zeta_m^{(k)}\right)^2 < 0. \end{cases} \tag{36}$$

In (36) $J_0(x)$ and $Y_0(x)$are Bessel functions of the first and second kind of order zero and $I_0(x)$ and $K_0(x)$ are Bessel functions of a purely imaginary argument of order zero and Macdonald function of order zero, respectively.

Thus, using (4), (29), (32), (35), and (36) we obtain the following dispersion equation from (23) and (27):

$$\det\left\|\beta_{ij}\right\| = 0, \quad i;j = 1,2,3,4,5,6,7,8. \tag{37}$$

The expression of β_{ij} are given in Appendix A by formulae (A1) and (A2).

4 Numerical results and discussion

Assume that $\rho^{(2)}/\rho^{(1)} = 1.0$, $\lambda^{(2)}/\mu^{(2)} = \lambda^{(1)}/\mu^{(1)} = 1.5$, consider the dispersion curves $c = c(kR)$ and analyze the influence of the non-dimensional shear-spring parameter F on these curves for various values of elongation parameters $\lambda_3^{(2)}$ and $\lambda_3^{(1)}$. To simplify the following discussions we introduce the following notation:

$$c_{20}^{(k)} = \sqrt{\frac{\mu^{(k)}}{\rho^{(k)}}}, \quad c_2^{(k)}(\lambda_3^{(k)}) = \sqrt{\frac{\omega'^{(k)}_{1313}}{\rho'^{(k)}}}, \tag{38}$$

where $c_{20}^{(k)} = c_2^{(k)}(1.0)$.

4.1 On the algorithm of the calculation

The numerical results of the dispersion of the considered wave propagation problem are obtained from the numerical solution to equation (38) which is solved by utilizing the well known "bisection method". In this case, for fixed values of the problem parameters for each value of *kR*, the roots of the dispersion equation with respect to $c/c_2^{(2)}$ are found.

In the present paper the main purpose of the numerical investigations is the study of the influence of shear-spring type imperfectness on the contact conditions between the inner and outer cylinders of the pre-strained compound cylinder on the fundamental modes. However, for construction of the dispersion curves corresponding to these modes it is necessary to use the certain N number roots of equation (37). In this case, the graphs of the dependencies among the roots $\left(c/c_2^{(2)}\right)_1$, $\left(c/c_2^{(2)}\right)_2, \ldots,$

$\left(c/c_2^{(2)}\right)_N$ and kR create the net on the plane $\left\{c/c_2^{(2)}, kR\right\}$. Note that, in general, the graph corresponding to the dependence between $\left(c/c_2^{(2)}\right)_n$ and kR, contains dispersive and non-dispersive parts related to various dispersion modes. Below, under construction of the dispersion curves, we mainly use the dispersive parts of these graphs.

In the present paper we analyze the dispersion of the first (fundamental) mode.

4.2 Numerical results related to perfect contact conditions

First, we consider the case where the contact conditions on the interface between the layers of the cylinder are perfect, i.e. the case where $F = 0$ in equation (27). Recall that Akbarov and Guliev (2009) made the corresponding analysis for the bi-material solid cylinder.

According to the procedure described in this paper, in the case where $F = 0$ we determine the following low and high wave number limits for the wave propagation velocity in the first mode:

$$\frac{c}{c_{20}^{(2)}} = \frac{e^{(2)}\left(\lambda_3^{(2)}\right)^2 \eta^{(2)} + e^{(1)}\left(\lambda_3^{(1)}\right)^2 \eta^{(1)}\mu^{(1)}/\mu^{(2)}}{\eta^{(2)} + \eta^{(1)}\rho^{(1)}/\rho^{(2)}}$$

as $kR \to 0$,

$$\frac{c}{c_{20}^{(2)}} = \min\left(c_R^{(1)}(\lambda_3^{(1)})\Big/c_{20}^{(2)}; c_R^{(2)}(\lambda_3^{(2)})\Big/c_{20}^{(2)}\right) \tag{39}$$

as $kR \to \infty$, where $c_R^{(2)}(\lambda_3^{(2)})\left(c_R^{(1)}(\lambda_3^{(1)})\right)$ is the Rayleigh wave velocity in the strained inner (outer) cylinder material and the notation

$$e^{(k)} = 2\left(1 + \lambda^{(k)}/\left(2\left(\lambda^{(k)} + \mu^{(k)}\right)\right)\right),$$

$$\eta^{(2)} = \frac{2 - h^{(2)}/R}{\left(1 + h^{(1)}/h^{(2)}\right)\left(2 + h^{(1)}/R - h^{(2)}/R\right)},$$

$$\eta^{(1)} = \frac{2 + h^{(1)}/R}{\left(1 + h^{(1)}/h^{(2)}\right)\left(2 + h^{(1)}/R - h^{(2)}/R\right)}. \tag{40}$$

is also used in (39). For the case under consideration the values of $c_R^{(k)}(\lambda_3^{(k)})(k = 1, 2)$ are determined from the following characteristic equation

$$\mathfrak{R}_k(c_R^{(k)}(\lambda_3^{(k)}), \lambda_3^{(k)}, \mu^{(k)}/\lambda^{(k)}) =$$

$$\left((X^{(k)})^2-\frac{2(\lambda_3^{(k)})^3}{\lambda_1^{(k)}+\lambda_3^{(k)}}\right)\frac{2(\lambda_3^{(k)})^2}{(\lambda_1^{(k)})^2(\lambda_1^{(k)}+\lambda_3^{(k)})}\left[\frac{1+2\mu^{(k)}/\lambda^{(k)}}{\lambda_3^{(k)}}\times\right.$$

$$\left.\left((X^{(k)})^2-(\lambda_3^{(k)})^3(2+\lambda^{(k)}/\mu^{(k)})\right)+\frac{\lambda^{(k)}}{\lambda_1^{(k)}\mu^{(k)}}\right]^2$$

$$-\frac{\mu^{(k)}}{\lambda^{(k)}}\frac{1+2\mu^{(k)}/\lambda^{(k)}}{\lambda_3^{(k)}}\left((X^{(k)})^2-(\lambda_3^{(k)})^3(2+\lambda^{(k)}/\mu^{(k)})\right)$$

$$\left((X^{(k)})^2-\frac{2(\lambda_3^{(k)})^3}{\lambda_1^{(k)}+\lambda_3^{(k)}}\right)\frac{2(\lambda_3^{(k)})^2}{(\lambda_1^{(k)})^2(\lambda_1^{(k)}+\lambda_3^{(k)})}+\left.\frac{4(\lambda_1^{(k)})^2\lambda_3^{(k)}}{(\lambda_1^{(k)}+\lambda_3^{(k)})^2}\right]^2=0, \tag{41}$$

where $X^{(k)}=c_R^{(k)}(\lambda_3^{(k)})/c_{20}^{(k)}$. Note that in the case where $\lambda_3^{(k)}=\lambda_1^{(k)}=\lambda_2^{(k)}=1.0$ this characteristic equation coincide with the corresponding one obtained in the classical linear theory of elasto-dynamics.

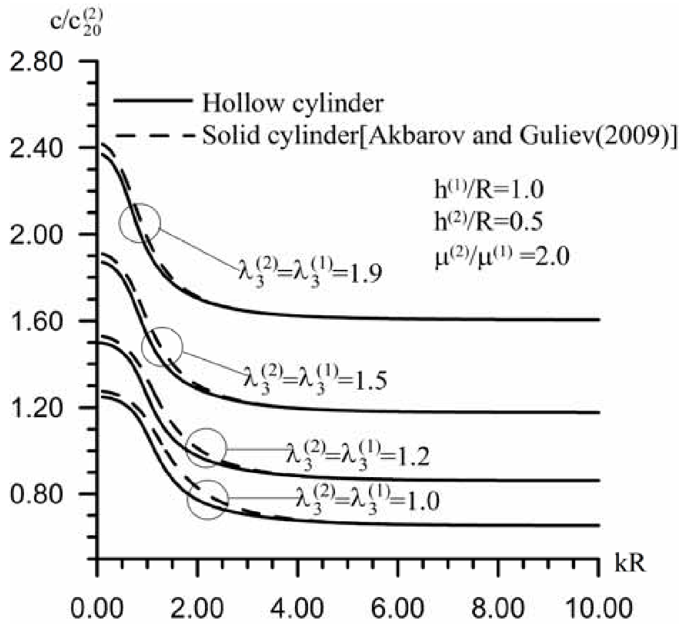

Figure 2: Dispersion curves obtained for the perfect contact conditions for the hollow and solid bi-material cylinders under initial stretching of those

Now we consider the numerical results, which are obtained with the numerical solution of the dispersion equation (37). In the present paper we consider the case where the material of the inner cylinder is stiffer than the material of the outer cylinder and $\mu^{(2)}/\mu^{(1)}$ =2.0. Moreover, all numerical results which will be discussed in the present subsection are obtained under $h^{(1)}/R$ =1.0. Moreover, we assume that $h^{(2)}/R$ =0.5 if otherwise not specified.

Fig. 2 shows the influence of the initial stretching of the layers of the cylinder on the dispersion curves. In Fig. 2 the corresponding results obtained by Akbarov and Guliev (2009) are also presented. The influence of the initial compression of the cylinder under consideration on the dispersion curves is illustrated by the graphs given in Fig. 3. Note that in Fig. 3 the corresponding results related to the bi-material solid cylinder which were not analyzed by Akbarov and Guliev (2009), are also given. Remember, that in Akbarov and Guliev's (2009) paper, the influence of the pre-stretching of the constituents of the cylinder on the wave propagation velocity was studied only. It follows from the results that the values of the wave propagation velocity obtained for the hollow bi-layered cylinder approaches the corresponding ones obtained for the bi-material solid cylinder. This conclusion agrees with the well-known mechanical consideration. Moreover, these results show that the initial stretching (compressing) of the cylinder causes to increase (to decrease) the wave propagation velocity. In this case the values of $c/c_{20}^{(2)}$ obtained for the hollow cylinder are less than the corresponding ones obtained for the solid cylinder. Because of this, we consider the case where $c_{20}^{(2)} > c_{20}^{(1)}$, $\mu^{(2)} > \mu^{(1)}$ and $\lambda_3^{(1)} = \lambda_3^{(2)}$.

Now we return to the analysis of the numerical results on the influences of the non-dimensional shear-spring parameter F. First we consider the case where the initial strains are absent in the constituents of the compound cylinder.

4.3 Numerical results related to the case where $\lambda_3^{(2)} = \lambda_3^{(1)} = 1.0$

Consider the dispersion curves given in Fig. 4. These curves are constructed for various values of the shear-spring parameter F which characterizes the degree of the imperfectness of the contact conditions.

The analyses of the foregoing numerical results show that as a result of the shear-spring type imperfectness of the contact conditions, instead of the dispersion curves corresponding to the fundamental dispersive mode constructed under satisfaction of perfect contact conditions (i.e. for $F = 0$) and illustrated in Figs. 2 and 3, two types of mode arise. The first (the second) appears below (over) the dispersion curve corresponding to the first mode constructed for $F = 0$. Throughout the discussion below the aforementioned first (second) type dispersion curve will be called the first (second) branch of the fundamental mode obtained under imperfect contact conditions, i.e. for $F > 0$.

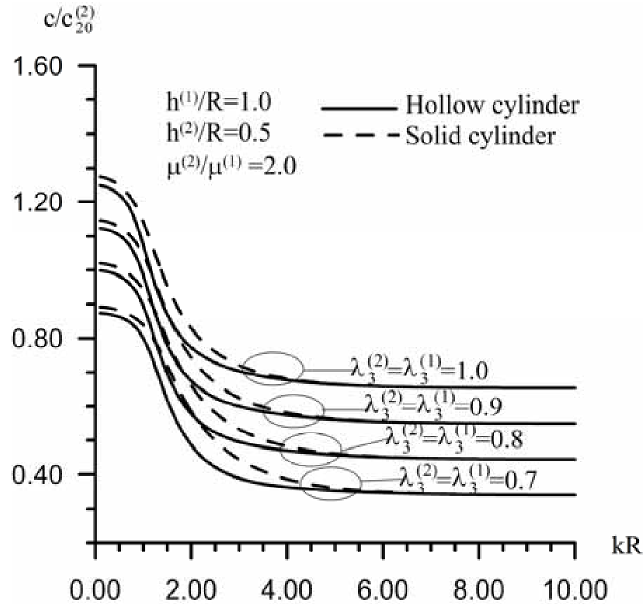

Figure 3: Dispersion curves obtained for the perfect contact conditions for the hollow and solid bi-material cylinders under initial compression of those

The graphs illustrated above are constructed under $\mu^{(2)}/\mu^{(1)} = 2$. For estimation how the change of $\mu^{(2)}/\mu^{(1)}$acts on the behavior of these graphs we consider Fig. 5 which shows dispersion curves constructed for various $\mu^{(2)}/\mu^{(1)}$ under perfect contact between the layers of the cylinder, i.e. under $F = 0$. It follows from Fig. 5 that the dispersion curves "moves" wholly down with $\mu^{(2)}/\mu^{(1)}$, i.e. the values of $c/c_{20}^{(2)}$decrease with $\mu^{(2)}/\mu^{(1)}$. Moreover, Fig. 5 shows that in the case where $\mu^{(2)}/\mu^{(1)} = 1$, i.e. the dispersion curves obtained for the homogeneous hollow cylinder coincide with the well-known results related to the classical Pochhammer-Chree problems for the hollow cylinder (see, for example, Eringen and Suhubi (1975)). According to the results given in Fig. 5 we can conclude that the graphs given in Fig. 4 must also "move" down wholly with $\mu^{(2)}/\mu^{(1)}$.

We denote the velocity of the wave propagation velocity for $F = 0$ with c, but the wave propagation velocity of the first (second) branch for $F > 0$ we denote with $c_{IF}(c_{IIF})$. It follows from the numerical results given in Fig. 4 that the following relation takes place:

$$c_{IF} < c < c_{IIF}. \tag{42}$$

Consider the low wave number limit of the wave propagation velocity as $kR \to 0$

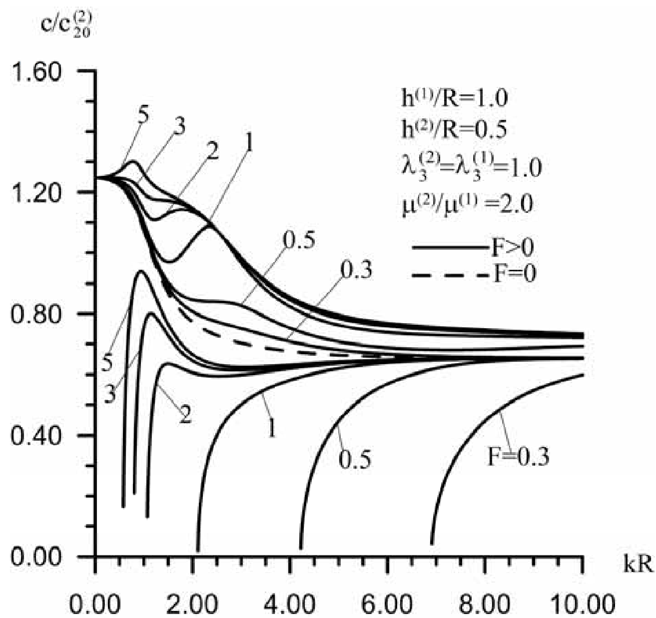

Figure 4: Dispersion curves obtained for the imperfect contact conditions for the hollow bi-layered cylinder

for both the first and the second branches of the dispersion curves. The numerical results show that the first branch of the dispersion curves has "cut off" values of kR (denoted by $(kR)_{cf}$), i.e. the dispersion curves related to the first branch appear after certain values of kR. In this case the values of $(kR)_{cf}$ depend on the non-dimensional shear-spring parameter F. According to the numerical results, we can conclude that:

$$(kR)_{cf} \to \infty \tag{43}$$

as $F \to 0$.

The wave propagation velocity in the second branch of the fundamental mode has a finite limit as $kR \to 0$ and this limit coincides with that obtained for the case where the contact conditions are perfect, i.e. for the case where $F = 0$. The specified limit is determined by the expression:

$$\frac{c_{IIF}}{c_{20}^{(2)}} = \frac{e^{(2)}\eta^{(2)} + e^{(1)}\eta^{(1)}\mu^{(1)}/\mu^{(2)}}{\eta^{(2)} + \eta^{(1)}\rho^{(1)}/\rho^{(2)}} \tag{44}$$

as $kR \to 0$, which coincides with the first expression in (39) under $\lambda_3^{(2)} = \lambda_3^{(1)} = 1.0$. This statement, i.e. the independence of the low wave number limit as $kR \to 0$ on

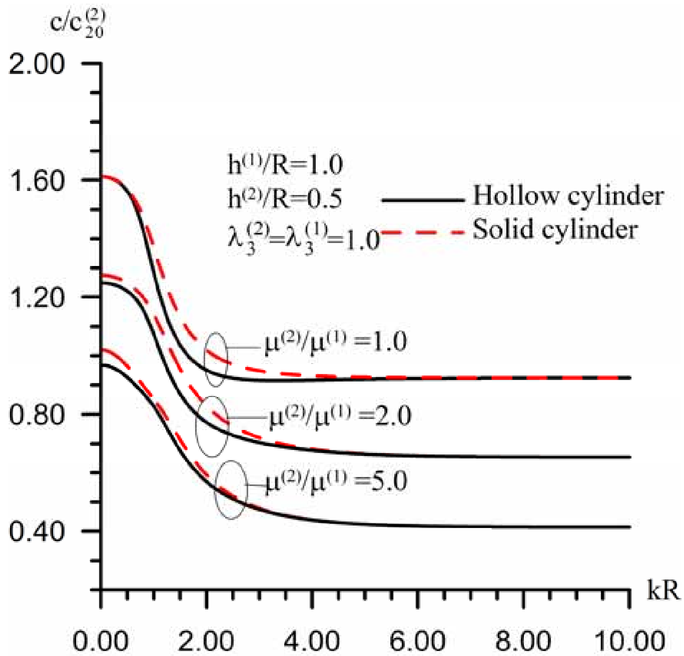

Figure 5: The influence of the ratio $\mu^{(2)}/\mu^{(1)}$on the character of the dispersion curves constructed under $F=0$

the imperfectness of the contact conditions agrees with the physical considerations and has been also pointed out in papers by Berger et al. (2000) and Kepceler (2010) for torsional wave propagation in a compounded cylinder.

Consider the high wave number limit values as $kR \to \infty$. It follows from Fig. 4 and other results (which are not given here) that for the values of $kR >> 10$ for the case under consideration the following high wave number limit values occur:

$$c_{IF} \to c_R^{(1)} - 0, \quad c_{IIF} \to c_R^{(1)} + 0 \tag{45}$$

as $kR \to \infty$.

Note that the expression (45) can be generalized for all possible values of the ratio $\mu^{(2)}/\mu^{(1)}$, as follows:

$$c_{IF} \to \min\left\{c_R^{(1)} - 0;\ c_R^{(2)} - 0;\ c_S - 0\right\}, \quad c_{IIF} \to \min\left\{c_R^{(1)} + 0;\ c_R^{(2)} + 0;\ c_S + 0\right\} \tag{46}$$

as $kR \to \infty$, where $c_R^{(1)}(c_R^{(2)})$ is the Rayleigh wave velocity in the inner (outer) cylinder material and c_S is the Stoneley wave velocity. The characteristic equation for the Rayleigh wave velocity is given above by equation (41), but the characteristic equation for the Stoneley wave velocity for the case under consideration is

$$\det\|\alpha_{nm}\| = 0, \quad n;m = 1,2,3,4 \tag{47}$$

The explicit expressions of $\alpha_{nm} = \alpha_{nm}(c_S(\lambda_3^{(1)}, \lambda_3^{(2)}), \mu^{(1)}/\lambda^{(1)}, \mu^{(2)}/\lambda^{(2)})$are given in Appendix B by formulae (B1) and (B2). Consequently, according to the foregoing discussions, we get

$$\det\left\|\beta_{ij}\right\| \Rightarrow \ \det\|\alpha_{nm}(c_S)\| \cdot \Re_1(c_R^{(1)}) \cdot \Re_2(c_R^{(2)}) \tag{48}$$

as $kR \to \infty$, where $\det\left\|\beta_{ij}\right\|$ is the right side of the dispersion equation (37), $\det\|\alpha_{nm}(c_S)\|$ is a right side of the characteristic equation (47) for Stoneley waves, but the expressions for $\Re_1(c_R^{(1)})$ and $\Re_2(c_R^{(2)})$are determined through (41).

Now we analyze the character of the dispersion curves. The dispersion curves obtained for the first branch show that after a certain value (denoted by F_{I*}) of the shear-spring parameter F, the dependence between c_{IF} and kR becomes non-monotonic. In other words, in the cases where $F > F_{I*}$ in the near right vicinity of $(kR)_{cf.}$, the values of c_{IF}(denoted by $(kR)_{Icf.}$) increase sharply with kR and have their maximum at a certain value of kR. So, we can write:

$$\left.\frac{dc_{IF}}{d(kR)}\right|_{kR=(kR)_{Icr.}} = 0. \tag{49}$$

In the cases where $kR > (kR)_{Icr.}$the values of c_{IF} decrease monotonically with kR.

Consider the behavior of the second branch of the dispersion curves. It follows from the foregoing results that before a certain value of kR(denoted by $(kR)_m$) the second branch of the fundamental mode obtained for the case where $F > 0$ merges with that obtained for the corresponding case where $F = 0$. The values of $(kR)_m$ depend on the shear-spring parameter F and the ratios $h^{(1)}/R$ and $h^{(2)}/R$. Analyses of the results given in Fig. 4 and other ones (which are not given here) show that the following estimation occurs:

$$(kR)_m \to 0 \tag{50}$$

as $F \to \infty$.

The results also show that the values of c_{IIF} increase with F. However, the character of the second branch of the dispersion curves becomes more complicated with an increase in F. There exist such values of kR under which the local maximum or minimum for the wave propagation velocity c_{IIF} appears. The values of

kR(denoted by $(kR)_{IIcr.}$) which correspond to these local maximums or minimums are determined from the following relation:

$$\left.\frac{dc_{IIF}}{d(kR)}\right|_{kR=(kR)_{IIcr.}} = 0. \tag{51}$$

It should be noted that the cases for which the relation (51) occurs, appear after certain values of the shear-spring parameter F. Moreover, we note that the existence of (49) and (51) type relations means that where $kR = (kR)_{Icr.}$or where $kR = (kR)_{IIcr.}$the group velocity of the wave propagation is equal to its phase velocity. Consequently, the point $kR = (kR)_{Icr.}$ or the point $kR = (kR)_{IIcr.}$separates the parts of the dispersion curves, which correspond to anomalous and normal dispersions.

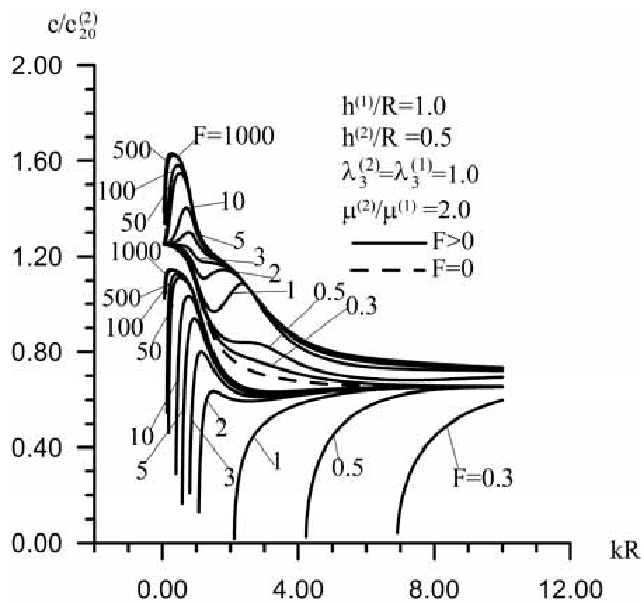

Figure 6: Limit dispersion curve obtained by increasing the shear-spring parameter F

As noted above, the case where $F = 0$ corresponds to the perfect interface conditions but the case where $F = \infty$ corresponds to the fully slipping interface conditions. It follows from the results discussed above that:

$$c_{IIF} \to c + 0 \tag{52}$$

as $F \to 0$.

This statement agrees with the well known mechanical considerations. Consider the behavior of the dispersion curves as $F \to \infty$. For this purpose analyze Fig. 6 which shows the dispersion curves constructed for the case where $0 \le F \le 1000$ when $\{h^{(1)}/R = 1.0; h^{(2)}/R = 0.5\}$. It follows from these results that the curves $c_{IF} = c_{IF}(kR)$ and $c_{IIF} = c_{IIF}(kR)$ approach their limits as F increases. In this case for each fixed kR, the velocities c_{IF} and c_{IIF} increase monotonically with F and the difference between the values of $c_{IF}/c_{20}^{(2)}$(or $c_{IIF}/c_{20}^{(2)}$) obtained in the cases where $F = 500$ and $F = 1000$, is less than 10^{-5}. Consequently, the results obtained when $F = 1000$, can be taken with very high accuracy as results corresponding to the fully slipping interface conditions.

With this we exhaust the discussions of the numerical results related to the case where the initial strains are absent in the layers of the hollow cylinder.

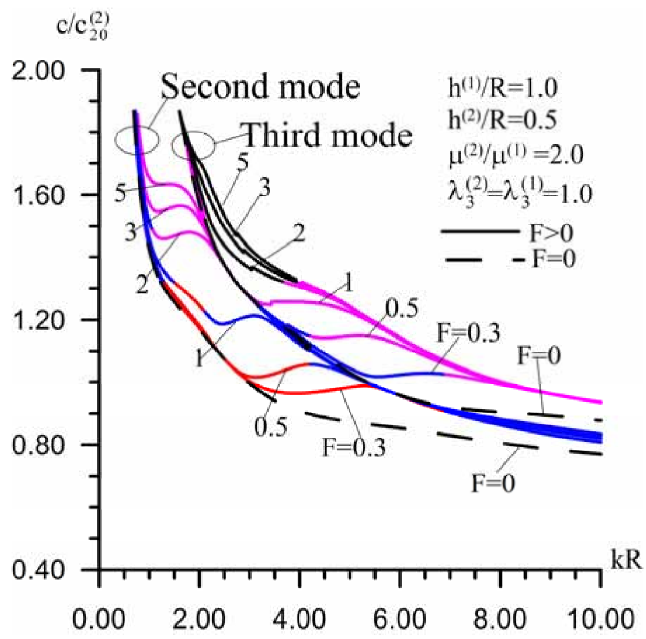

Figure 7: Dispersion curves related to the second and third modes constructed for various values of the parameter F

All the results discussed above relate to the dispersion curves of the fundamental (the first mode) of the axisymmetric wave propagation under consideration. It is also interest, how the shear-spring parameter Finfluences on the dispersion curves higher modes. For this purpose we consider dispersion curves constructed for various $F$for the second and third modes and illustrated in Fig. 7. It follows from these results that the considered type imperfectness of the contact conditions between the layers of the cylinder change also significantly the character of the dispersion curves of the higher modes. Nevertheless, the dispersion curves of the higher modes have only one branch similar to that obtained within the scope of the perfect contact conditions. Note that the detail investigations of the influence of the parameter F on the dispersion curves of the higher modes may be subject of the separate paper which will be made by the authors in future.

4.4 Numerical results related to the pre-strained case

The influence of the initial strains on the dispersion curves in the case where the contact conditions are perfect, i.e. in the case where $F = 0$, is illustrated by the graphs given in Figs. 2 and 3. Thus, consider the numerical results which illustrate the influence of the initial strains of the layers of the cylinder on the dispersion curves in the case where the contact on the interface of the layers of the cylinder is imperfect, i.e. the case where $F > 0$. These numerical results are given in Figs 8 and 9 for the first and second branches respectively of the fundamental mode. Note that to improve the illustrations the dispersion curves constructed under initial stretching (Figs.8a and 9a) and under initial compressing (Fig. 8b and 9b) of the cylinder are given separately. It follows from the foregoing numerical results that the considered type initial strains in the compound cylinder do not change (in the qualitative sense) the character of the influence of the imperfectness of the interface conditions on the character of the dispersion curves. Consequently, the considered type initial strains cause an increase (under initial stretching) or a decrease (under initial compression) in the values of the wave propagation velocity. However, in this case the relation (43) must be rewritten as follows:

$$c_{IF}(\lambda_3^{(1)}, \lambda_3^{(2)}) < c(\lambda_3^{(1)}, \lambda_3^{(2)}) < c_{IIF}(\lambda_3^{(1)}, \lambda_3^{(2)}). \tag{53}$$

Here $c_{IF}(\lambda_3^{(1)}, \lambda_3^{(2)})$, $c_{IIF}(\lambda_3^{(1)}, \lambda_3^{(2)})$ and $c(\lambda_3^{(1)}, \lambda_3^{(2)})$ are the values of c_{IF}, c_{IIF} and c respectively in the pre-strained case.

At the same time, the numerical results show that the relation (44) also holds for the pre-strained case. Nevertheless, the initial strains significantly change the values of $(kR)_{cf.}$so that under initial stretching, the values of $(kR)_{cf.}$decrease with $\lambda_3^{(1)}(=\lambda_3^{(2)})$.

Consider the influence of $\mu^{(2)}/\mu^{(1)}$ on the dispersion curves constructed under $F = 0$. These curves are given in Fig. 10 under initial stretching (Fig. 10a) and under initial compressing (Fig. 10b) of the layers of the cylinder. It follows from these

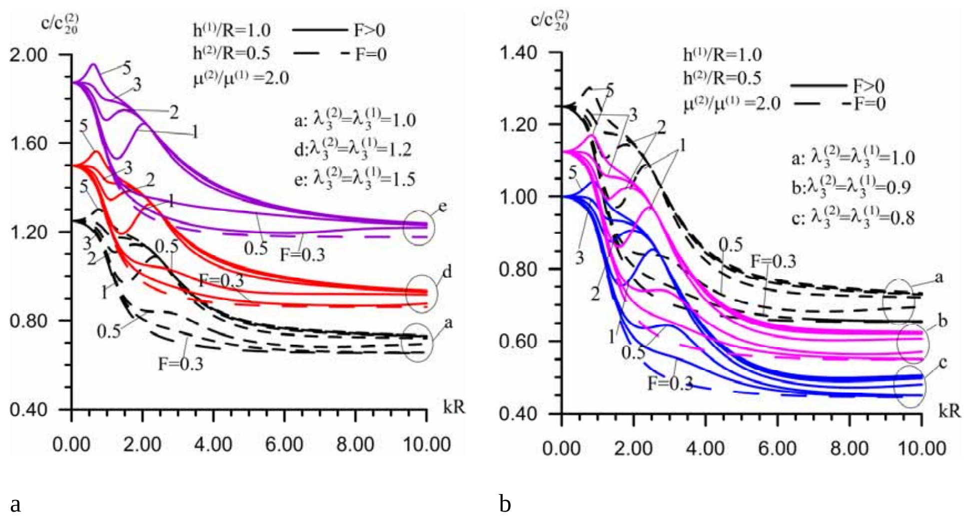

Figure 8: The influence of the initial stretching (a) and compressing (b) of the cylinder on the second branch of the dispersion curves.

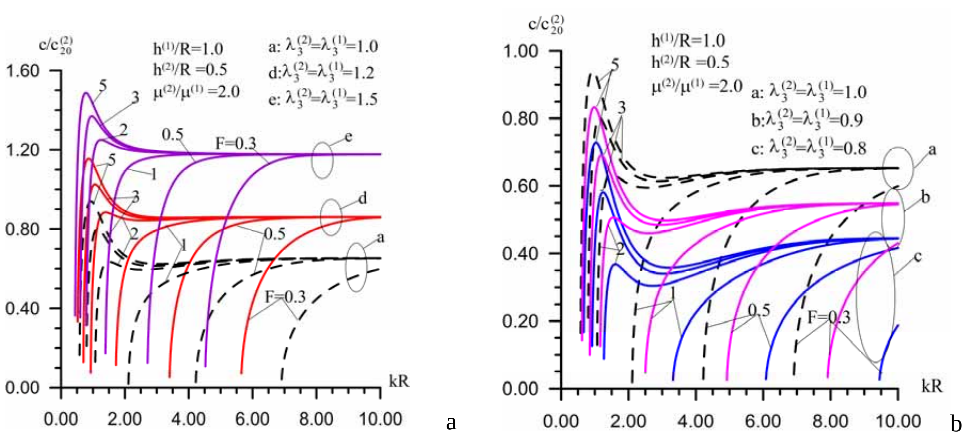

Figure 9: The influence of the initial stretching (a) and compressing (b) of the cylinder on the first branch of the dispersion curves

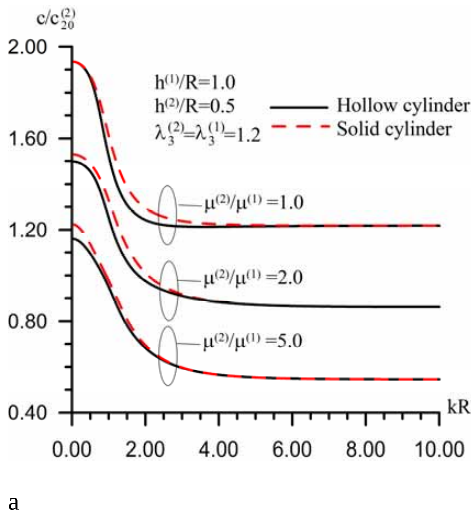

Figure 10: The influence of the ratio $\mu^{(2)}/\mu^{(1)}$ on the character of the dispersion curves obtained for the initially stretched (a) and initially compressed hollow cylinder under $F=0$

graphs that, as in the non-prestrained case, the dispersion curves "moves" down wholly with $\mu^{(2)}/\mu^{(1)}$. Note that the dispersion curves constructed in the case where $\mu^{(2)}/\mu^{(1)}=1$ correspond to that which relate to the finitely pre-deformed homogeneous hollow cylinder. To the authors best knowledge, there is not investigation published in past on the wave propagation in the finitely pre-deformed hollow cylinder, the elasticity relations for which are describes through the harmonic potential. Consequently, the results given in Fig. 10 are also new ones in the noted above sense. Moreover, according to these results we can conclude that an increase in the values of $\mu^{(2)}/\mu^{(1)}$the dispersion curves obtained under $F>0$ must also "move" down wholly.

The graphs given in Figs. 6 and 7 illustrate that in the pre-strained case, the low wave number limit as $(kR)\to 0$ of the wave propagation velocity related to the second branch of the fundamental mode does not depend on the shear-spring parameter F and this limit value is determined by the first expression in (39). From the numerical results it also follows that in the pre-strained case the expression (45) for the high wave number limit for the wave propagation velocities must be changed with the following one:

$$c_{IF}(\lambda_3^{(1)},\lambda_3^{(2)}) \to \min\left\{c_R^{(1)}(\lambda_3^{(1)})-0,\ c_R^{(2)}(\lambda_3^{(2)})-0,\ c_S(\lambda_3^{(1)},\lambda_3^{(2)})-0\right\},$$

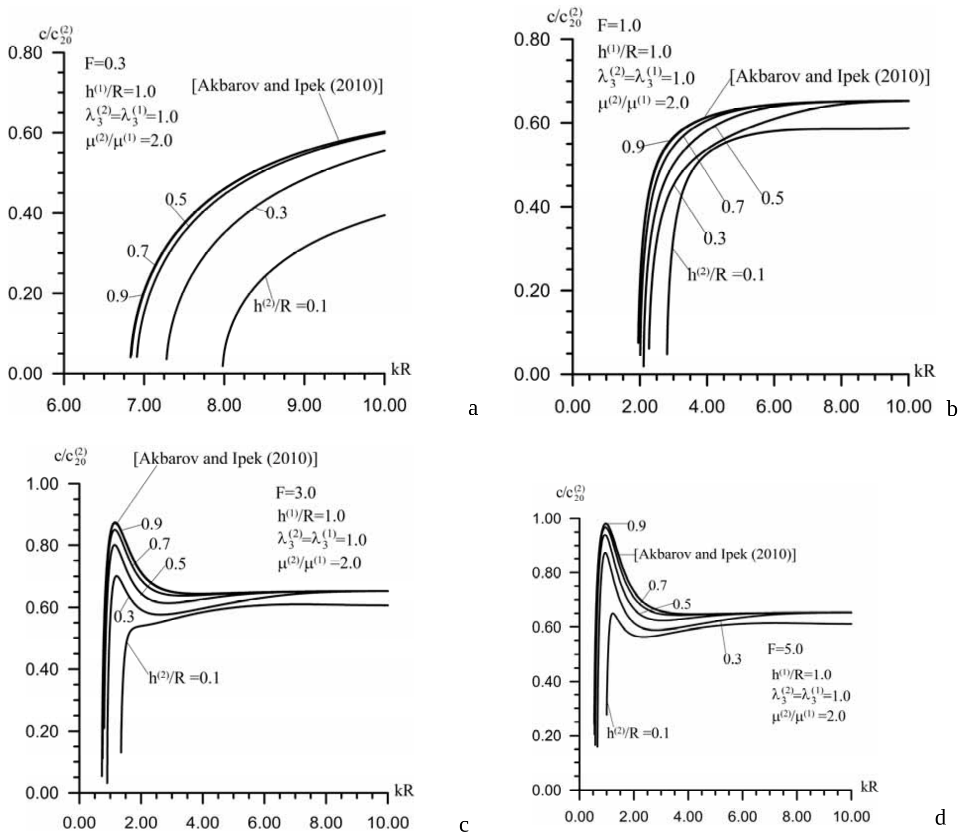

Figure 11: The influence of $h^{(2)}/R$ on the dispersion curves of the first branch in the cases where $F=0.3$ (a), 1.0 (b), 3 (c) and 5(d) under$\lambda_3^{(1)}=\lambda_3^{(2)}$ =1.0

$$c_{IIF}(\lambda_3^{(1)},\lambda_3^{(2)}) \to \min\left\{c_R^{(1)}(\lambda_3^{(1)})+0,\; c_R^{(2)}(\lambda_3^{(2)})+0,\; c_S(\lambda_3^{(1)},\lambda_3^{(2)})+0\right\} \tag{54}$$

as $kR \to \infty$, where $c_S(\lambda_3^{(1)},\lambda_3^{(2)})$ in (53) is the Stoneley wave velocity in the pre-strained case.

Moreover, the analyses of the numerical results allow us to write the following relation:

$$c_{IIF}(\lambda_3^{(1)},\lambda_3^{(2)}) \to c(\lambda_3^{(1)},\lambda_3^{(2)})+0 \tag{55}$$

as $F \to 0$, and conclude that the values of $(kR)_{I\,cr.}$ and $(kR)_{II\,cr.}$ decrease with the parameter $\lambda_3^{(1)}(=\lambda_3^{(2)})$.

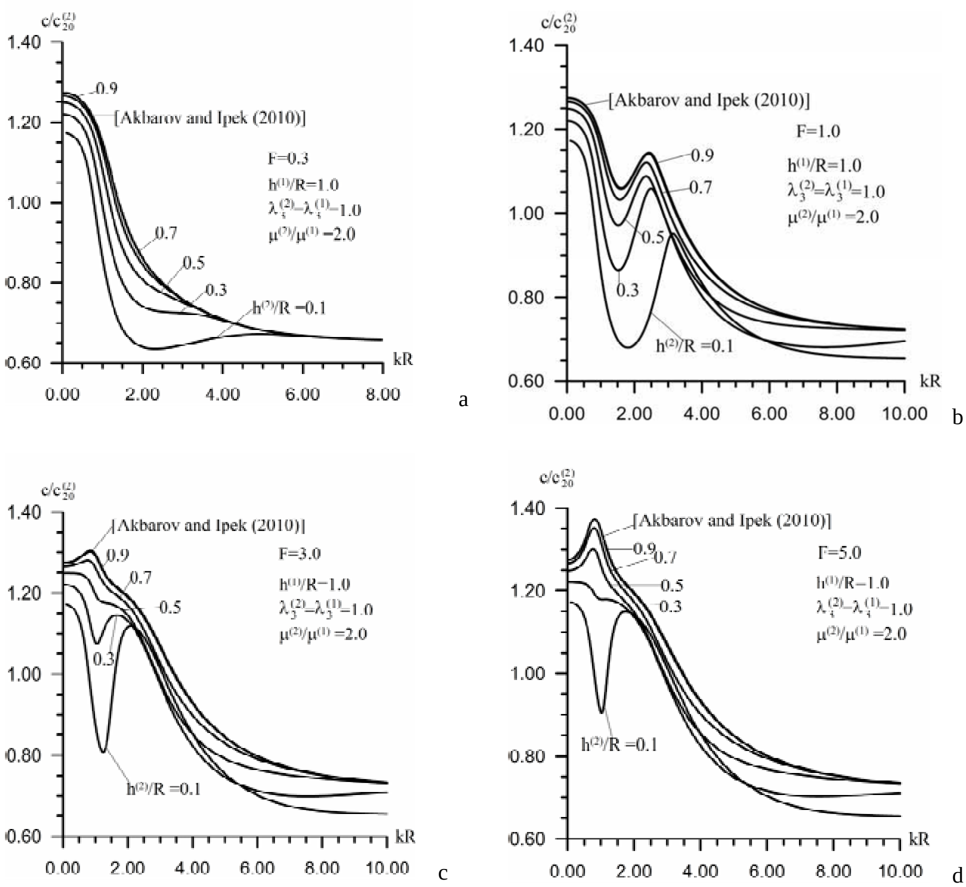

Figure 12: The influence of $h^{(2)}/R$ on the dispersion curves of the second branch in the cases where $F = 0.3$ (a), 1.0 (b), 3 (c) and 5(d) under$\lambda_3^{(1)} = \lambda_3^{(2)}$ =1.0

4.5 Comparison of the obtained numerical results with the corresponding ones obtained for the bi-material solid cylinder

At first, we note that in the qualitative sense all the foregoing numerical results are similar with those obtained for the bi-material solid cylinder (Akbarov and Ipek (2010)). The difference between the results obtained in the authors' present and previous papers is caused by the parameter $h^{(2)}/R$, which characterizes the thickness of the inner hollow cylinder. Now we consider how the change in the ratio $h^{(2)}/R$ influences the dispersion curves under imperfect contact conditions between the layers of the cylinder. For this purpose, we analyze the graphs given in Figs. 11 and 12. These graphs are constructed for the first and second branches of the fundamental mode respectively in the cases where $F = 0.3$(Figs. 11a and 12a), 1.0 (Figs. 11b and 12b), 3.0 (Figs. 11c and 12c) and 5.0 (Figs. 11d and 12d) in the absence of the initial strains in the constituents of the cylinder, i.e. under $\lambda_3^{(1)} = \lambda_3^{(2)} = 1.0$.

According to mechanical considerations, the results obtained for the bi-layered hollow cylinder must approach the corresponding ones obtained for the bi-material solid cylinder with $h^{(2)}/R$. We denote the wave propagation velocity of the first and second branches respectively in the hollow (solid) bi-material cylinder through c_{Ih} (c_{Is}) and c_{IIh} (c_{IIs}) respectively. It follows from the results given in Figs. 11 and 12 that the relation:

$$c_{Ih} < c_{Is} \text{ and } c_{Ih} \rightarrow c_{Is} - 0 \tag{56}$$

as $h^{(2)}/R \rightarrow 1$ for all considered kR, occurs for the case under consideration. The comparison between the values of c_{IIh} and c_{IIs} shows that the relation between them depends on the values of $h^{(2)}/R$, kR and the shear-spring parameter F. For relatively small values of F, for instance in the case where $F = 0.3$, it can be written that:

$$c_{IIh} < c_{IIs} \text{ and } c_{IIh} \rightarrow c_{IIs} - 0 \tag{57}$$

as $h^{(2)}/R \rightarrow 1$ *for all considered kR*.

However, in the cases where $F \geq 3$ the relation (56) holds under certain restrictions on the values of kR and the range of this restriction depends on the values of $h^{(2)}/R$. At the same time, Fig. 12 shows that the character of the dispersion curves of the second branch depends significantly on the thickness of the inner hollow cylinder, i.e. on the values of $h^{(2)}/R$. For example, in the case where $F \geq 3$ for relatively small values of $h^{(2)}/R$ under $kR = (kR)_{IIcr.}$ (in relation (51)) the value of c_{IIh} has its local minimum, but after a certain $h^{(2)}/R$ the value of c_{IIh} has its local maximum. Note that the similar type of differences between c_{IIh} and c_{IIs} are also obtained for the pre-strained bi-material cylinder.

5 Conclusions

Thus, in the present paper within the scope of the piecewise homogeneous body model utilizing the 3D linearized theory of elastic waves in initially stressed bodies, the effect of the imperfectness of the contact conditions on the dispersion of the longitudinal axisymmetric waves in the pre-strained bi-layered hollow cylinder, is studied. It is assumed that the materials of the constituents are hyper-elastic compressible ones and the elasticity relations of these materials are described by the harmonic potential. The shear-spring type imperfectness of the contact conditions is considered and the degree of this imperfectness is estimated through the shear-spring parameter F. The cases where $F = 0$ and $F = \infty$ correspond to the perfect and the fully slipping interface conditions, respectively. The solution method for the formulated corresponding eigen-value problem and the algorithm for constructing the dispersion curves are developed. From the numerical results the following conclusions are reached:

1. as a result of the shear-spring type imperfectness of the contact conditions two branches of the fundamental mode appear, the first of them disappears but the second approaches the dispersion curve obtained for the perfect contact conditions;

2. the shear-spring type imperfectness of the contact conditions change also significantly the character of the dispersion curves of the higher modes, however, these dispersion curves have one branches only similar to that obtained within the scope of the perfect contact conditions;

3. the dispersion curves of the foregoing two branches of the fundamental mode approach the corresponding limit dispersion curve related to the fully slipping interface conditions as the shear-spring parameter F increases;

4. before a certain value of kR(denoted by $(kR)_m$) the second branch of the fundamental mode obtained for the case where $F > 0$ merges with that obtained for the case where $F = 0$ and the values of $(kR)_m$ depend not only on the shear-spring parameter F but also on the ratios $h^{(1)}/R$ and $h^{(2)}/R$: the values of $(kR)_m$ decrease with F, but increase with a decrease in $h^{(1)}/R$, as well as with a decrease in $h^{(2)}/R$;

5. the shear-spring type imperfectness of the contact conditions does not change the low and high wave number limits;

6. the wave propagation velocity in the first (second) branch of the fundamental dispersion mode is less (greater) than that obtained in the perfect case;

7. there exists “cut off” values for kR(denoted by $(kR)_{cf.}$) for the first branch of the dispersion curve of the fundamental mode and $(kR)_{cf.} \to 0$ as $F \to \infty$, as well as $(kR)_{cf.} \to \infty$ as $F \to 0$;

8. the initial strains of the layers of the compound cylinder qualitatively change only the influence of the considered imperfectness of the interface conditions on the behavior of the dispersion curves;

9. the wave propagation velocity obtained for the hollow layered cylinder satisfies the relations (55) and (56) for the case under consideration;

10. the character of the dispersion curves of the second branch depends significantly on the thickness of the inner hollow cylinder.

Although the discussed numerical results are obtained for the selected values of the problem parameters, they also have a general meaning for the estimation of the influence of the imperfectness of the interface conditions on the dispersion of the axisymmetric longitudinal waves in the bi-layered hollow cylinder.

Appendix A

We write the expressions for calculation of the terms β_{ij} which enter the dispersion equation (37)

$$\beta_{11}\left(s_2^{(2)},\chi_{2h^{(2)}}^{(2)}\right)=$$
$$\begin{cases} \frac{\omega'^{(2)}_{1111}}{\lambda_3^{(2)}}\left(-\left(s_2^{(2)}\right)^2\frac{1}{2}\left(J_2\left(\chi_{2h^{(2)}}^{(2)}\right)-J_0\left(\chi_{2h^{(2)}}^{(2)}\right)\right)\right)+\frac{\omega'^{(2)}_{1122}}{\lambda_3^{(2)}\eta}s_2^{(2)}J_1\left(\chi_{2h^{(2)}}^{(2)}\right)+\frac{\omega'^{(2)}_{1133}}{2\lambda_1^{(2)}}\times \\ \left(\beta_1^{(2)}\left(s_2^{(2)}\right)^2\left(J_2\left(\chi_{2h^{(2)}}^{(2)}\right)-J_0\left(\chi_{2h^{(2)}}^{(2)}\right)\right)-\frac{s_2^{(2)}}{\eta}J_1\left(\chi_{2h^{(2)}}^{(2)}\right)-\beta_2^{(2)}J_0\left(\chi_{2h^{(2)}}^{(2)}\right)\right) \\ if\ \left(\zeta_2^{(2)}\right)^2>0, \\ \frac{\omega'^{(2)}_{1111}}{\lambda_3^{(2)}}\left(-\left(s_2^{(2)}\right)^2\frac{1}{2}\left(I_2\left(\chi_{2h^{(2)}}^{(2)}\right)+I_0\left(\chi_{2h^{(2)}}^{(2)}\right)\right)\right)-\frac{\omega'^{(2)}_{1122}}{\lambda_3^{(2)}\eta}s_2^{(2)}I_1\left(\chi_{2h^{(2)}}^{(2)}\right)+\frac{\omega'^{(2)}_{1133}}{2\lambda_1^{(2)}}\times \\ \left(\beta_1^{(2)}\left(s_2^{(2)}\right)^2\left(I_2\left(\chi_{2h^{(2)}}^{(2)}\right)+I_0\left(\chi_{2h^{(2)}}^{(2)}\right)\right)+\frac{s_2^{(2)}}{\eta}I_1\left(\chi_{2h^{(2)}}^{(2)}\right)-\beta_2^{(2)}I_0\left(\chi_{2h^{(2)}}^{(2)}\right)\right) \\ if\ \left(\zeta_2^{(2)}\right)^2<0, \end{cases}$$

$$\beta_{21}\left(s_2^{(2)},\chi_{2h^{(2)}}^{(2)}\right)=$$
$$\begin{cases} -\frac{\omega'^{(2)}_{1313}}{\lambda_1^{(2)}}s_2^{(2)}J_1\left(\chi_{2h^{(2)}}^{(2)}\right)+\frac{\omega'^{(2)}_{1331}}{\lambda_3^{(2)}4}\left(\beta_1^{(2)}\left(\left(s_2^{(2)}\right)^3\left(3J_1\left(\chi_{2h^{(2)}}^{(2)}\right)-J_3\left(\chi_{2h^{(2)}}^{(2)}\right)\right)+\right.\right. \\ \left.\left.\frac{s_2^{(2)}}{\eta^2}J_1\left(\chi_{2h^{(2)}}^{(2)}\right)+\frac{\left(s_2^{(2)}\right)^2}{\eta 2}\left(J_2\left(\chi_{2h^{(2)}}^{(2)}\right)-J_0\left(\chi_{2h^{(2)}}^{(2)}\right)\right)+\beta_2^{(2)}s_2^{(2)}J_1\left(\chi_{2h^{(2)}}^{(2)}\right)\right)\right) \\ if\ \left(\zeta_2^{(2)}\right)^2>0; \\ \frac{\omega'^{(2)}_{1313}}{\lambda_1^{(2)}}s_2^{(2)}I_1\left(\chi_{2h^{(2)}}^{(2)}\right)+\frac{\omega'^{(2)}_{1331}}{\lambda_3^{(2)}4}\left(\beta_1^{(2)}\left(\left(s_2^{(2)}\right)^3\left(3I_1\left(\chi_{2h^{(2)}}^{(2)}\right)+I_3\left(\chi_{2h^{(2)}}^{(2)}\right)\right)-\right.\right. \\ \left.\left.\frac{s_2^{(2)}}{\eta^2}I_1\left(\chi_{2h^{(2)}}^{(2)}\right)+\frac{\left(s_2^{(2)}\right)^2}{2\eta}\left(I_2\left(\chi_{2h^{(2)}}^{(2)}\right)+I_0\left(\chi_{2h^{(2)}}^{(2)}\right)\right)-\beta_2^{(2)}s_2^{(2)}I_1\left(\chi_{2h^{(2)}}^{(2)}\right)\right)\right) \\ if\ \left(\zeta_2^{(2)}\right)^2<0, \end{cases}$$

$$\beta_{13}\left(s_2^{(2)},\chi_{2h^{(2)}}^{(2)}\right)=$$
$$\begin{cases}
\frac{\omega'^{(2)}_{1111}}{\lambda_3^{(2)}}\left(-\left(s_2^{(2)}\right)^2\frac{1}{2}\left(Y_2\left(\chi_{2h^{(2)}}^{(2)}\right)-Y_0\left(\chi_{2h^{(2)}}^{(2)}\right)\right)\right)+\frac{\omega'^{(2)}_{1122}}{\lambda_3^{(2)}\eta}s_2^{(1)}Y_1\left(\chi_{2h^{(2)}}^{(2)}\right)+\frac{\omega'^{(2)}_{1133}}{2\lambda_1^{(2)}}\times\\
\left(\beta_1^{(2)}\left(s_2^{(2)}\right)^2\left(Y_2\left(\chi_{2h^{(2)}}^{(2)}\right)-Y_0\left(\chi_{2h^{(2)}}^{(2)}\right)\right)-\frac{s_2^{(2)}}{\eta}Y_1\left(\chi_{2h^{(2)}}^{(2)}\right)-\beta_2^{(2)}Y_0\left(\chi_{2h^{(2)}}^{(2)}\right)\right)\\
if\ \left(\zeta_2^{(2)}\right)^2>0,\\
\frac{\omega'^{(2)}_{1111}}{\lambda_3^{(2)}}\left(\left(s_2^{(2)}\right)^2\frac{1}{2}\left(K_2\left(\chi_{2h^{(2)}}^{(2)}\right)+K_0\left(\chi_{2h^{(2)}}^{(2)}\right)\right)\right)+\frac{\omega'^{(2)}_{1122}}{\lambda_3^{(2)}\eta}s_2^{(2)}K_1\left(\chi_{2h^{(2)}}^{(2)}\right)+\frac{\omega'^{(2)}_{1133}}{2\lambda_1^{(2)}}\times\\
\left(\beta_1^{(2)}\left(s_2^{(2)}\right)^2\left(I_2\left(\chi_{2h^{(2)}}^{(2)}\right)+J_0\left(\chi_{2h^{(2)}}^{(2)}\right)\right)+\frac{s_2^{(2)}}{\eta}I_1\left(\chi_{2h^{(2)}}^{(2)}\right)-\beta_2^{(2)}I_0\left(\chi_{2h^{(2)}}^{(2)}\right)\right)\\
if\ \left(\zeta_2^{(2)}\right)^2<0,
\end{cases}$$

$$\beta_{23}\left(s_2^{(2)},\chi_{2h^{(2)}}^{(2)}\right)=$$
$$\begin{cases}
-\frac{\omega'^{(2)}_{1313}}{\lambda_1^{(2)}}s_2^{(2)}Y_1\left(\chi_{2h^{(2)}}^{(2)}\right)+\frac{\omega'^{(2)}_{1331}}{\lambda_3^{(2)}4}\left(\beta_1^{(2)}\left(\left(s_2^{(2)}\right)^3\left(3Y_1\left(\chi_{2h^{(2)}}^{(2)}\right)-Y_3\left(\chi_{2h^{(2)}}^{(2)}\right)\right)+\right.\right.\\
\left.\left.\frac{s_2^{(2)}}{\eta^2}Y_1\left(\chi_{2h^{(2)}}^{(2)}\right)+\frac{\left(s_2^{(2)}\right)^2}{2\eta}\left(Y_2\left(\chi_{2h^{(2)}}^{(2)}\right)-Y_0\left(\chi_{2h^{(2)}}^{(2)}\right)\right)+\beta_2^{(2)}s_2^{(2)}Y_1\left(\chi_{2h^{(2)}}^{(2)}\right)\right)\right)\\
if\ \left(\zeta_2^{(2)}\right)^2>0;\\
-\frac{\omega'^{(2)}_{1313}}{\lambda_1^{(1)}}s_2^{(2)}K_1\left(\chi_{2h^{(2)}}^{(2)}\right)+\frac{\omega'^{(2)}_{1331}}{\lambda_3^{(1)}4}\left(\beta_1^{(2)}\left(\left(s_2^{(2)}\right)^3\left(3K_1\left(\chi_{2h^{(2)}}^{(2)}\right)+K_3\left(\chi_{2h^{(2)}}^{(2)}\right)\right)+\right.\right.\\
\left.\left.\frac{s_2^{(2)}}{\eta^2}K_1\left(\chi_{2h^{(2)}}^{(2)}\right)+\frac{\left(s_2^{(2)}\right)^2}{2\eta}\left(K_2\left(\chi_{2h^{(2)}}^{(2)}\right)+K_0\left(\chi_{2h^{(2)}}^{(2)}\right)\right)+\beta_2^{(2)}s_2^{(2)}K_1\left(\chi_{2h^{(2)}}^{(2)}\right)\right)\right)\\
if\ \left(\zeta_2^{(2)}\right)^2<0,
\end{cases}$$

$$\beta_{n2}=\beta_{n1}\left(s_3^{(2)},\chi_{3h^{(2)}}^{(2)}\right),\quad \beta_{n4}=\beta_{n3}\left(s_3^{(2)},\chi_{3h^{(2)}}^{(2)}\right),\quad \beta_{n5}=\beta_{n6}=\beta_{n7}=\beta_{n8}=0,$$
$$n=1,2,$$

$$\beta_{31}\left(s_2^{(2)},\chi_2^{(2)}\right)=$$
$$\begin{cases} \frac{\omega'^{(2)}_{1111}}{\lambda_3^{(2)}}\left(-\left(s_2^{(2)}\right)^2\frac{1}{2}\left(J_2\left(\chi_2^{(2)}\right)-J_0\left(\chi_2^{(2)}\right)\right)\right)+\frac{\omega'^{(2)}_{1122}}{\lambda_3^{(2)}\eta}s_2^{(2)}J_1\left(\chi_2^{(2)}\right)+\frac{\omega'^{(2)}_{1133}}{2\lambda_1^{(2)}}\times \\ \left(\beta_1^{(2)}\left(s_2^{(2)}\right)^2\left(J_2\left(\chi_2^{(2)}\right)-J_0\left(\chi_2^{(2)}\right)\right)-\frac{s_2^{(2)}}{\eta}J_1\left(\chi_2^{(2)}\right)-\beta_2^{(2)}J_0\left(\chi_2^{(2)}\right)\right) \\ if\ \left(\zeta_2^{(2)}\right)^2>0, \\ \frac{\omega'^{(2)}_{1111}}{\lambda_3^{(2)}}\left(-\left(s_2^{(2)}\right)^2\frac{1}{2}\left(I_2\left(\chi_2^{(2)}\right)+I_0\left(\chi_2^{(2)}\right)\right)\right)-\frac{\omega'^{(2)}_{1122}}{\lambda_3^{(2)}\eta}s_2^{(2)}I_1\left(\chi_2^{(2)}\right)+\frac{\omega'^{(2)}_{1133}}{2\lambda_1^{(2)}}\times \\ \left(\beta_1^{(2)}\left(s_2^{(2)}\right)^2\left(I_2\left(\chi_2^{(2)}\right)+I_0\left(\chi_2^{(2)}\right)\right)+\frac{s_2^{(2)}}{\eta}I_1\left(\chi_2^{(2)}\right)-\beta_2^{(2)}I_0\left(\chi_2^{(2)}\right)\right) \\ if\ \left(\zeta_2^{(2)}\right)^2<0, \end{cases}$$

$$\beta_{41}\left(s_2^{(2)},\chi_2^{(2)}\right)=$$
$$\begin{cases} -\frac{\omega'^{(2)}_{1313}}{\lambda_1^{(2)}}s_2^{(2)}J_1\left(\chi_2^{(2)}\right)+\frac{\omega'^{(2)}_{1331}}{\lambda_3^{(2)}4}\left(\beta_1^{(2)}\left(\left(s_2^{(2)}\right)^3\left(3J_1\left(\chi_2^{(2)}\right)-J_3\left(\chi_2^{(2)}\right)\right)+\right.\right. \\ \left.\left.\frac{s_2^{(2)}}{\eta^2}J_1\left(\chi_2^{(2)}\right)+\frac{\left(s_2^{(2)}\right)^2}{\eta 2}\left(J_2\left(\chi_2^{(2)}\right)-J_0\left(\chi_2^{(2)}\right)\right)+\beta_2^{(2)}s_2^{(2)}J_1\left(\chi_2^{(2)}\right)\right)\right) \\ if\ \left(\zeta_2^{(2)}\right)^2>0; \\ \frac{\omega'^{(2)}_{1313}}{\lambda_1^{(2)}}s_2^{(2)}I_1\left(\chi_2^{(2)}\right)+\frac{\omega'^{(2)}_{1331}}{\lambda_3^{(2)}4}\left(\beta_1^{(2)}\left(\left(s_2^{(2)}\right)^3\left(3I_1\left(\chi_2^{(2)}\right)+I_3\left(\chi_2^{(2)}\right)\right)-\right.\right. \\ \left.\left.\frac{s_2^{(2)}}{\eta^2}I_1\left(\chi_2^{(2)}\right)+\frac{\left(s_2^{(2)}\right)^2}{\eta 2}\left(I_2\left(\chi_2^{(2)}\right)+I_0\left(\chi_2^{(2)}\right)\right)-\beta_2^{(2)}s_2^{(2)}I_1\left(\chi_2^{(2)}\right)\right)\right) \\ if\ \left(\zeta_2^{(2)}\right)^2<0, \end{cases}$$

$$\beta_{51}\left(s_2^{(2)},\chi_2^{(2)}\right)=\begin{cases} -s_2^{(2)}J_1\left(\chi_2^{(2)}\right) & if\ \left(\zeta_2^{(2)}\right)^2>0, \\ s_2^{(2)}I_1\left(\chi_2^{(2)}\right) & if\ \left(\zeta_2^{(2)}\right)^2<0, \end{cases}$$

$$\beta_{61}\left(s_2^{(2)},\chi_2^{(2)}\right)=-\frac{FR}{\mu^{(2)}}\beta_{41}\left(s_2^{(2)},\chi_2^{(2)}\right)+$$
$$\begin{cases}\left(-\beta_1^{(2)}\left(\zeta_2^{(2)}\right)^2-\beta_2^{(2)}\right)J_0\left(\chi_2^{(2)}\right) & if\ \left(\zeta_2^{(2)}\right)^2>0;\\ \left(-\beta_1^{(2)}\left(\zeta_2^{(2)}\right)^2-\beta_2^{(2)}\right)I_0\left(\chi_2^{(2)}\right) & if\ \left(\zeta_2^{(2)}\right)^2<0,\end{cases}$$

$$\beta_{33}\left(s_2^{(2)},\chi_2^{(2)}\right)=$$
$$\begin{cases}\frac{\omega'^{(2)}_{1111}}{\lambda_3^{(2)}}\left(-\left(s_2^{(2)}\right)^2\frac{1}{2}\left(Y_2\left(\chi_2^{(2)}\right)-Y_0\left(\chi_2^{(2)}\right)\right)\right)+\frac{\omega'^{(2)}_{1122}}{\lambda_3^{(2)}\eta}s_2^{(2)}Y_1\left(\chi_2^{(2)}\right)+\frac{\omega'^{(2)}_{1133}}{2\lambda_1^{(2)}}\times\\ \left(\beta_1^{(2)}\left(s_2^{(2)}\right)^2\left(Y_2\left(\chi_2^{(2)}\right)-Y_0\left(\chi_2^{(2)}\right)\right)-\frac{s_2^{(2)}}{\eta}Y_1\left(\chi_2^{(2)}\right)-\beta_2^{(2)}Y_0\left(\chi_2^{(2)}\right)\right)\\ if\ \left(\zeta_2^{(2)}\right)^2>0,\\ \frac{\omega'^{(2)}_{1111}}{\lambda_3^{(2)}}\left(\left(s_2^{(2)}\right)^2\frac{1}{2}\left(K_2\left(\chi_2^{(2)}\right)+K_0\left(\chi_2^{(2)}\right)\right)\right)+\frac{\omega'^{(2)}_{1122}}{\lambda_3^{(2)}\eta}s_2^{(2)}K_1\left(\chi_2^{(2)}\right)+\frac{\omega'^{(2)}_{1133}}{2\lambda_1^{(2)}}\times\\ \left(\beta_1^{(2)}\left(s_2^{(2)}\right)^2\left(I_2\left(\chi_2^{(2)}\right)+I_0\left(\chi_2^{(2)}\right)\right)+\frac{s_2^{(2)}}{\eta}I_1\left(\chi_2^{(2)}\right)-\beta_2^{(2)}I_0\left(\chi_2^{(2)}\right)\right)\\ if\ \left(\zeta_2^{(2)}\right)^2<0,\end{cases}$$

$$\beta_{43}\left(s_2^{(2)},\chi_2^{(2)}\right)=$$
$$\begin{cases}-\frac{\omega'^{(2)}_{1313}}{\lambda_1^{(2)}}s_2^{(2)}Y_1\left(\chi_2^{(2)}\right)+\frac{\omega'^{(2)}_{1331}}{\lambda_3^{(2)}4}\left(\beta_1^{(2)}\left(\left(s_2^{(2)}\right)^3\left(3Y_1\left(\chi_2^{(2)}\right)-Y_3\left(\chi_2^{(2)}\right)\right)+\right.\right.\\ \left.\left.\frac{s_2^{(2)}}{\eta^2}Y_1\left(\chi_2^{(2)}\right)+\frac{\left(s_2^{(2)}\right)^2}{2\eta}\left(Y_2\left(\chi_2^{(2)}\right)-Y_0\left(\chi_2^{(2)}\right)\right)+\beta_2^{(2)}s_2^{(2)}Y_1\left(\chi_2^{(2)}\right)\right)\right)\\ if\ \left(\zeta_2^{(2)}\right)^2>0;\\ -\frac{\omega'^{(2)}_{1313}}{\lambda_1^{(2)}}s_2^{(2)}K_1\left(\chi_2^{(2)}\right)+\frac{\omega'^{(2)}_{1331}}{\lambda_3^{(2)}4}\left(\beta_1^{(2)}\left(\left(s_2^{(2)}\right)^3\left(3K_1\left(\chi_2^{(2)}\right)+K_3\left(\chi_2^{(2)}\right)\right)+\right.\right.\\ \left.\left.\frac{s_2^{(2)}}{\eta^2}K_1\left(\chi_2^{(2)}\right)+\frac{\left(s_2^{(2)}\right)^2}{\eta 2}\left(K_2\left(\chi_2^{(2)}\right)+K_0\left(\chi_2^{(2)}\right)\right)+\beta_2^{(2)}s_2^{(2)}K_1\left(\chi_2^{(2)}\right)\right)\right)\\ if\ \left(\zeta_2^{(2)}\right)^2<0,\end{cases}$$

$$\beta_{53}\left(s_2^{(2)},\chi_2^{(2)}\right)=\begin{cases}-s_2^{(2)}Y_1\left(\chi_2^{(2)}\right) & if\ \left(\zeta_2^{(2)}\right)^2>0;\\ -s_2^{(2)}K_1\left(\chi_2^{(2)}\right) & if\ \left(\zeta_2^{(2)}\right)^2<0,\end{cases}\quad ,$$

$$\beta_{63}\left(s_2^{(2)},\chi_2^{(2)}\right) = -\frac{FR}{\mu^{(2)}}\beta_{43}\left(s_2^{(2)},\chi_2^{(2)}\right) + \begin{cases} \left(-\beta_1^{(2)}\left(\zeta_2^{(2)}\right)^2 - \beta_2^{(2)}\right) Y_0\left(\chi_2^{(2)}\right) & if\ \left(\zeta_2^{(2)}\right)^2 > 0; \\ \left(-\beta_1^{(2)}\left(\zeta_2^{(2)}\right)^2 - \beta_2^{(2)}\right) K_0\left(\chi_2^{(2)}\right) & if\ \left(\zeta_2^{(2)}\right)^2 < 0, \end{cases},$$

$$\beta_{n2} = \beta_{n1}\left(s_3^{(2)},\chi_3^{(2)}\right), \quad \beta_{n4} = \beta_{n3}\left(s_3^{(2)},\chi_3^{(2)}\right), \quad n = 3,4,5,6$$

$$\beta_{35}\left(s_2^{(1)},\chi_2^{(1)}\right) = \begin{cases} \frac{\omega'^{(1)}_{1111}}{\lambda_3^{(1)}}\left(-\left(s_2^{(1)}\right)^2 \frac{1}{2}\left(J_2\left(\chi_2^{(1)}\right) - J_0\left(\chi_2^{(1)}\right)\right)\right) + \frac{\omega'^{(1)}_{1122}}{\lambda_3^{(1)}\eta} s_2^{(1)} J_1\left(\chi_2^{(1)}\right) + \frac{\omega'^{(1)}_{1133}}{2\lambda_1^{(1)}} \times \\ \left(\beta_1^{(1)}\left(s_2^{(1)}\right)^2\left(J_2\left(\chi_2^{(1)}\right) - J_0\left(\chi_2^{(1)}\right)\right) - \frac{s_2^{(1)}}{\eta} J_1\left(\chi_2^{(1)}\right) - \beta_2^{(1)} J_0\left(\chi_2^{(1)}\right)\right) \\ if\ \left(\zeta_2^{(1)}\right)^2 > 0, \\ \frac{\omega'^{(1)}_{1111}}{\lambda_3^{(1)}}\left(-\left(s_2^{(1)}\right)^2 \frac{1}{2}\left(I_2\left(\chi_2^{(1)}\right) + I_0\left(\chi_2^{(1)}\right)\right)\right) - \frac{\omega'^{(1)}_{1122}}{\lambda_3^{(1)}\eta} s_2^{(1)} I_1\left(\chi_2^{(1)}\right) + \frac{\omega'^{(1)}_{1133}}{2\lambda_1^{(1)}} \times \\ \left(\beta_1^{(1)}\left(s_2^{(1)}\right)^2\left(I_2\left(\chi_2^{(1)}\right) + I_0\left(\chi_2^{(1)}\right)\right) + \frac{s_2^{(1)}}{\eta} I_1\left(\chi_2^{(1)}\right) - \beta_2^{(1)} I_0\left(\chi_2^{(1)}\right)\right) \\ if\ \left(\zeta_2^{(1)}\right)^2 < 0, \end{cases}$$

$$\beta_{37}\left(s_2^{(1)},\chi_2^{(1)}\right) = \begin{cases} \frac{\omega'^{(1)}_{1111}}{\lambda_3^{(1)}}\left(-\left(s_2^{(1)}\right)^2 \frac{1}{2}\left(Y_2\left(\chi_2^{(1)}\right) - Y_0\left(\chi_2^{(1)}\right)\right)\right) + \frac{\omega'^{(1)}_{1122}}{\lambda_3^{(1)}\eta} s_2^{(1)} Y_1\left(\chi_2^{(1)}\right) + \frac{\omega'^{(1)}_{1133}}{2\lambda_1^{(1)}} \times \\ \left(\beta_1^{(1)}\left(s_2^{(1)}\right)^2\left(Y_2\left(\chi_2^{(1)}\right) - Y_0\left(\chi_2^{(1)}\right)\right) - \frac{s_2^{(1)}}{\eta} Y_1\left(\chi_2^{(1)}\right) - \beta_2^{(1)} Y_0\left(\chi_2^{(1)}\right)\right) \\ if\ \left(\zeta_2^{(1)}\right)^2 > 0, \\ \frac{\omega'^{(1)}_{1111}}{\lambda_3^{(1)}}\left(\left(s_2^{(1)}\right)^2 \frac{1}{2}\left(K_2\left(\chi_2^{(1)}\right) + K_0\left(\chi_2^{(1)}\right)\right)\right) + \frac{\omega'^{(1)}_{1122}}{\lambda_3^{(1)}\eta} s_2^{(1)} K_1\left(\chi_2^{(1)}\right) + \frac{\omega'^{(1)}_{1133}}{2\lambda_1^{(1)}} \times \\ \left(\beta_1^{(1)}\left(s_2^{(1)}\right)^2\left(I_2\left(\chi_2^{(1)}\right) + I_0\left(\chi_2^{(1)}\right)\right) + \frac{s_2^{(1)}}{\eta} I_1\left(\chi_2^{(1)}\right) - \beta_2^{(1)} I_0\left(\chi_2^{(1)}\right)\right) \\ if\ \left(\zeta_2^{(1)}\right)^2 < 0, \end{cases}$$

$$\beta_{45}\left(s_2^{(1)},\chi_2^{(1)}\right)=$$
$$\begin{cases} -\frac{\omega'^{(1)}_{1313}}{\lambda_1^{(1)}}s_2^{(1)}J_1\left(\chi_2^{(1)}\right)+\frac{\omega'^{(1)}_{1331}}{\lambda_3^{(1)}4}\left(\beta_1^{(1)}\left(\left(s_2^{(1)}\right)^3\left(3J_1\left(\chi_2^{(1)}\right)-J_3\left(\chi_2^{(1)}\right)\right)+\right.\right. \\ \left.\left.\frac{s_2^{(1)}}{\eta^2}J_1\left(\chi_2^{(1)}\right)+\frac{\left(s_2^{(1)}\right)^2}{2\eta}\left(J_2\left(\chi_2^{(1)}\right)-J_0\left(\chi_2^{(1)}\right)\right)+\beta_2^{(1)}s_2^{(1)}J_1\left(\chi_2^{(1)}\right)\right) \\ if\ \left(\zeta_2^{(1)}\right)^2>0; \\ \frac{\omega'^{(1)}_{1313}}{\lambda_1^{(1)}}s_2^{(1)}I_1\left(\chi_2^{(1)}\right)+\frac{\omega'^{(1)}_{1331}}{\lambda_3^{(1)}4}\left(\beta_1^{(1)}\left(\left(s_2^{(1)}\right)^3\left(3I_1\left(\chi_2^{(1)}\right)+I_3\left(\chi_2^{(1)}\right)\right)-\right.\right. \\ \left.\left.\frac{s_2^{(1)}}{\eta^2}I_1\left(\chi_2^{(1)}\right)+\frac{\left(s_2^{(1)}\right)^2}{2\eta}\left(I_2\left(\chi_2^{(1)}\right)+I_0\left(\chi_2^{(1)}\right)\right)-\beta_2^{(1)}s_2^{(1)}I_1\left(\chi_2^{(1)}\right)\right) \\ if\ \left(\zeta_2^{(1)}\right)^2<0, \end{cases}$$

$$\beta_{47}\left(s_2^{(1)},\chi_2^{(1)}\right)=$$
$$\begin{cases} -\frac{\omega'^{(1)}_{1313}}{\lambda_1^{(1)}}s_2^{(1)}Y_1\left(\chi_2^{(1)}\right)+\frac{\omega'^{(1)}_{1331}}{\lambda_3^{(1)}4}\left(\beta_1^{(1)}\left(\left(s_2^{(1)}\right)^3\left(3Y_1\left(\chi_2^{(1)}\right)-Y_3\left(\chi_2^{(1)}\right)\right)+\right.\right. \\ \left.\left.\frac{s_2^{(1)}}{\eta^2}Y_1\left(\chi_2^{(1)}\right)+\frac{\left(s_2^{(1)}\right)^2}{\eta 2}\left(Y_2\left(\chi_2^{(1)}\right)-Y_0\left(\chi_2^{(1)}\right)\right)+\beta_2^{(1)}s_2^{(1)}Y_1\left(\chi_2^{(1)}\right)\right) \\ if\ \left(\zeta_2^{(1)}\right)^2>0; \\ -\frac{\omega'^{(1)}_{1313}}{\lambda_1^{(1)}}s_2^{(1)}K_1\left(\chi_2^{(1)}\right)+\frac{\omega'^{(1)}_{1331}}{\lambda_3^{(1)}4}\left(\beta_1^{(1)}\left(\left(s_2^{(1)}\right)^3\left(3K_1\left(\chi_2^{(1)}\right)+K_3\left(\chi_2^{(1)}\right)\right)+\right.\right. \\ \left.\left.\frac{s_2^{(1)}}{\eta^2}K_1\left(\chi_2^{(1)}\right)+\frac{\left(s_2^{(1)}\right)^2}{\eta 2}\left(K_2\left(\chi_2^{(1)}\right)+K_0\left(\chi_2^{(1)}\right)\right)+\beta_2^{(1)}s_2^{(1)}K_1\left(\chi_2^{(1)}\right)\right) \\ if\ \left(\zeta_2^{(1)}\right)^2<0, \end{cases}$$

$$\beta_{36}=\beta_{35}\left(s_3^{(1)},\chi_3^{(1)}\right),\quad \beta_{38}=\beta_{37}\left(s_3^{(1)},\chi_3^{(1)}\right),$$

$$\beta_{46}=\beta_{45}\left(s_3^{(1)},\chi_3^{(1)}\right),\quad \beta_{48}=\beta_{47}\left(s_3^{(1)},\chi_3^{(1)}\right),$$

$$\beta_{55}\left(s_2^{(1)},\chi_2^{(1)}\right)=\begin{cases} -s_2^{(1)}J_1\left(\chi_2^{(1)}\right) & if\ \left(\zeta_2^{(1)}\right)^2>0, \\ s_2^{(1)}I_1\left(\chi_2^{(1)}\right) & if\ \left(\zeta_2^{(1)}\right)^2<0, \end{cases} \quad ,$$

$$\beta_{65}\left(s_2^{(1)},\chi_2^{(1)}\right)=\begin{cases}\left(-\beta_1^{(1)}\left(\zeta_2^{(1)}\right)^2-\beta_2^{(1)}\right)J_0\left(\chi_2^{(1)}\right) & if\ \left(\zeta_2^{(1)}\right)^2>0;\\ \left(-\beta_1^{(1)}\left(\zeta_2^{(1)}\right)^2-\beta_2^{(1)}\right)I_0\left(\chi_2^{(1)}\right) & if\ \left(\zeta_2^{(1)}\right)^2<0,\end{cases}$$

$$\beta_{57}\left(s_2^{(1)},\chi_2^{(1)}\right)=\begin{cases}-s_2^{(1)}Y_1\left(\chi_2^{(1)}\right) & if\ \left(\zeta_2^{(1)}\right)^2>0;\\ -s_2^{(1)}K_1\left(\chi_2^{(1)}\right) & if\ \left(\zeta_2^{(1)}\right)^2<0,\end{cases},$$

$$\beta_{67}\left(s_2^{(1)},\chi_2^{(1)}\right)=\begin{cases}\left(-\beta_1^{(1)}\left(\zeta_2^{(1)}\right)^2-\beta_2^{(1)}\right)Y_0\left(\chi_2^{(1)}\right) & if\ \left(\zeta_2^{(1)}\right)^2>0;\\ \left(-\beta_1^{(1)}\left(\zeta_2^{(1)}\right)^2-\beta_2^{(1)}\right)K_0\left(\chi_2^{(1)}\right) & if\ \left(\zeta_2^{(1)}\right)^2<0,\end{cases}$$

$$\beta_{56}=\beta_{55}\left(s_3^{(1)},\chi_3^{(1)}\right),\quad \beta_{58}=\beta_{57}\left(s_3^{(1)},\chi_3^{(1)}\right),$$

$$\beta_{66}=\beta_{65}\left(s_3^{(1)},\chi_3^{(1)}\right),\quad \beta_{68}=\beta_{67}\left(s_3^{(1)},\chi_3^{(1)}\right),$$

$$\beta_{75}\left(s_2^{(1)},\chi_{2h^{(1)}}^{(1)}\right)=$$

$$\begin{cases}\frac{\omega'^{(1)}_{1111}}{\lambda_3^{(1)}}\left(-\left(s_2^{(1)}\right)^2\frac{1}{2}\left(J_2\left(\chi_{2h^{(1)}}^{(1)}\right)-J_0\left(\chi_{2h^{(1)}}^{(1)}\right)\right)\right)+\frac{\omega'^{(1)}_{1122}}{\lambda_3^{(1)}\eta}s_2^{(1)}J_1\left(\chi_{2h^{(1)}}^{(1)}\right)+\frac{\omega'^{(1)}_{1133}}{2\lambda_1^{(1)}}\times\\ \left(\beta_1^{(1)}\left(s_2^{(1)}\right)^2\left(J_2\left(\chi_{2h^{(1)}}^{(1)}\right)-J_0\left(\chi_{2h^{(1)}}^{(1)}\right)\right)-\frac{s_2^{(1)}}{\eta}J_1\left(\chi_{2h^{(1)}}^{(1)}\right)-\beta_2^{(1)}J_0\left(\chi_{2h^{(1)}}^{(1)}\right)\right)\\ if\ \left(\zeta_2^{(1)}\right)^2>0,\\ \frac{\omega'^{(1)}_{1111}}{\lambda_3^{(1)}}\left(-\left(s_2^{(1)}\right)^2\frac{1}{2}\left(I_2\left(\chi_{2h^{(1)}}^{(1)}\right)+I_0\left(\chi_{2h^{(1)}}^{(1)}\right)\right)\right)-\frac{\omega'^{(1)}_{1122}}{\lambda_3^{(1)}\eta}s_2^{(1)}I_1\left(\chi_{2h^{(1)}}^{(1)}\right)+\frac{\omega'^{(1)}_{1133}}{2\lambda_1^{(1)}}\times\\ \left(\beta_1^{(1)}\left(s_2^{(1)}\right)^2\left(I_2\left(\chi_{2h^{(1)}}^{(1)}\right)+I_0\left(\chi_{2h^{(1)}}^{(1)}\right)\right)+\frac{s_2^{(1)}}{\eta}I_1\left(\chi_{2h^{(1)}}^{(1)}\right)-\beta_2^{(2)}I_0\left(\chi_{2h^{(1)}}^{(1)}\right)\right)\\ if\ \left(\zeta_2^{(1)}\right)^2<0,\end{cases}$$

$$\beta_{85}\left(s_2^{(1)},\chi_{2h^{(1)}}^{(1)}\right)=$$
$$\begin{cases}
-\frac{\omega'^{(1)}_{1313}}{\lambda_1^{(1)}}s_2^{(1)}J_1\left(\chi_{2h^{(1)}}^{(1)}\right)+\frac{\omega'^{(1)}_{1331}}{\lambda_3^{(1)}4}\left(\beta_1^{(1)}\left(\left(s_2^{(1)}\right)^3\left(3J_1\left(\chi_{2h^{(1)}}^{(1)}\right)-J_3\left(\chi_{2h^{(1)}}^{(1)}\right)\right)+\right.\right.\\
\left.\left.\frac{s_2^{(1)}}{\eta^2}J_1\left(\chi_{2h^{(1)}}^{(1)}\right)+\frac{\left(s_2^{(1)}\right)^2}{2\eta}\left(J_2\left(\chi_{2h^{(1)}}^{(1)}\right)-J_0\left(\chi_{2h^{(1)}}^{(1)}\right)\right)+\beta_2^{(1)}s_2^{(1)}J_1\left(\chi_{2h^{(1)}}^{(1)}\right)\right)\right.\\
if\ \left(\zeta_2^{(1)}\right)^2>0;\\
\frac{\omega'^{(1)}_{1313}}{\lambda_1^{(1)}}s_2^{(1)}I_1\left(\chi_{2h^{(1)}}^{(1)}\right)+\frac{\omega'^{(1)}_{1331}}{\lambda_3^{(1)}4}\left(\beta_1^{(1)}\left(\left(s_2^{(1)}\right)^3\left(3I_1\left(\chi_{2h^{(1)}}^{(1)}\right)+I_3\left(\chi_{2h^{(1)}}^{(1)}\right)\right)-\right.\right.\\
\left.\left.\frac{s_2^{(1)}}{\eta^2}I_1\left(\chi_{2h^{(1)}}^{(1)}\right)+\frac{\left(s_2^{(1)}\right)^2}{2\eta}\left(I_2\left(\chi_{2h^{(1)}}^{(1)}\right)+I_0\left(\chi_{2h^{(1)}}^{(1)}\right)\right)-\beta_2^{(1)}s_2^{(1)}I_1\left(\chi_{2h^{(1)}}^{(1)}\right)\right)\right.\\
if\ \left(\zeta_2^{(1)}\right)^2<0,
\end{cases}$$

$$\beta_{77}\left(s_2^{(1)},\chi_{2h^{(1)}}^{(1)}\right)=$$
$$\begin{cases}
\frac{\omega'^{(2)}_{1111}}{\lambda_3^{(2)}}\left(-\left(s_2^{(2)}\right)^2\frac{1}{2}\left(Y_2\left(\chi_{2h^{(2)}}^{(2)}\right)-Y_0\left(\chi_{2h^{(2)}}^{(2)}\right)\right)\right)+\frac{\omega'^{(2)}_{1122}}{\lambda_3^{(2)}\eta}s_2^{(1)}Y_1\left(\chi_{2h^{(2)}}^{(2)}\right)+\frac{\omega'^{(2)}_{1133}}{2\lambda_1^{(2)}}\times\\
\left(\beta_1^{(2)}\left(s_2^{(2)}\right)^2\left(Y_2\left(\chi_{2h^{(2)}}^{(2)}\right)-Y_0\left(\chi_{2h^{(2)}}^{(2)}\right)\right)-\frac{s_2^{(2)}}{\eta}Y_1\left(\chi_{2h^{(2)}}^{(2)}\right)-\beta_2^{(2)}Y_0\left(\chi_{2h^{(2)}}^{(2)}\right)\right)\\
if\ \left(\zeta_2^{(2)}\right)^2>0,\\
\frac{\omega'^{(2)}_{1111}}{\lambda_3^{(2)}}\left(\left(s_2^{(2)}\right)^2\frac{1}{2}\left(K_2\left(\chi_{2h^{(2)}}^{(2)}\right)+K_0\left(\chi_{2h^{(2)}}^{(2)}\right)\right)\right)+\frac{\omega'^{(2)}_{1122}}{\lambda_3^{(2)}\eta}s_2^{(2)}K_1\left(\chi_{2h^{(2)}}^{(2)}\right)+\frac{\omega'^{(2)}_{1133}}{2\lambda_1^{(2)}}\times\\
\left(\beta_1^{(2)}\left(s_2^{(2)}\right)^2\left(I_2\left(\chi_{2h^{(2)}}^{(2)}\right)+J_0\left(\chi_{2h^{(2)}}^{(2)}\right)\right)+\frac{s_2^{(2)}}{\eta}I_1\left(\chi_{2h^{(2)}}^{(2)}\right)-\beta_2^{(2)}I_0\left(\chi_{2h^{(2)}}^{(2)}\right)\right)\\
if\ \left(\zeta_2^{(2)}\right)^2<0,
\end{cases}$$

$$\beta_{87}\left(s_2^{(1)},\chi_{2h^{(1)}}^{(1)}\right)=$$
$$\begin{cases} -\frac{\omega'^{(1)}_{1313}}{\lambda_1^{(1)}}s_2^{(1)}Y_1\left(\chi_{2h^{(1)}}^{(1)}\right)+\frac{\omega'^{(1)}_{1331}}{\lambda_3^{(1)}4}\left(\beta_1^{(1)}\left(\left(s_2^{(1)}\right)^3\left(3Y_1\left(\chi_{2h^{(1)}}^{(1)}\right)-Y_3\left(\chi_{2h^{(1)}}^{(1)}\right)\right)+\right.\right. \\ \left.\left.\frac{s_2^{(1)}}{\eta^2}Y_1\left(\chi_{2h^{(1)}}^{(1)}\right)+\frac{\left(s_2^{(1)}\right)^2}{2\eta}\left(Y_2\left(\chi_{2h^{(1)}}^{(1)}\right)-Y_0\left(\chi_{2h^{(1)}}^{(1)}\right)\right)+\beta_2^{(1)}s_2^{(1)}Y_1\left(\chi_{2h^{(1)}}^{(1)}\right)\right)\right. \\ if\ \left(\zeta_2^{(1)}\right)^2>0; \\ -\frac{\omega'^{(1)}_{1313}}{\lambda_1^{(1)}}s_2^{(1)}K_1\left(\chi_{2h^{(1)}}^{(1)}\right)+\frac{\omega'^{(1)}_{1331}}{\lambda_3^{(1)}4}\left(\beta_1^{(1)}\left(\left(s_2^{(1)}\right)^3\left(3K_1\left(\chi_{2h^{(1)}}^{(1)}\right)+K_3\left(\chi_{2h^{(1)}}^{(1)}\right)\right)+\right.\right. \\ \left.\left.\frac{s_2^{(1)}}{\eta^2}K_1\left(\chi_{2h^{(1)}}^{(1)}\right)+\frac{\left(s_2^{(1)}\right)^2}{2\eta}\left(K_2\left(\chi_{2h^{(1)}}^{(1)}\right)+K_0\left(\chi_{2h^{(1)}}^{(1)}\right)\right)+\beta_2^{(1)}s_2^{(1)}K_1\left(\chi_{2h^{(1)}}^{(1)}\right)\right)\right. \\ if\ \left(\zeta_2^{(1)}\right)^2<0, \end{cases}$$

$$\beta_{n6}=\beta_{n5}\left(s_3^{(1)},\chi_{3h^{(1)}}^{(1)}\right),\quad \beta_{n8}=\beta_{n7}\left(s_3^{(1)},\chi_{3h^{(1)}}^{(1)}\right),$$
$$\beta_{n1}=\beta_{n2}=\beta_{n3}=\beta_{n4}=0,\quad n=7,8. \tag{A1}$$

In relation (A1) the following notation is used:

$$\chi_2^{(n)}=kR\lambda_1^{(n)}\left|\zeta_2^{(n)}\right|,\quad \chi_3^{(n)}=kR\lambda_1^{(n)}\left|\zeta_3^{(n)}\right|,$$
$$s_2^{(n)}=\left|\zeta_2^{(n)}\right|,\quad s_3^{(n)}=\left|\zeta_3^{(n)}\right|,\quad n=1,2,$$

$$\chi_{2h^{(2)}}^{(n)}=kR\left(1-\frac{h^{(2)}}{R}\right)\lambda_1^{(n)}\left|\zeta_2^{(n)}\right|,\quad \chi_{3h^{(2)}}^{(n)}=kR\left(1-\frac{h^{(2)}}{R}\right)\lambda_1^{(n)}\left|\zeta_3^{(n)}\right|,$$

$$\chi_{2h^{(1)}}^{(n)}=kR\left(1+\frac{h^{(1(}}{R}\right)\lambda_1^{(n)}\left|\zeta_2^{(n)}\right|,\quad \chi_{3h^{(1)}}^{(n)}=kR\left(1+\frac{h^{(1)}}{R}\right)\lambda_1^{(n)}\left|\zeta_3^{(n)}\right|,$$

$$\beta_1^{(n)}=\frac{\lambda_1^{(n)}}{\lambda_3^{(n)}}\frac{\omega'^{(n)}_{1111}}{\left(\omega'^{(n)}_{1133}+\omega'^{(n)}_{1313}\right)},$$
$$\beta_2^{(n)}=\frac{\lambda_3^{(n)}}{\lambda_1^{(n)}}\frac{\omega'^{(n)}_{3113}}{\left(\omega'^{(n)}_{1133}+\omega'^{(n)}_{1313}\right)}-\rho'^{(n)}c^2\frac{\lambda_3^{(n)}}{\lambda_1^{(n)}}\left(\omega'^{(n)}_{1133}+\omega'^{(n)}_{1313}\right)^{-1}. \tag{A2}$$

Appendix B

We write the expressions for calculation of the terms α_{nm} which enter the dispersion equation (47)

$$\alpha_{11} = \alpha_1^{(2)}(\omega'^{(2)}_{1133} + \omega'^{(2)}_{1313}), \quad \alpha_{12} = \alpha_2^{(2)}(\omega'^{(2)}_{1133} + \omega'^{(2)}_{1313}),$$

$$\alpha_{13} = -\alpha_1^{(1)}(\omega'^{(1)}_{1133} + \omega'^{(1)}_{1313}), \quad \alpha_{14} = -\alpha_2^{(1)}(\omega'^{(1)}_{1133} + \omega'^{(1)}_{1313}),$$

$$\alpha_{21} = (\alpha_1^{(2)})^2\omega'^{(2)}_{3113} - \omega'^{(2)}_{3333} + \rho'^{(2)}c_S^2, \quad \alpha_{22} = (\alpha_2^{(2)})^2\omega'^{(2)}_{3113} - \omega'^{(2)}_{3333} + \rho'^{(2)}c_S^2,$$

$$\alpha_{23} = -(\alpha_1^{(1)})^2\omega'^{(1)}_{3113} + \omega'^{(1)}_{3333} - \rho'^{(1)}c_S^2, \quad \alpha_{24} = -(\alpha_2^{(1)})^2\omega'^{(1)}_{3113} + \omega'^{(1)}_{3333} - \rho'^{(1)}c_S^2,$$

$$\alpha_{31} = \alpha_1^{(2)}\left\{\omega'^{(2)}_{1111}\omega'^{(2)}_{3113}(\alpha_1^{(2)})^2 - \left[\omega'^{(2)}_{3333}\omega'^{(2)}_{1111} - \omega'^{(2)}_{1133}(\omega'^{(2)}_{1133} + \omega'^{(2)}_{1313}) + \rho'^{(2)}\omega'^{(2)}_{1111}c_S^2\right\},$$

$$\alpha_{32} = \alpha_2^{(2)}\left\{\omega'^{(2)}_{1111}\omega'^{(2)}_{3113}(\alpha_2^{(2)})^2 - \left[\omega'^{(2)}_{3333}\omega'^{(2)}_{1111} - \omega'^{(2)}_{1133}(\omega'^{(2)}_{1133} + \omega'^{(2)}_{1313}) + \rho'^{(2)}\omega'^{(2)}_{1111}c_S^2\right\},$$

$$\alpha_{33} = -\alpha_1^{(1)}\left\{\omega'^{(1)}_{1111}\omega'^{(1)}_{3113}(\alpha_1^{(1)})^2 - \left[\omega'^{(1)}_{3333}\omega'^{(1)}_{1111} - \omega'^{(1)}_{1133}(\omega'^{(1)}_{1133} + \omega'^{(1)}_{1313}) + \rho'^{(1)}\omega'^{(1)}_{1111}c_S^2\right\},$$

$$\alpha_{34} = -\alpha_2^{(1)}\left\{\omega'^{(1)}_{1111}\omega'^{(1)}_{3113}(\alpha_2^{(1)})^2 - \left[\omega'^{(1)}_{3333}\omega'^{(1)}_{1111} - \omega'^{(1)}_{1133}(\omega'^{(1)}_{1133} + \omega'^{(1)}_{1313}) + \rho'^{(1)}\omega'^{(1)}_{1111}c_S^2\right\}$$

$$\alpha_{41} = (\rho'^{(2)}\omega'^{(2)}_{1313}c_S^2 - \alpha_1^{(2)}\omega'^{(2)}_{3113}\omega'^{(2)}_{1133} - \omega'^{(2)}_{1313}\omega'^{(2)}_{3333}),$$

$$\alpha_{42} = (\rho'^{(2)}\omega'^{(2)}_{1313}c_S^2 - \alpha_2^{(2)}\omega'^{(2)}_{3113}\omega'^{(2)}_{1133} - \omega'^{(2)}_{1313}\omega'^{(2)}_{3333}),$$

$$\alpha_{43} = -(\rho'^{(1)}\omega'^{(1)}_{1313}c_S^2 - \alpha_1^{(1)}\omega'^{(1)}_{3113}\omega'^{(1)}_{1133} - \omega'^{(1)}_{1313}\omega'^{(1)}_{3333}),$$

$$\alpha_{44} = -(\rho'^{(1)}\omega'^{(1)}_{1313}c_S^2 - \alpha_2^{(1)}\omega'^{(1)}_{3113}\omega'^{(1)}_{1133} - \omega'^{(1)}_{1313}\omega'^{(1)}_{3333}), \tag{B1}$$

where $\alpha_1^{(k)}$ and $\alpha_2^{(k)}$ $k = 1,2$ are the positive roots of the equation

$$(\alpha^{(k)})^4\omega'^{(k)}_{1111}\omega'^{(k)}_{3113} + (\alpha^{(k)})^2\left[\omega'^{(k)}_{1111}(\rho'^{(k)}c_S^2 - \omega'^{(k)}_{3333}) + \omega'^{(k)}_{3113}(\rho'^{(k)}c_S^2 - \omega'^{(k)}_{3113}) + \right.$$

$$\left.(\omega'^{(k)}_{1133} + \omega'^{(k)}_{1313})^2\right] + (\rho'^{(k)}c_S^2 - \omega'^{(k)}_{3333})(\rho'^{(k)}c_S^2 - \omega'^{(k)}_{3113}) = 0. \tag{B2}$$

Note that in (B1) and (B2) the $\rho'^{(k)}$,$\omega'^{(k)}_{3311}$,.., $\omega'^{(k)}_{3333}$are determined through the expressions (21) and (22).

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