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The Influence of Third Order Elastic Constants on Axisymmetric Wave Propagation Velocity in the Two-Layered Pre-Stressed Hollow Cylinder

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Abstract: By the use of the Murnaghan potential the influence of third order elastic constants on axisymmetric longitudinal wave propagation velocity in a pre-stressed two-layered circular hollow cylinder is investigated. This investigation is carried out within the scope of the piecewise homogeneous body model by utilizing the first version of the small initial deformation theory of the Three-dimensional Linearized Theory of Elastic Waves in Initially Stressed Bodies. Numerical results are obtained and analyzed for the cases where the material of the outer hollow cylinder material is aluminum, but the material of the inner cylinder is steel (Case 1) and tungsten (Case 2). These results are obtained not only for the case where the initial uniaxial stress is a stretching one, but also for the case where the initial uniaxial stress is a compressing one. According to these results, it is established that the third order elastic constants of the selected materials influence not only quantitatively, but also qualitatively the axisymmetric wave propagation velocity in the initially stressed two-layered hollow cylinder. It is also established that the mentioned influence in Case 2 is more significant than that in Case 1.

Keywords: initial stress, axisymmetric wave propagation velocity, third order elastic constants, two-layered hollow cylinder, dispersion

1 Introduction

The study of wave propagation and its dispersion in initially stressed elastic bodies is of interest in a number of physical and mechanical areas, such as geophysics, electrical devices, earthquake engineering, composite materials, ultrasonic non-destructive stress analysis of solids and others. Therefore a large number of investigations have been made in this field. Note that almost all these investigations

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were made by utilizing the so called Three-dimensional Linearized Theory of Elastic Waves in Initially Stressed Bodies (TLTEWISB). The relations and equations of the TLTEWISB are obtained from the exact relations and equations of the non-linear theory of elastodynamics by linearization with respect to small dynamical perturbations. The general questions of the TLTEWISB have been elaborated in many investigations such as in works by Biot (1965), Truestell (1961), Eringen and Suhubi (1975), Guz (2004) and others. It should be noted that there are some versions of the TDLTEWISB which were developed in the monograph by Guz (2004). These versions of the TLTEWISB are distinguished from each other with respect to the magnitude of the initial strains. The version of the TLTEWISB developed for high-elastic materials, according to which the initial strains in the bodies are determined within the scope of the non-linear theory of elasticity without any restrictions on the magnitude of the initial strains, is called the large (or finite) initial deformation version. The version of the TLTEWISB, according to which an initial stress-strain state in bodies is determined within the scope of the geometrically non-linear theory of elasticity and under which changes to the elementary areas and volumes as a result of the initial deformation are not taken into account, is called the first version of the small initial deformation theory of the TDLTEWISB. The second version of the small initial deformation theory of the TDLTEWISB is the version, according to which an initial stress-strain state in bodies is determined within the scope of the classical linear theory of elasticity.

Now we review briefly the investigations related to wave propagation in pre-stressed circular cylinders. Note that the first attempt in this field was made in a paper by Green (1961) in which torsional wave propagation in an initially stretched circular solid cylinder was studied. An attempt to investigate axisymmetric wave propagation in an initially twisted circular cylinder was done in a study by Demiray and Suhubi (1970). It was established that the initial twisting of the circular cylinder causes the coupled wave propagation field to occur between the axisymmetric torsional and longitudinal waves. In other words, it was established that in the initially twisted circular cylinder the axisymmetric torsional and longitudinal waves cannot be propagated separately. However, in the paper by Demiray and Suhubi (1970), as an example of numerical results, only the approximate analytical expression for perturbation of the torsional oscillation frequency caused by the initial twisting is given.

Longitudinal wave propagation in initially stretched circular cylinders was a subject of many investigations in papers such as Belward (1976), Guz et al (1975) and Kushnir (1979). However, in these investigations the object of the research, as in the foregoing investigations, was a homogeneous circular cylinder. Note that the studies in works by Green (1961), Demiray and Suhubi (1970) and Belward (1976) were made within the scope of the finite initial deformation version, but the works by Guz et al (1975) and Kushnir (1979) were within the scope of the first version of the small initial deformation theory of the TLTEWISB.

Until ten years ago there hadn't been any investigations on wave propagation in pre-stressed compound cylinders. These investigations were started with a work by Akbarov and Guz (2004) in which axisymmetric wave dispersion in a pre-stressed bi-material compound solid cylinder is studied. The investigations were made within the scope of the piecewise homogeneous body model by utilizing the first version of the small initial deformation theory of the TLTEWISB. It is assumed that the elasticity relations of the cylinders' materials are given through the Murnaghan potential described in a monograph by Murnaghan (1951). Below we will again turn to the work by Akbarov and Guz (2004), but now we consider a brief review of the investigations which are further developments of this work. The first such development was made in a paper by Akbarov and Guliev (2009) for the bi-material compound solid circular cylinder fabricated from high elastic materials and the corresponding investigations were carried out by utilizing the large (or finite) initial deformation version of the TLTEWISB. The materials of the constituents were assumed to be compressible and their elasticity relations were given by the harmonic-type potential. With the same assumptions, the influence of the finite initial strains on the axisymmetric wave dispersion in a circular cylinder, embedded in a compressible elastic medium, was studied in a work by Akbarov and Guliev (2010). Moreover, in a paper by Akbarov et al (2010) the problem considered in works by Akbarov and Guliev (2009, 2010) was developed for the case where the materials of the components of the system were incompressible and their stress-strain relations were given through the Treloar potential. Numerical results, regarding the influence of the initial strains in the cylinder and embedded body on the wave dispersion, were presented and discussed. Note that in the foregoing papers related to the compound cylinders, it was assumed that complete contact conditions were satisfied for the interface surface between the constituents. However, in papers by Akbarov and Ipek (2010, 2012) this condition was refuted and, within the scope of the assumptions accepted in the paper by Akbarov and Guliev (2009), the influence of the imperfectness of the mentioned interface conditions on the dispersion of the axisymmetric longitudinal waves in the bi-material compound cylinder was studied.

A paper by Akbarov et al (2011a) within the scope of the second version of the small initial deformation theory of the TDLTEWISB investigated the dispersion relations of axisymmetric wave propagation in an initially twisted bi-material compound cylinder. It was assumed that the constituents of the compound cylinder were isotropic and homogeneous and, in particular, it was established that as a result of the existence of the initial twisting, at least in one constituent of the considered compound cylinder, the axisymmetric longitudinal and torsional waves could not be propagated separately, i.e. new axisymmetric waves, which differ from the axisymmetric torsional and longitudinal waves, must appear.

In papers by Ozturk and Akbarov (2008, 2009a, 2009b), within the scope of the second version of the small initial deformation theory of the TDLTEWISB, the axisymmetric torsional wave propagation in the initially uni-axially stretched bi-material compound cylinder was investigated. The elasticity relations for the components of the compound cylinder were obtained from the Murnaghan potential. It should be noted that in all the foregoing investigations related to the torsional wave propagation in the pre-stressed bi-layered compound cylinder it was assumed that complete contact conditions were satisfied with respect to the contact surface between the inner and outer cylinders. In a paper by Kepceler (2010) the problems considered in the papers by Ozturk and Akbarov (2008, 2009a, 2009b) were examined for the case where the specified contact conditions were imperfect and the numerical results on the effects of this imperfection on the influence of the initial stresses on the wave propagation velocity are presented and discussed. In a paper by Cilli and Ozturk (2010), the torsional wave propagation in a pre-stretched multilayered solid cylinder was studied within the scope of the assumptions used in papers by Ozturk and Akbarov (2008, 2009a, 2009b).

Finally, in a paper by Akbarov et al (2011b) by utilizing the finite initial deformation version of the TLTEWISB within the scope of the piecewise homogeneous body model, torsional wave dispersion in a pre-strained three-layered (sandwich) hollow cylinder was studied. The mechanical relations of the materials of the cylinders are described through their harmonic potential.

This completes the review of the related investigations, from which it follows that the investigations related to axisymmetric longitudinal wave propagation in pre-stressed bi-material compound cylinders were made within the scope of the following assumptions:

1. a bi-material compound cylinder is a solid compound;
2. these investigations (except the paper by Akbarov and Guz (2004)) were made by utilizing the finite initial deformation version of the TLTEWISB;
3. under these investigations the elasticity relations of the materials of the constituents are described by the harmonic potential (for compressible materials) or by the Treloar potential (for the incompressible materials).

Therefore, according to the last assumption, the results obtained in these papers can be applied mainly to the cylinders made from polymer type materials. However these results cannot be applied to the compound cylinders made from metals such as aluminum, steel, tungsten etc. This situation can be explained by experimental data detailed in the monograph by Guz (2004) and in other references listed therein, according to which, the absolute values of the influence of the initial stretching and the initial compressing stresses on the wave propagation velocity in the pre-stressed bodies fabricated from the foregoing metals differ from each other in the quantitative sense. Consequently, in a theoretical sense such experimental results can be described only by employing the strain energy function (potential) containing not only the first and the second algebraic invariants of Green's strain tensor (which are contained by the aforementioned harmonic potential), but also the third order algebraic invariant of Green's strain tensor. The Murnaghan potential can be taken as an example of such a potential. Note that the coefficients of the terms which enter the expression of this potential contain the third order powers of the components of Green's strain tensor which are called the third order elastic constants. Note that these third order elastic constants enter into the linearized elasticity relations of the TLTEWISB and through these constants the aforementioned effect on the difference between the absolute values of the influence of the initial stretching and the initial compressing stresses on the wave propagation velocity in the pre-stressed bodies is described and estimated. Therefore, theoretical investigations on the influence of the third order elastic constants on the longitudinal wave propagation in the compound cylinders have a great significance not only in the sense of the fundamental investigations but also in the sense of their application. The first attempt in this field was made in the paper by Akbarov and Guz (2004) in which the subject of the investigation was a bi-material compound solid cylinder. Moreover, in the paper by Akbarov and Guz (2004), numerical results were obtained for the case where the initial stresses were stretching ones and the main attention was not focused on the influence of the third order elastic constants on the axisymmetric wave propagation velocity.

Taking the foregoing discussions into account in the present paper by the use of the Murnaghan potential, the influence of the third order elastic constants on the axisymmetric wave propagation velocity in the pre-stressed two-layered circular hollow cylinder is investigated. This investigation is carried out within the scope of the piecewise homogeneous body model by utilizing the first version of the small initial deformation theory of the TLTEWISB. Numerical results are obtained and analyzed for the case where the initial uniaxial stress is a stretching one as well as for the case where the initial uniaxial stress is a compressing one. These results are obtained for the specifically-selected materials, such as steel, aluminum and tungsten.

2 Formulation of the problem

We consider the two-layered hollow circular cylinder shown in Fig. 1 and assume that in the natural state the radius of the connecting surface of the inner and outer cylinders is R and the thickness of the inner and outer cylinders is $h^{(1)}$and $h^{(2)}$, respectively. In the natural state we determine the position of the points of the cylinders by the Lagrange coordinates in the cylindrical system of coordinates $Or\theta z$. The values related to the inner and outer hollow cylinders will be denoted by the upper indices (1) and (2), respectively. Furthermore, we denote the values related to the initial state by an additional upper index 0. Assume that the cylinders have an infinite length in the direction of the Oz axis.

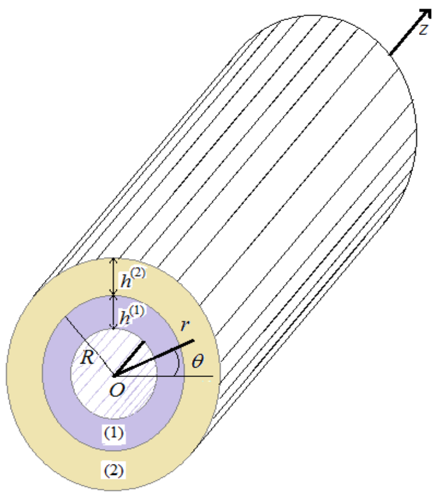

Figure 1: The geometry of the two-layered hollow cylinder

We propose that the cylinders (before the compounding) be stretched separately along the Oz axis and that in each of them the homogeneous axisymmetric initial stress state appears. However, the results which will be discussed below, can also relate to the case where the cylinders are stretched after the compounding. In this case, as a result of the difference in Poisson's coefficients of the cylinders' materials, the inhomogeneous initial stresses acting on the areas which are parallel to the Oz axis arise. But the values of these inhomogeneous stresses are much smaller than the values of the homogeneous initial stresses acting on the areas which are perpendicular to the Oz axis and therefore the inhomogeneous initial stresses can be neglected for consideration. With the initial state of the cylinders we associate the Lagrange cylindrical system of coordinates $O'r'\theta'z'$. Thus, according to the above, the initial state in the cylinders can be written as follows:

$$u_r^{(k),0} = (\lambda_1^{(k)} - 1)r, \quad u_z^{(k),0} = (\lambda_3^{(k)} - 1)z, \quad u_\theta^{(k),0} = 0, \quad k = 1,2, \tag{1}$$

where $\lambda_1^{(k)}$ and $\lambda_3^{(k)}$ are constants.

Based on the above, let us investigate the axisymmetric wave propagation in the $O'z'$ axis direction in the compound cylinder. We make this investigation by the use of coordinates r' and z' in the framework of the first version of the small initial deformation theory of the TLTEWISB. According to this theory, it is assumed that the values $\delta_m^{(k)} = 1 - \lambda_m^{(k)}$ and shears are smaller than unity and thus can be neglected under the linearization procedure. How this theory is constructed was analyzed by Guz (2004) in detail.

It follows from (1) that

$$h'^{(1)} = \lambda_1^{(1)} h^{(1)}, \quad h'^{(2)} = \lambda_1^{(2)} h^{(2)}, \quad R' = \lambda_1^{(1)} R,$$

$$r' = \lambda_1^{(1)} r \text{ for } R - h^{(1)} \le r \le R, \quad r' = \lambda_1^{(2)} r \text{ for } R \le r \le R + h^{(2)}. \tag{2}$$

Below, the values related to the system of coordinates associated with the initial state, i.e. with $O'r'\theta'z'$ are denoted by the upper prime.

Thus, based on Guz (2004), we write the basic relations of the TLTEWISB for the axisymmetrical case.

The equation of motion:

$$\frac{\partial}{\partial r'} T'^{(k)}_{r'r'} + \frac{\partial}{\partial z'} T'^{(k)}_{r'z'} + \frac{1}{r'}(T'^{(k)}_{r'r'} - T'^{(k)}_{\theta'\theta'}) = \rho'^{(k)} \frac{\partial^2}{\partial t^2} u'^{(k)}_{r'},$$

$$\frac{\partial}{\partial r'} T'^{(k)}_{z'r'} + \frac{1}{r'} T'^{(k)}_{z'r'} + \frac{\partial}{\partial z'} T'^{(k)}_{z'z'} = \rho'^{(k)} \frac{\partial^2}{\partial t^2} u'^{(k)}_{z'}, \tag{3}$$

The mechanical relations:

$$T'^{(k)}_{r'r'} = \omega'^{(k)}_{1111} \frac{\partial u'^{(k)}_{r'}}{\partial r'} + \omega'^{(\underline{k})}_{1122} \frac{u'^{(k)}_{r'}}{r'} + \omega'^{(\underline{k})}_{1133} \frac{\partial u'^{(k)}_{z'}}{\partial z'},$$

$$T'^{(k)}_{\theta'\theta'} = \omega'^{(k)}_{2211} \frac{\partial u'^{(k)}_{r'}}{\partial r'} + \omega'^{(k)}_{2222} \frac{u'^{(k)}_{r'}}{r'} + \omega'^{(k)}_{2233} \frac{\partial u'^{(k)}_{z'}}{\partial z'},$$

$$T'^{(k)}_{r'z'} = \omega'^{(k)}_{1313} \frac{\partial u'^{(k)}_{r'}}{\partial z'} + \omega'^{(k)}_{1331} \frac{\partial u'^{(k)}_{z'}}{\partial r'},$$

$$T'^{(\underline{k})}_{z'r'} = \omega'^{(\underline{k})}_{3113} \frac{\partial u'^{(\underline{k})}_{r'}}{\partial z'} + \omega'^{(\underline{k})}_{3131} \frac{\partial u'^{(\underline{k})}_{z'}}{\partial r'},$$

$$T'^{(k)}_{z'z'} = \omega'^{(k)}_{3311}\frac{\partial u'^{(k)}_{r'}}{\partial r'} + \omega'^{(k)}_{3322}\frac{u'^{(k)}_{r'}}{r'} + \omega'^{(k)}_{3333}\frac{\partial u'^{(k)}_{z'}}{\partial z'}. \tag{4}$$

In (3) and (4) through $T'^{(k)}_{r'r'}, \ldots, T'^{(k)}_{z'z'}$ perturbation of the components of the Kirchhoff stress tensor are denoted; the notation $u'^{(k)}_{r'}$ and$u'^{(k)}_{z'}$ shows the perturbation of the components of the displacement vector. The constants $\omega'^{(k)}_{1111}, \ldots, \omega'^{(k)}_{3333}$ and $\rho'^{(k)}$ in (3) and (4) are determined through the mechanical constants of the cylinders' materials and through the initial stress state. Note that for the considered initial stress state, expression of these constants is given through those in the system of coordinates $Or\theta z$ (we denote them by $\omega^{(k)}_{1111}, \ldots, \omega^{(k)}_{3333}$) with the following formulae:

$$\omega'^{(k)}_{1111} = (\lambda^{(k)}_1)^2\omega^{(k)}_{1111}, \quad \omega'^{(k)}_{1331} = (\lambda^{(k)}_1)^2\omega^{(k)}_{1331}, \quad \omega'^{(k)}_{3333} = (\lambda^{(k)}_3)^2\omega^{(k)}_{3333},$$

$$\omega'^{(k)}_{3113} = (\lambda^{(k)}_3)^2\omega^{(k)}_{3113}, \quad \rho'^{(k)} = \frac{\rho^{(k)}}{\lambda^{(k)}_1\lambda^{(k)}_2\lambda^{(k)}_3}, \omega'^{(k)}_{3113} = (\lambda^{(k)}_3)^2\omega^{(k)}_{3113},$$

$$\omega'^{(k)}_{1133} = \lambda^{(k)}_1\lambda^{(k)}_3\omega^{(k)}_{1133}, \quad \omega'^{(k)}_{1313} = \lambda^{(k)}_1\lambda^{(k)}_3\omega^{(k)}_{1313}. \tag{5}$$

As has been noted above, in the present investigation we assume that the elasticity relations of the cylinders' materials are given by the Murnaghan (1951) potential. This potential is given as follows:

$$\Phi = \frac{1}{2}\lambda A_1^2 + \mu A_2 + \frac{a}{3}A_1^3 + bA_1A_2 + \frac{c}{3}A_3. \tag{6}$$

In (6), λ and μ are Lamé's constants, a, b and c are the third order elastic constants; and A_1, A_2 and A_3 are the first, second and third algebraic invariants of Green's strain tensor respectively. For the case under consideration the expressions of these invariants are the following:

$$A_1 = \varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz},$$

$$A_2 = \varepsilon_{rr}^2 + 2\varepsilon_{rz}^2 + \varepsilon_{\theta\theta}^2 + \varepsilon_{zz}^2,$$

$$A_3 = \varepsilon_{rr}^3 + \varepsilon_{\theta\theta}^3 + \varepsilon_{zz}^3 + 3\varepsilon_{rz}^2(\varepsilon_{rr} + \varepsilon_{zz}). \tag{7}$$

In (7), ε_{rr}, $\varepsilon_{\theta\theta}$, ε_{zz} and ε_{rz} are the components of Green's strain tensor and these components are determined through the components of the displacement vector by the following formulae:

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} + \frac{1}{2}\left(\frac{\partial u_r}{\partial r}\right)^2 + \frac{1}{2}\left(\frac{\partial u_z}{\partial r}\right)^2,$$

$$\varepsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{2}\frac{u_r^2}{r^2},$$

$$\varepsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} + \frac{\partial u_r}{\partial r}\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\frac{\partial u_z}{\partial z}\right),$$

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z} + \frac{1}{2}\left(\frac{\partial u_z}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial u_r}{\partial r}\right)^2. \tag{8}$$

In this case the components $\sigma_{rr},\ldots,\ \sigma_{zz}$ of the symmetric stress tensor are determined as follows:

$$\sigma_{rr} = \frac{\partial}{\partial \varepsilon_{rr}}\Phi, \quad \sigma_{\theta\theta} = \frac{\partial}{\partial \varepsilon_{\theta\theta}}\Phi, \quad \sigma_{zz} = \frac{\partial}{\partial \varepsilon_{zz}}\Phi, \quad \sigma_{rz} = \frac{1}{2}\left(\frac{\partial}{\partial \varepsilon_{rz}} + \frac{\partial}{\partial \varepsilon_{zr}}\right)\Phi. \tag{9}$$

Note that the expressions (5)-(9) are written in the arbitrary system of cylindrical coordinates without any restriction related to the association of this system with the natural or initial state of the considered compound cylinders.

For the considered case, the relations between the perturbation of the Kirchhoff stress tensor and perturbation of the components of the ordinary symmetric tensor of stresses can be written as follows:

$$T'^{(k)}_{r'r'} = \lambda_1^{(k)}\sigma'^{(k)}_{r'r'},$$

$$T'^{(k)}_{\theta'\theta'} = \lambda_1^{(k)}\sigma'^{(k)}_{\theta'\theta'},$$

$$T'^{(k)}_{r'z'} = \sigma'^{(k)}_{r'z'} + \sigma_{zz}^{(k),0}\frac{\partial u'^{(k)}_{r'}}{\partial z'},$$

$$T'^{(k)}_{z'r'} = \sigma'^{(k)}_{z'r'},$$

$$T'^{(k)}_{z'z'} = \lambda_3^{(k)}\sigma'^{(k)}_{z'z'} + \sigma_{zz}^{(k),0}\frac{\partial u'^{(k)}_{z'}}{\partial z'}. \tag{10}$$

By linearization of the equation (9) and taking (5), (10) and (1) into account we obtain the following expressions for the constants $\omega_{1111}^{(k)},\ldots,\omega_{3333}^{(k)}$ in (5):

$$\omega_{1111}^{(k)} = \omega_{2222}^{(k)} = \lambda_1^{(k)2}A_{11}^{(k)}, \quad \omega_{3333}^{(k)} = (\lambda_3^{(k)})^2A_{33}^{(k)} + \sigma_{zz}^{(k),0},$$

$$\omega_{1133}^{(k)} = \omega_{2233}^{(k)} = \lambda_1^{(k)}\lambda_3^{(k)}A_{13}^{(k)}, \quad \omega_{1122}^{(k)} = (\lambda_1^{(k)})^2A_{12}^{(k)},$$

$$\omega_{3113}^{(k)} = \omega_{3223}^{(k)} = \lambda_1^{(k)2}\mu_{13}^{(k)} + \sigma_{zz}^{(k),0}, \quad \omega_{1331}^{(k)} = \omega_{2332}^{(k)} = (\lambda_3^{(k)})^2\mu_{13}^{(k)},$$

$$\omega_{1221}^{(k)} = \omega_{2112}^{(k)} = (\lambda_1^{(k)})^2\mu_{12}^{(k)}, \quad \omega_{1313}^{(k)} = \omega_{2323}^{(k)} = \lambda_1^{(k)}\lambda_3^{(k)}\mu_{13}^{(k)},$$

$$\omega_{1213}^{(k)} = (\lambda_1^{(k)})^2 \mu_{12}^{(k)}, \tag{11}$$

where

$$(\lambda_1^{(k)})^2 = 1 - \frac{\lambda^{(k)}}{3K_0^{(k)}\mu^{(k)}}\sigma_{zz}^{(k),0}, \quad (\lambda_3^{(k)})^2 = 1 + \frac{2\left(\lambda^{(k)} + \mu^{(k)}\right)}{3K_0^{(k)}\mu^{(k)}}\sigma_{zz}^{(k),0},$$

$$A_{11}^{(k)} =$$

$$(\lambda^{(k)}+2\mu^{(k)})\left[1 + \frac{\sigma_{zz}^{(k),0}}{\left(\lambda^{(k)}+2\mu^{(k)}\right)3K_0^{(k)}}\left(2a^{(k)} - \frac{\lambda^{(k)}-\mu^{(k)}}{\mu^{(k)}}2b^{(k)} - \frac{\lambda^{(k)}}{\mu^{(k)}}c^{(k)}\right)\right],$$

$$A_{33}^{(k)} = (\lambda^{(k)}+2\mu^{(k)})\left[1 + \frac{\sigma_{zz}^{(k),0}}{\left(\lambda^{(k)}+2\mu^{(k)}\right)3K_0^{(k)}}\left(2a^{(k)} + \right.\right.$$

$$\left.\left.\frac{2\lambda^{(k)}+3\mu^{(k)}}{\mu^{(k)}}2b^{(k)} + \frac{\lambda^{(k)}+\mu^{(k)}}{\mu^{(k)}}2c^{(k)}\right)\right],$$

$$\mu_{13}^{(k)} = \mu^{(k)}\left[1 + \frac{\sigma_{zz}^{(k),0}}{3K_0^{(k)}\mu^{(k)}}\left(b^{(k)} + \frac{1}{4}\frac{\lambda^{(k)}+2\mu^{(k)}}{\mu^{(k)}}c^{(k)}\right)\right],$$

$$A_{13}^{(k)} = \lambda^{(k)}\left[1 + \frac{\sigma_{zz}^{(k),0}}{3K_0^{(k)}\lambda^{(k)}}\left(2a^{(k)} + \frac{\lambda^{(k)}+2\mu^{(k)}}{\mu^{(k)}}b^{(k)}\right)\right],$$

$$A_{12}^{(k)} = \lambda^{(k)}\left[1 + \frac{\sigma_{zz}^{(k),0}}{3K_0^{(k)}\lambda^{(k)}}\left(a^{(k)} - \frac{\lambda^{(k)}}{\mu^{(k)}}b^{(k)}\right)\right],$$

$$\mu_{12}^{(k)} = \mu^{(k)}\left[1 + \frac{\sigma_{zz}^{(k),0}}{3K_0^{(k)}\lambda^{(k)}}\left(b^{(k)} - \frac{\lambda^{(k)}}{2\mu^{(k)}}c^{(k)}\right)\right]. \tag{12}$$

Thus, the wave propagation in the considered compound cylinder will be investigated by the use of the equations (3)-(5), (11) and (12). In this case we will assume that the following contact and boundary conditions are satisfied:

$$T'^{(1)}_{r'r'}\Big|_{r'=R'} = T'^{(2)}_{r'r'}\Big|_{r'=R'}, \quad T'^{(1)}_{r'z'}\Big|_{r'=R'} = T'^{(2)}_{r'z'}\Big|_{r'=R'}, \quad u'^{(1)}_{r'}\Big|_{r'=R'} = u'^{(2)}_{r'}\Big|_{r'=R'},$$

$$u'^{(1)}_{z'}\Big|_{r'=R'} = u'^{(2)}_{z'}\Big|_{r'=R'}, \quad T'^{(2)}_{r'r'}\Big|_{r'=R'+h'^{(2)}} = 0, \quad T'^{(2)}_{r'z'}\Big|_{r'=R'+h'^{(2)}} = 0.$$

$$T'^{(1)}_{r'r'}\Big|_{r'=R'-h'^{(1)}} = 0, \quad T'^{(1)}_{r'z'}\Big|_{r'=R'-h'^{(1)}} = 0. \tag{13}$$

This completes formulation of the problem and consideration of the governing field equations.

3 Solution procedure and obtaining the dispersion equation

Substituting (4) in (3), we obtain the following equation of motion in displacement terms:

$$\omega'^{(k)}_{1111}\frac{\partial^2 u'^{(k)}_{r'}}{\partial r'^2}+\omega'^{(k)}_{1122}\frac{\partial}{\partial r'}\left(\frac{u'^{(k)}_{r'}}{r'}\right)+\left(\omega'^{(k)}_{1133}+\omega'^{(k)}_{1331}\right)\frac{\partial^2 u'^{(k)}_{z'}}{\partial r'\partial z'}+$$

$$\omega'^{(k)}_{1313}\frac{\partial^2 u'^{(k)}_{r'}}{\partial z'^2}+\frac{1}{r'}\left(\omega'^{(k)}_{1111}-\omega'^{(k)}_{2211}\right)\times$$

$$\frac{\partial u'^{(k)}_{r'}}{\partial r'}+\left(\omega'^{(k)}_{1122}-\omega'^{(k)}_{2222}\right)\frac{u'^{(k)}_{r'}}{r'^2}+\left(\omega'^{(k)}_{1133}-\omega'^{(k)}_{2233}\right)\frac{1}{r'}\frac{\partial u'^{(k)}_{z'}}{\partial z'}=\rho'^{(k)}\frac{\partial^2 u'^{(k)}_{r'}}{\partial t^2},$$

$$\omega'^{(k)}_{1313}\frac{\partial^2 u'^{(k)}_{r'}}{\partial r'\partial z'}+\omega'^{(k)}_{3131}\frac{\partial^2 u'^{(k)}_{z'}}{\partial r'^2}+\frac{1}{r'}\omega'^{(k)}_{3113}\frac{\partial u'^{(k)}_{r'}}{\partial z'}+\frac{1}{r'}\omega'^{(k)}_{3131}\frac{\partial u'^{(k)}_{z'}}{\partial r'}+\omega'^{(k)}_{3311}\frac{\partial^2 u'^{(k)}_{r'}}{\partial z'\partial r'}+$$

$$\omega'^{(k)}_{3322}\frac{1}{r'}\frac{\partial u'^{(k)}_{r'}}{\partial z'}+\omega'^{(k)}_{3333}\frac{\partial^2 u'^{(k)}_{z'}}{\partial z'^2}=\rho'^{(k)}\frac{\partial^2 u'^{(k)}_{z'}}{\partial t^2}. \tag{14}$$

According to Guz (2004), we use the following representation for the displacements:

$$u'^{(k)}_{r'}=-\frac{\partial^2}{\partial r'\partial z'}\mathrm{X}^{(k)},$$

$$u'^{(k)}_{z'}=\frac{1}{\omega'^{(k)}_{1133}+\omega'^{(k)}_{1313}}\left(\omega'^{(k)}_{1111}\Delta'_1+\omega'^{(k)}_{3113}\frac{\partial^2}{\partial z'^2}-\rho'^{(k)}\frac{\partial^2}{\partial t^2}\right)\mathrm{X}^{(k)}, \tag{15}$$

where $\mathrm{X}^{(k)}$ satisfies the following equation:

$$\left[\left(\Delta'_1+\xi'^{(k)2}_{2}\frac{\partial^2}{\partial y_3'^2}\right)\left(\Delta'_1+\xi'^{(k)2}_{3}\frac{\partial^2}{\partial y_3'^2}\right)-\right.$$

$$\rho'^{(k)}\left(\frac{\omega'^{(k)}_{1111}+\omega'^{(k)}_{1331}}{\omega'^{(k)}_{1111}\omega'^{(k)}_{1331}}\Delta'_1+\frac{\omega'^{(k)}_{3333}+\omega'^{(k)}_{3113}}{\omega'^{(k)}_{1111}\omega'^{(k)}_{1331}}\times\frac{\partial^2}{\partial z'^2}\right)\frac{\partial^2}{\partial t^2}+$$

$$\left.\frac{\rho'^{(k)}}{\omega'^{(k)}_{1111}\omega'^{(k)}_{1331}}\frac{\partial^4}{\partial t^4}\right]\mathrm{X}^{(k)}=0. \tag{16}$$

In (15) and (16) the following notation is used:

$$\Delta'_1=\frac{d^2}{dr'^2}+\frac{1}{r'}\frac{d}{dr'},\quad (\xi'^{(k)}_{2,3})^2=d^{(k)}\pm\left[d^{(k)2}-\omega'^{(k)}_{3333}\omega'^{(k)}_{3113}\left(\omega'^{(k)}_{1111}\omega'^{(k)}_{1331}\right)^{-1}\right]^{\frac{1}{2}},$$

$$d^{(k)} = \left(2\omega'^{(k)}_{1111}\omega'^{(k)}_{1331}\right)^{-1}\left[\omega'^{(k)}_{1111}\omega'^{(k)}_{3333} + \omega'^{(k)}_{1331}\omega'^{(k)}_{3113} - \left(\omega'^{(k)}_{1133} + \omega'^{(k)}_{1313}\right)\right]. \tag{17}$$

We represent the function $\mathrm{X}^{(m)} = \mathrm{X}^{(m)}(r', z', t)$ as

$$\mathrm{X}^{(m)} = \mathrm{X}_1^{(m)}(r')\cos\left(kz' - \omega t\right), \quad m = 1, 2. \tag{18}$$

Substituting (18) in (16) and doing some manipulations we obtain the following equation for $\mathrm{X}_1^{(m)}(r')$:

$$\left(\Delta_1' + \zeta'^{(m)2}_{2}\right)\left(\Delta_1' + \zeta'^{(m)2}_{3}\right)\mathrm{X}_1^{(m)}(r') = 0. \tag{19}$$

The constants $\zeta'^{(k)}_{2,3}$ are determined from the following equation:

$$(\lambda_1^{(m)})^4\omega_{1111}^{(m)}\omega_{1313}^{(m)}(\zeta'^{(m)})^4 - k^2(\zeta'^{(m)})^2\Big[(\lambda_1^{(m)})^2\omega_{1111}^{(m)}\left(\rho'^{(m)}c^2 - (\lambda_3^{(m)})^2\omega^{(m)}_{3333}\right) +$$
$$(\lambda_1^{(m)})^2\omega_{1331}^{(m)}\left(\rho^{(m)}c^2 - (\lambda_3^{(m)})^2\omega^{(m)}_{3113}\right) + (\lambda_1^{(m)})^2(\lambda_3^{(m)})^2\left(\omega_{1133}^{(m)} + \omega_{1313}^{(m)}\right)^2\Big] +$$
$$k^4\left(\rho'^{(m)}c^2 - \lambda_3^{(m)}\omega_{3333}^{(m)}\right)\left(\rho'^{(m)}c^2 - (\lambda_3^{(m)})^2\,\omega_{3113}^{(m)}\right) = 0\,, \tag{20}$$

where $c = \omega/k$, i.e. c is the phase velocity of the propagating wave. We determine the following expression for $\mathrm{X}_1^{(m)}(r')$ from equations (19) and (20):

$$\mathrm{X}_{(1)}^{(m)}(r') = B_2^{(m)}E_0^{(m)}(kr'\zeta'^{(m)}_{2}) + B_3^{(m)}E_0^{(m)}(kr'\zeta'^{(m)}_{3}) +$$
$$D_2^{(m)}F_0^{(m)}(kr'\zeta'^{(m)}_{2}) + D_3^{(m)}F_0^{(m)}(kr'\zeta'^{(m)}_{3}), \tag{21}$$

where

$$E_0^{(n)}\left(\zeta'^{(n)}_{m}kr'\right) = \begin{cases} J_0\left(\zeta'^{(n)}_{m}kr'\right) & \text{if } \left(\zeta'^{(n)}_{m}\right)^2 > 0 \\ I_0\left(\left|\zeta'^{(n)}_{m}\right|kr'\right) & \text{if } \left(\zeta'^{(n)}_{m}\right)^2 < 0 \end{cases}, \tag{22}$$

$$F_0^{(n)}\left(\zeta'^{(n)}_{m}kr'\right) = \begin{cases} Y_0\left(\zeta'^{(n)}_{m}kr'\right) \text{if } \left(\zeta'^{(n)}_{m}\right)^2 > 0 \\ K_0\left(\left|\zeta'^{(n)}_{m}\right|kr'\right) \text{if } \left(\zeta'^{(n)}_{m}\right)^2 < 0 \end{cases}, \tag{23}$$

In (22) and (23) $J_0(x)$ and $Y_0(x)$ are Bessel functions of the first and second kind of the zeroth order; $I_0(x)$ and $K_0(x)$ are Bessel functions of a purely imaginary argument in the zeroth order and Macdonald function in the zeroth order, respectively. Note that the expressions for the functions $E_0^{(n)}(x)$ and $F_0^{(n)}(x)$ for the other cases which are not considered in (22) and (23), (i.e. for the cases where $\zeta'^{(n)}_2 = 0$ or $\zeta'^{(n)}_3 = 0$), can easily be determined according to the well-known procedure. As these cases do not appear in our investigations, the corresponding expressions for $E_0^{(m)}(x)$and $F_0^{(m)}(x)$ are not given here.

Thus, using the expressions (18), (20)-(23), (15), (4) and (5) we obtain the following dispersion equation from (13):

$$\det\left\|\alpha_{ij}\right\| = 0, \quad i,j = 1,2,3,4,5,6,7,8, \tag{24}$$

where

$$\alpha_{ij} = \alpha_{ij}\left(kR', \frac{c}{c_2^{(1)}}, \frac{\mu^{(1)}}{\mu^{(2)}}, \frac{h'^{(1)}}{R'}, \frac{h'^{(2)}}{R'}, \frac{a^{(1)}}{\mu^{(1)}}, \frac{b^{(1)}}{\mu^{(1)}}, \frac{c^{(1)}}{\mu^{(1)}}, \frac{\sigma_{zz}^{0(1)}}{\mu^{(1)}}, \frac{a^{(2)}}{\mu^{(2)}}, \frac{b^{(2)}}{\mu^{(2)}}, \frac{c^{(2)}}{\mu^{(2)}}, \frac{\sigma_{zz}^{0(2)}}{\mu^{(2)}}\right), \tag{25}$$

where $c_2^{(1)} = \sqrt{\mu^{(1)}/\rho^{(1)}}$. The explicit expressions of α_{ij} $(i = 1,2,...,8,\ j = 1,2,...,8)$ are given in the Appendix through the equations (A1) and (A2). Thus the dispersion equation for the considered wave propagation problem has been derived in the form (24), (25), (A1) and (A2).

4 Numerical results and discussions

The numerical results are obtained for steel (St), tungsten (Tg) and aluminum (Al). Note that the material of the internal hollow cylinder is selected as steel (St) or tungsten (Tg), but the material of the external hollow cylinder is selected as aluminum (Al). All mechanical characteristics of these materials and their notation, which will be used below, are given in Table 1. The values of the velocity of wave dilatation and bar velocity for these materials are given in Table 2. Note that the data given in Table 1 are selected according to Guz (2004) and Guz and Makhort (2000).

We assume that $\sigma_{zz}^{(1),0}/\mu^{(1)} = \sigma_{zz}^{(2),0}/\mu^{(2)}$ and introduce the notation:

$$\eta = \frac{\sigma_{zz}^{(1),0}}{\mu^{(1)}} = \frac{\sigma_{zz}^{(2),0}}{\mu^{(2)}}, \quad n = 1,2. \tag{26}$$

for estimation of the initial stresses in the constituents.

We analyze the numerical results related to the first (fundamental) mode only and assume that the material of the outer cylinder is aluminum (Al), but with respect to

Table 1: The values of elastic constants of selected materials

Materials	Density	Young's moduli	Pois.'s ratio	Third order elastic constants		
Steel (St)	$\rho_{St} \times 10^{-3} = 7.795 kg/m^3$	$E_{St} \times 10^{-4} = 19.6 MPa$	$\nu_{St} = 0.27$	$a^{(1)}_{St} \times 10^{-5} =$ 3.19 MPa	$b^{(1)}_{St} \times 10^{-5} =$ -3.03MPa	$c^{(1)}_{St} \times 10^{-5} =$ -7.84 Mpa
Tungsten (Tg)	$\rho_{Tg} \times 10^{-3} = 19.3 kg/m^3$	$E_{Tg} \times 10^{-4} = 34.3 MPa$	$\nu_{Tg} = 0.28$	$a^{(1)}_{Tg} \times 10^{-5} =$ -10.75 MPa	$b^{(1)}_{Tg} \times 10^{-5} =$ -14.3 Mpa	$c^{(1)}_{Tg} \times 10^{-5} =$ -49.6 Mpa
Aluminum (Al)	$\rho_{Al} \times 10^{-3} = 2.77 kg/m^3$	$E_{Al} \times 10^{-4} = 7.28 MPa$	$\nu_{Al} = 0.30$	$a^{(2)}_{Al} \times 10^{-5} =$ 0.62 MPa	$b^{(2)}_{Al} \times 10^{-5} =$ -0.49 Mpa	$c^{(2)}_{Al} \times 10^{-5} =$ -3.43 MPa

Table 2: The wave velocity of the selected materials

Materials	Velocity of wave of dilatation	Bar velocity	The values of velocity ratio
Steel (St)	$c^{(1)}_{2,St} \times 10^{-3} = 3.152$ m/s	$c^{(1)}_{b,St} \times 10^{-3} = 5.025$ m/s	$c^{(1)}_{b,St}/c^{(1)}_{2,St} = 1.594$
Tungsten (Tg)	$c^{(1)}_{2,Tg} \times 10^{-3} = 2.63$ m/s	$c^{(1)}_{b,Tg} \times 10^{-3} = 4.219$ m/s	$c^{(1)}_{b,Tg}/c^{(1)}_{2,Tg} = 1.604$
Aluminum (Al)	$c^{(2)}_{2,Al} \times 10^{-3} = 3.179$ m/s	$c^{(2)}_{b,Al} \times 10^{-3} = 5.126$ m/s	$c^{(2)}_{b,Al}/c^{(2)}_{2,Al} = 1.612$

the material of the inner hollow cylinder we consider the following two cases: Case 1. The material of the inner hollow cylinder is steel (St); Case 2. The material of the inner hollow cylinder is tungsten (Tg).

For testing of the algorithm and PC programs, first, we consider the case where $\eta = 0$, i.e. the case where the initial stresses in the constituents of the cylinder are absent and analyze the dispersion curves given in Fig. 2 which are constructed for Case 1 for $h^{(2)}/R = 0.1$ under various values of $h^{(1)}/R$. According to mechanical considerations, an increase in the values of $h^{(1)}/R$ must cause to approach the dispersion curves corresponding to the ones obtained for the solid compound cylinder. This consideration is confirmed by the graphs given in Fig. 2. Moreover, the dispersion curves given in Fig. 2 agree in the quantitative sense with the dispersion curves obtained for the hollow cylinder and detailed in a monograph by Rose (2004) and other references listed therein.

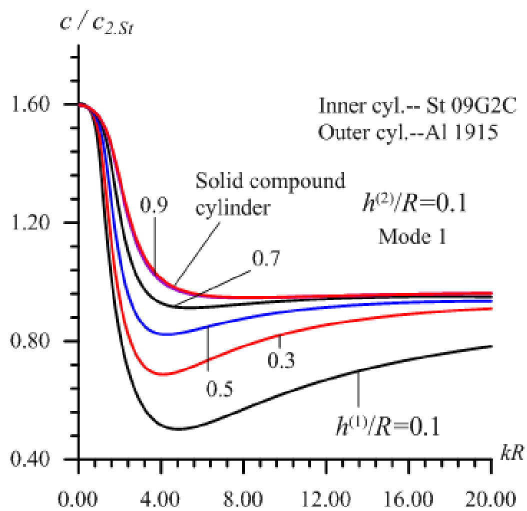

Figure 2: Dispersion curves constructed in Case 1 for various values of $h^{(1)}/R$ under $h^{(2)}/R = 0.1$.

Thus the foregoing results can be taken as validation of the employed algorithm and programs which are used for the numerical solution to the dispersion equation (24). Note that this algorithm is based on the well-known “bisection method”. In this case, for fixed values of the problem parameters for each value of kR, the roots of the equation (24) with respect to c, are found.

Now we turn to consideration of the numerical results related to the influence of the initial stresses of the cylinder under consideration on the wave propagation velocity.

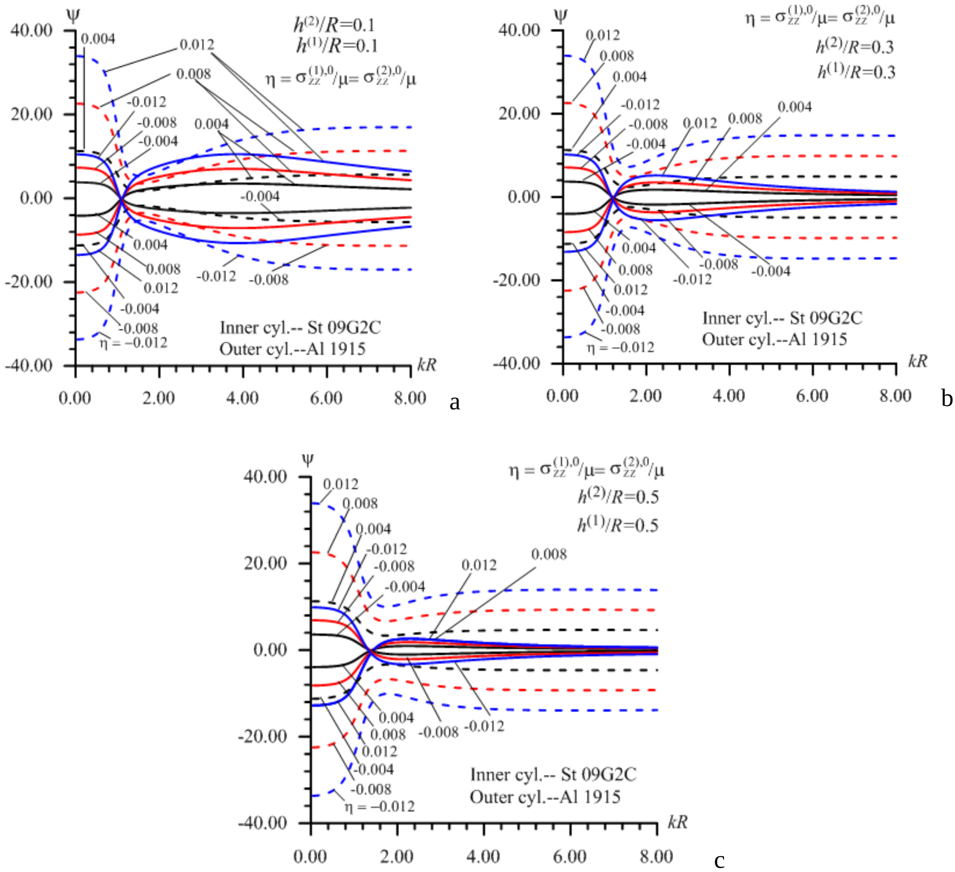

Figure 3: The graphs of the dependence between the parameter ψ (27) and kR constructed in Case 1 under $h^{(1)}/R = h^{(2)}/R = 0.1$(a), 0.3 (b) and 0.5 (c). Solid (dashed) lines correspond to the case where the influence of the third order elastic constants is taken (is not taken) into account.

For the quantitative estimation of this influence we introduce the parameter:

$$\psi = \frac{10^3 \times (c|_{\eta \neq 0} - c|_{\eta=0})}{c_2^{(1)}} \tag{27}$$

where $c_2^{(1)}$ is the dilatational wave velocity in the inner cylinder material, and investigate the dependencies between the parameter ψ and kR obtained for various values of η, $h^{(2)}/R$ and $h^{(1)}/R$ in the aforementioned Cases 1 and 2. Graphs of

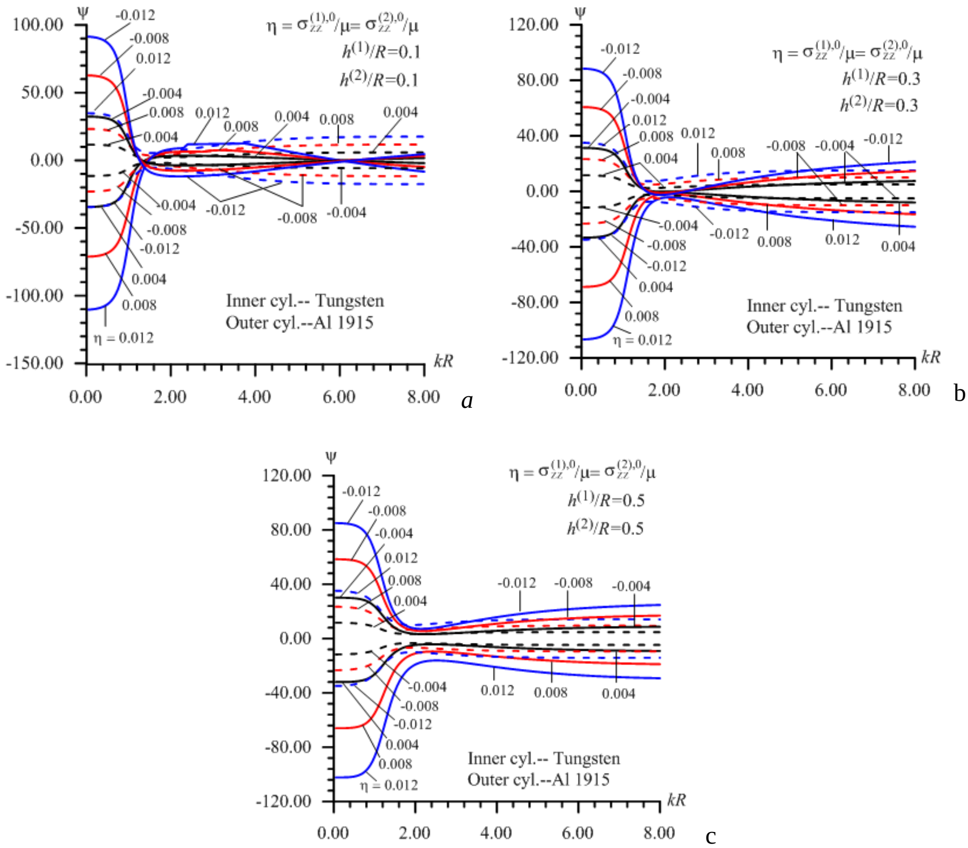

Figure 4: The graphs of the dependence between the parameter ψ (27) and kR constructed in Case 2 under $h^{(1)}/R = h^{(2)}/R = 0.1$(a), 0.3 (b) and 0.5 (c). Solid (dashed) lines correspond to the case where the influence of the third order elastic constants is taken (is not taken) into account.

these dependencies are given in Figs. 3 (for Case 1) and 4 (for Case 2) for the values of $h^{(2)}/R = h^{(1)}/R = 0.1$ (Figs. 3a and 4a), 0.3 (Figs. 3b and 4b) and 0.5 (Figs. 3c and 4c). Note that the graphs illustrated in Figs. 3 and 4 are constructed for various values of the initial stretching and compressing stresses, i.e. for various values of the parameter η(26) under $\eta > 0$ and under $\eta < 0$. Moreover, note that the results given in Figs. 3 and 4 are obtained in both cases where the influence of the third order elastic constants on the wave propagation velocity is taken into account (the graphs related to this case are drawn by the solid lines) and not taken into account (the graphs related to this case are drawn by the dashed lines).

According to the graphs given in Figs. 3 and 4, we can make the following conclusions for the case where the influence of the third order elastic constants on the wave propagation velocity in the pre-stressed two-layered hollow cylinder is not taken into account:

- an initial stretching (compressing) of the cylinder causes an increase (a decrease) in the wave propagation velocity, in other words

$$\begin{array}{l} \psi > 0 \;\; if \;\; \eta > 0 \\ \psi < 0 \;\; if \;\; \eta < 0 \end{array}; \tag{28}$$

- low wave number limit values of ψ as $kR \to 0$ are its absolute maximum;

- the graphs of the dependence between ψ and kR constructed under $\eta > 0$ (i.e. under the initial stretching of the cylinder) are symmetric to those constructed under $\eta < 0$ (i.e. under the initial compressing of the cylinder), in other words,

$$\psi(\eta) = -\psi(-\eta); \tag{29}$$

- for high wave number limit values of ψ as $kR \to \infty$, we can write the following relation:

$$\psi \to \min\left\{ \left| c_R^{(1)}\Big|_{\eta\neq 0} - c_R^{(1)}\Big|_{\eta=0} \right| / c_2^{(1)}\,;\, \left| c_R^{(2)}\Big|_{\eta\neq 0} - c_R^{(2)}\Big|_{\eta=0} \right| / c_2^{(1)};\right.$$

$$\left. \left| c_S|_{\eta\neq 0} - c_S|_{\eta=0} \right| / c_2^{(1)} \right\} \text{ as } kR \to \infty \tag{30}$$

where $c_R^{(m)}$ is the Rayleigh wave velocity in the m– th material and c_S is the Stoneley wave velocity for a selected pair of materials.

Also, according to the graphs given in Figs. 3 and 4, we can make the following conclusions for the case where the influence of the third order elastic constants on the wave propagation velocity in the pre-stressed two-layered hollow cylinder is taken into account:

- in Case 1, the character of the influence of the initial stresses on the wave propagation velocity depends on the values of kR, i.e. there exists such a value of kR (denoted by $(kR)_*$) with respect to which the following relation occurs:

$$\begin{array}{l} \psi \geq 0 \;\; if \;\; \eta < 0 \\ \psi \leq 0 \;\; if \;\; \eta > 0 \end{array} \text{under } 0 < kR \leq (kR)_*$$

and

$$\begin{matrix} \psi < 0 \ \ if \ \ \eta < 0 \\ \psi > 0 \ \ if \ \ \eta > 0 \end{matrix} \text{under } kR > (kR)_*. \tag{31}$$

The equality in the relation (31) occurs in the case where $kR = (kR)_*$. This means that in the case where $kR = (kR)_*$ the existence of the initial stresses in the cylinder under consideration does not influence the wave propagation velocity. Moreover, the relation (31) means that in the case where $0 < kR \leq (kR)_*$ ($kR > (kR)_*$) initial stretching stresses cause a decrease (an increase), but initial compressing stresses cause an increase (a decrease) in the wave propagation velocity;

- the foregoing conclusion and the relation (31) also take place in Case 2 for the relatively small values of $|\eta|$ and of $h^{(2)}/R$ $(= h^{(1)}/R\,)$, but for the moderately greater values of $|\eta|$ and of $h^{(2)}/R$ $(= h^{(1)}/R\,)$ the relation (31) is violated, for instance, in the case where $h^{(2)}/R\ = h^{(1)}/R = 0.5$, the relation:

$$\begin{matrix} \psi < 0 \ \ if \ \ \eta > 0 \\ \psi > 0 \ \ if \ \ \eta < 0 \end{matrix} \tag{32}$$

takes place for each value of kR instead of the relation (31);

- in Case 1 (in Case 2) the influence of the third order elastic constants causes a decrease (an increase) in the absolute values of the parameter ψ with respect to the corresponding values of ψ obtained in the case where the influence of the third order elastic constants is not taken into account;

- the magnitude of the aforementioned influence on the absolute values of the parameter ψ in Case 2 is more significant than that in Case 1;

- in the case where the third order elastic constants are taken into account the relation (29) is violated and the relation (33)

$$\psi(\eta) \neq -\psi(-\eta) \tag{33}$$

occurs instead of the relation (29);

- the relation (30) which relates to the high wave number limit values of the parameter ψ also takes place for the case where the third order elastic constants are taken into account.

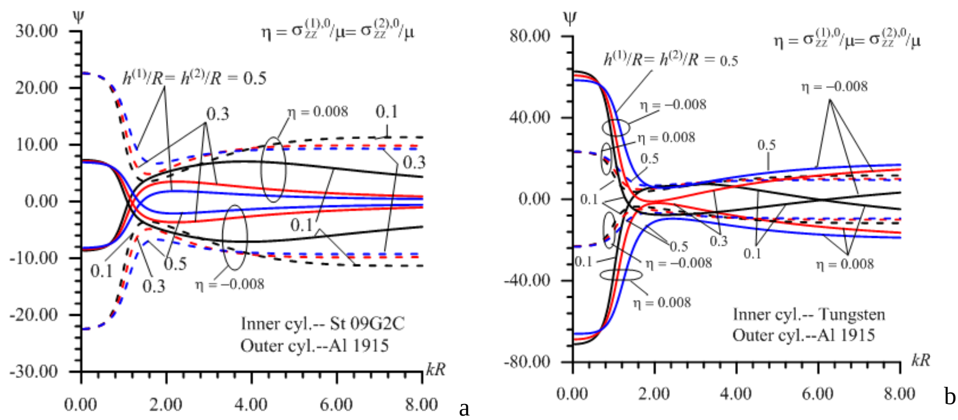

Figure 5: The influence of the ratio $h^{(1)}/R(=h^{(2)}/R)$on the dependence between the parameter ψ (27) and kR: (a) Case 1, (b) Case 2. Solid (dashed) lines correspond to the case where the influence of the third order elastic constants is taken (is not taken) into account.

Now we consider how the values of $h^{(2)}/R(=h^{(1)}/R)$ affect the influence of the initial stresses on the wave propagation velocity. For this purpose we consider the graphs given in Fig. 5 which also illustrate the dependence between the parameter ψ (27) and kR in Case 1 (Fig. 5a) and in Case 2 (Fig. 5b) under $|\eta| = 0.008$ for various values of $h^{(2)}/R(=h^{(1)}/R)$. It follows from Fig. 5 (especially from Fig. 5b) that, in the case where $h^{(1)}/R = h^{(2)}/R$ there exists such a value of kR, denoted by $(kR)'$, according to which, we can write the following relation:

$$|\psi| \text{ increases with decreasing } h^{(1)}/R(=h^{(2)}/R) \text{ if } kR < (kR)',$$

$$|\psi| \text{ decreases with decreasing } h^{(1)}/R(=h^{(2)}/R) \text{ if } kR > (kR)'. \tag{34}$$

Consequently, an increase in the values of $h^{(1)}/R(=h^{(2)}/R)$causes a decrease in the low wave number limit value of $|\psi|$ as $kR \to 0$. Note that the relation (34) is observed distinctly in Case 2 (Fig. 5b) when taking the influence of the third order elastic constants into account. We attempt to explain the relation (34) with the data given in Table 1, according to which, the absolute values of the third order elastic constants of the tungsten, which is the inner cylinder material, are significantly greater than those of the aluminum, which is the outer cylinder material. As the influence of the third order elastic constants of each constituent of the two-layered hollow cylinder on the low wave number limit values of $|\psi|$ depends significantly on its volumetric fraction in the cylinder under consideration and as the dominant part of this influence is caused by the tungsten, the values of $|\psi|$ as $kR \to 0$ therefore decrease with increasing $h^{(1)}/R(=h^{(2)}/R)$. This is because the volumetric fraction of the inner cylinder material (tungsten), which is equal to $0.5(1-0.5h^{(1)}/R)$, decreases, but the volumetric fraction of the outer cylinder material (aluminum), which is equal to $0.5(1+0.5h^{(2)}/R)$, increases with $h^{(1)}/R(=h^{(2)}/R)$.

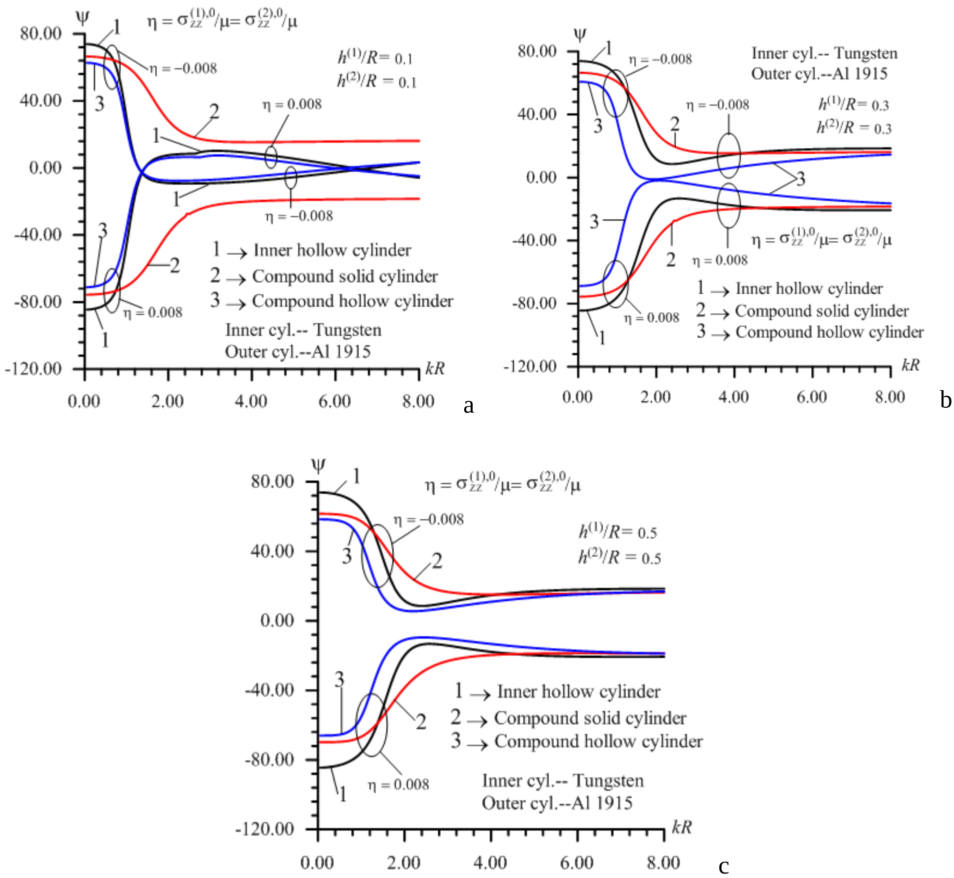

Figure 6: Comparison of the graphs of the dependence between the parameter ψ (27) and kR constructed in Case 2 for the inner hollow cylinder, solid compound cylinder and hollow compound cylinder under $h^{(1)}/R = h^{(2)}/R = 0.1$(a), 0.3 (b) and 0.5 (c). Solid (dashed) lines correspond to the case where the influence of the third order elastic constants is taken (is not taken) into account.

For an explanation of the contribution of the inner hollow cylinder on the values of the parameter ψ we consider the graphs given in Fig. 6 which are constructed for

Case 2 when taking the influence of the third order elastic constants into account in the case where $|\eta| = 0.008$. In this figure the graphs of the dependence between the parameter ψ and kR constructed for the compound two-layered hollow cylinder; the corresponding compound solid cylinder; and the single hollow cylinder are given for the cases where $h^{(1)}/R = h^{(2)}/R = 0.1$ (Fig. 6a), $h^{(1)}/R = h^{(2)}/R = 0.3$ (Fig. 6b) and $h^{(1)}/R = h^{(2)}/R = 0.5$ (Fig. 6c). It follows from these results that the difference between the low wave number limit values of ψ obtained for the compound two-layered hollow cylinder and for the single inner hollow cylinder increases with $h^{(1)}/R(= h^{(2)}/R)$. Moreover, introducing the notation $\psi^*_{c.h.c.}$, $\psi^*_{c.s.c.}$ and $\psi^*_{s.h.c.}$ for the low wave number limit values of the parameter ψ related to the compound hollow cylinder, the compound solid cylinder and the single inner hollow cylinder respectively, we can write the following relation:

$$\psi^*_{c.h.c} < \psi^*_{c.s.c} < \psi^*_{s.h.c}. \tag{35}$$

Note that the relation (35) agrees with well-known mechanical considerations.

5 Conclusions

Thus, in the present paper, by the use of the Murnaghan potential, the influence of the third order elastic constants on the axisymmetric wave propagation velocity in the pre-stressed two-layered circular hollow cylinder has been investigated. This investigation has been carried out within the scope of the piecewise homogeneous body model by utilizing the first version of the small initial deformation theory of the TLTEWISB. Numerical results have been obtained and analyzed for the cases where the material of the outer hollow cylinder material is aluminum, but the material of the inner cylinder material is steel (Case 1) and tungsten (Case2). According to these results, it has been established that the third order elastic constants of the selected materials influence not only quantitatively, but also qualitatively, the axisymmetric wave propagation velocity in the initially stressed two-layered hollow cylinder. The details of these results have been described in the previous section.

The obtained numerical results can be used for non-destructive determination of residual stresses in the bi-material elastic systems fabricated from the type of materials considered above.

Appendix

We write the explicit expressions of α_{ij} $(i = 1,2,...,8,\ j = 1,2,...,8)$ which enter the dispersion equation (24). To simply these expressions we introduce the following notation:

$$\delta_2^{(m)} = \zeta'^{(m)}_2 kR'\lambda_1^{(m)}, \quad \delta_3^{(m)} = \zeta'^{(m)}_3 kR'\lambda_1^{(m)}, \quad m = 1,2,$$

$$\delta_{2h}^{(1)} = \zeta'^{(1)}_2 kR'(1-\frac{h'^{(1)}}{R'})\lambda_1^{(1)}, \quad \delta_{3h}^{(1)} = \zeta'^{(1)}_3 kR'(1-\frac{h'^{(1)}}{R'})\lambda_1^{(1)},$$

$$\delta_{2h}^{(2)} = \zeta'^{(2)}_2 kR'(1+\frac{h'^{(2)}}{R'})\lambda_1^{(2)}, \quad \delta_{3h}^{(2)} = \zeta'^{(2)}_3 kR'(1+\frac{h'^{(2)}}{R'})\lambda_1^{(2)},$$

$$\beta_1^{(m)} = \frac{\omega'^{(m)}_{1111}\lambda_1^{(m)}}{\lambda_3^{(m)}(\omega'^{(m)}_{1133}+\omega'^{(m)}_{1313})},$$

$$\beta_2^{(m)} = \frac{\omega^{(m)}_{3113}\lambda_3^{(m)}}{\lambda_1^{(m)}(\omega^{(m)}_{1133}+\omega^{(m)}_{1313})} - c^2\rho^{(m)}\frac{(\lambda_1^{(m)})^3\lambda_3^{(m)}}{(\omega^{(m)}_{1133}+\omega^{(m)}_{1313})}. \tag{A1}$$

Using the notation (A1) we get

$$\alpha_{11} = \begin{cases} \frac{\omega'^{(1)}_{1111}}{2\lambda_3^{(1)}}(-(\zeta'^{(1)}_2)^2(I_2(\delta_{2h}^{(1)})+I_0(\delta_{2h}^{(1)})) - \frac{\omega'^{(1)}_{1122}}{\lambda_3^{(1)}}\frac{(\zeta'^{(1)}_2)^2}{\delta_{2h}^{(1)}}I_1(\delta_{2h}^{(1)}) + \frac{\omega'^{(1)}_{1133}}{\lambda_1^{(1)}}(\beta_1^{(1)}\times \\ \frac{\omega'^{(1)}_{1111}}{2\lambda_3^{(1)}}(-(\zeta'^{(1)}_2)^2(J_2(\delta_{2h}^{(1)})-J_0(\delta_{2h}^{(1)})) + \frac{\omega'^{(1)}_{1122}}{\lambda_3^{(1)}}\frac{(\zeta'^{(1)}_2)^2}{\delta_{2h}^{(1)}}J_1(\delta_{2h}^{(1)}) + \frac{\omega'^{(1)}_{1133}}{\lambda_1^{(1)}}(\beta_1^{(1)}\times \end{cases}$$

$$\begin{cases} (\frac{(\zeta_2^{(1)})^2}{2}(I_2(\delta_{2h}^{(1)})+I_0(\delta_{2h}^{(1)})) + \frac{(\zeta_2^{(1)})^2}{\delta_{2h}^{(1)}}I_1(\delta_{2h}^{(1)}) - \beta_2^{(1)}I_0(\delta_{2h}^{(1)})) & if\,(\zeta_2^{(1)})^2<0 \\ (\frac{(\zeta_2^{(1)})^2}{2}(J_2(\delta_{2h}^{(1)})-J_0(\delta_{2h}^{(1)})) - \frac{(\zeta_2^{(1)})^2}{\delta_{2h}^{(1)}}J_1(\delta_{2h}^{(1)}) - \beta_2^{(1)}J_0(\delta_{2h}^{(1)})) & if\,(\zeta_2^{(1)})^2>0 \end{cases}$$

$$\alpha_{12} = \begin{cases} \frac{\omega'^{(1)}_{1111}}{2\lambda_3^{(1)}}(-(\zeta'^{(1)}_2)^2(K_2(\delta_{2h}^{(1)})+K_0(\delta_{2h}^{(1)})) + \frac{\omega'^{(1)}_{1122}}{\lambda_3^{(1)}}\frac{(\zeta'^{(1)}_2)^2}{\delta_{2h}^{(1)}}K_1(\delta_{2h}^{(1)}) + \frac{\omega'^{(1)}_{1133}}{\lambda_1^{(1)}}(\beta_1^{(1)}\times \\ \frac{\omega'^{(1)}_{1111}}{2\lambda_3^{(1)}}(-(\zeta'^{(1)}_2)^2(Y_2(\delta_{2h}^{(1)})-Y_0(\delta_{2h}^{(1)})) + \frac{\omega'^{(1)}_{1122}}{\lambda_3^{(1)}}\frac{(\zeta'^{(1)}_2)^2}{\delta_{2h}^{(1)}}Y_1(\delta_{2h}^{(1)}) + \frac{\omega'^{(1)}_{1133}}{\lambda_1^{(1)}}(\beta_1^{(1)}\times \end{cases}$$

$$\begin{cases} (\frac{(\zeta_2^{(1)})^2}{2}(K_2(\delta_{2h}^{(1)})+K_0(\delta_{2h}^{(1)})) - \frac{(\zeta_2^{(1)})^2}{\delta_{2h}^{(1)}}K_1(\delta_{2h}^{(1)}) - \beta_2^{(1)}K_0(\delta_{2h}^{(1)})) & if\,(\zeta_2^{(1)})^2<0 \\ (\frac{(\zeta_2^{(1)})^2}{2}(Y_2(\delta_{2h}^{(1)})-Y_0(\delta_{2h}^{(1)})) - \frac{(\zeta_2^{(1)})^2}{\delta_{2h}^{(1)}}Y_1(\delta_{2h}^{(1)}) - \beta_2^{(1)}Y_0(\delta_{2h}^{(1)})) & if\,(\zeta_2^{(1)})^2>0 \end{cases}$$

$$\alpha_{13} = \begin{cases} \frac{\omega'^{(1)}_{1111}}{2\lambda_3^{(1)}}(-(\zeta'^{(1)}_3)^2(I_2(\delta_{3h}^{(1)})+I_0(\delta_{3h}^{(1)})) - \frac{\omega'^{(1)}_{1122}}{\lambda_3^{(1)}}\frac{(\zeta'^{(1)}_3)^2}{\delta_{3h}^{(1)}}I_1(\delta_{3h}^{(1)}) + \frac{\omega'^{(1)}_{1133}}{\lambda_1^{(1)}}(\beta_1^{(1)}\times \\ \frac{\omega'^{(1)}_{1111}}{2\lambda_3^{(1)}}(-(\zeta'^{(1)}_3)^2(J_2(\delta_{3h}^{(1)})-J_0(\delta_{3h}^{(1)})) + \frac{\omega'^{(1)}_{1122}}{\lambda_3^{(1)}}\frac{(\zeta'^{(1)}_3)^2}{\delta_{3h}^{(1)}}J_1(\delta_{3h}^{(1)}) + \frac{\omega'^{(1)}_{1133}}{\lambda_1^{(1)}}(\beta_1^{(1)}\times \end{cases}$$

$$\begin{cases} (\frac{(\zeta_3^{(1)})^2}{2}(I_2(\delta_{3h}^{(1)})+I_0(\delta_{3h}^{(1)})) + \frac{(\zeta_3^{(1)})^2}{\delta_{3h}^{(1)}}I_1(\delta_{3h}^{(1)}) - \beta_2^{(1)}I_0(\delta_{3h}^{(1)})) & if\,(\zeta_3^{(1)})^2<0 \\ (\frac{(\zeta_3^{(1)})^2}{2}(J_2(\delta_{3h}^{(1)})-J_0(\delta_{3h}^{(1)})) - \frac{(\zeta_3^{(1)})^2}{\delta_{3h}^{(1)}}J_1(\delta_{3h}^{(1)}) - \beta_2^{(1)}J_0(\delta_{3h}^{(1)})) & if\,(\zeta_3^{(1)})^2>0 \end{cases}$$

$$\alpha_{14} = \begin{cases} \frac{\omega'^{(1)}_{1111}}{2\lambda_3^{(1)}}(-(\zeta'^{(1)}_3)^2(K_2(\delta_{3h}^{(1)})+K_0(\delta_{3h}^{(1)})) + \frac{\omega'^{(1)}_{1122}}{\lambda_3^{(1)}}\frac{(\zeta'^{(1)}_3)^2}{\delta_{3h}^{(1)}}K_1(\delta_{3h}^{(1)}) + \frac{\omega'^{(1)}_{1133}}{\lambda_1^{(1)}}(\beta_1^{(1)}\times \\ \frac{\omega'^{(1)}_{1111}}{2\lambda_3^{(1)}}(-(\zeta'^{(1)}_3)^2(Y_2(\delta_{3h}^{(1)})-Y_0(\delta_{3h}^{(1)})) + \frac{\omega'^{(1)}_{1122}}{\lambda_3^{(1)}}\frac{(\zeta'^{(1)}_3)^2}{\delta_{3h}^{(1)}}Y_1(\delta_{3h}^{(1)}) + \frac{\omega'^{(1)}_{1133}}{\lambda_1^{(1)}}(\beta_1^{(1)}\times \end{cases}$$

$$\begin{cases} (\frac{(\zeta_3^{(1)})^2}{2}(K_2(\delta_{3h}^{(1)})+K_0(\delta_{3h}^{(1)})) - \frac{(\zeta_3^{(1)})^2}{\delta_{3h}^{(1)}}K_1(\delta_{3h}^{(1)}) - \beta_2^{(1)}K_0(\delta_{3h}^{(1)})) & if\,(\zeta_3^{(1)})^2<0 \\ (\frac{(\zeta_3^{(1)})^2}{2}(Y_2(\delta_{3h}^{(1)})-Y_0(\delta_{3h}^{(1)})) - \frac{(\zeta_3^{(1)})^2}{\delta_{3h}^{(1)}}Y_1(\delta_{3h}^{(1)}) - \beta_2^{(1)}Y_0(\delta_{3h}^{(1)})) & if\,(\zeta_3^{(1)})^2>0 \end{cases}$$

$$\alpha_{21} = \begin{cases} \frac{\omega'^{(1)}_{1313}}{\lambda_1^{(1)}}\zeta'^{(1)}_2 I_1(\delta_{2h}^{(1)}) + \frac{\omega'^{(1)}_{1331}}{\lambda_3^{(1)}}(\beta_1^{(1)}((\zeta'^{(1)}_2)^3(3I_1(\delta_{2h}^{(1)})+I_3(\delta_{2h}^{(1)}))/4- \\ -\frac{\omega'^{(1)}_{1313}}{\lambda_1^{(1)}}\zeta'^{(1)}_2 J_1(\delta_{2h}^{(1)}) + \frac{\omega'^{(1)}_{1331}}{\lambda_3^{(1)}}(\beta_1^{(1)}((\zeta'^{(1)}_2)^3(3J_1(\delta_{2h}^{(1)})-J_3(\delta_{2h}^{(1)}))/4+ \end{cases}$$

$$\begin{cases} \frac{(\zeta'^{(1)}_2)^3}{(\delta_{2h}^{(1)})^2}I_1(\delta_{2h}^{(1)}) + \frac{(\zeta'^{(1)}_2)^3}{2\delta_{2h}^{(1)}}(I_2(\delta_{2h}^{(1)})+I_0(\delta_{2h}^{(1)})) - \beta_2^{(1)}\zeta'^{(1)}_2 I_1(\delta_{2h}^{(1)})) & if\,(\zeta_2^{(1)})^2<0 \\ \frac{(\zeta'^{(1)}_2)^3}{(\delta_{2h}^{(1)})^2}J_1(\delta_{2h}^{(1)}) + \frac{(\zeta'^{(1)}_2)^3}{2\delta_{2h}^{(1)}}(J_2(\delta_{2h}^{(1)})-J_0(\delta_{2h}^{(1)})) + \beta_2^{(1)}\zeta'^{(1)}_2 J_1(\delta_{2h}^{(1)})) & if\,(\zeta_2^{(1)})^2>0 \end{cases}$$

$$\alpha_{22} = \begin{cases} -\frac{\omega'^{(1)}_{1313}}{\lambda_1^{(1)}}\zeta'^{(1)}_2 K_1(\delta_{2h}^{(1)}) + \frac{\omega'^{(1)}_{1331}}{\lambda_3^{(1)}}(\beta_1^{(1)}((\zeta'^{(1)}_2)^3(-3K_1(\delta_{2h}^{(1)})-K_3(\delta_{2h}^{(1)}))/4+ \\ -\frac{\omega'^{(1)}_{1313}}{\lambda_1^{(1)}}\zeta'^{(1)}_2 Y_1(\delta_{2h}^{(1)}) + \frac{\omega'^{(1)}_{1331}}{\lambda_3^{(1)}}(\beta_1^{(1)}((\zeta'^{(1)}_2)^3(3Y_1(\delta_{2h}^{(1)})-Y_3(\delta_{2h}^{(1)}))/4+ \end{cases}$$

$$\begin{cases} \frac{(\zeta'^{(1)}_2)^3}{(\delta_{2h}^{(1)})^2}K_1(\delta_{2h}^{(1)}) + \frac{(\zeta'^{(1)}_2)^3}{2\delta_{2h}^{(1)}}(K_2(\delta_{2h}^{(1)})+K_0(\delta_{2h}^{(1)})) + \beta_2^{(1)}\zeta'^{(1)}_2 K_1(\delta_{2h}^{(1)})) & if\,(\zeta_2^{(1)})^2<0 \\ \frac{(\zeta'^{(1)}_2)^3}{(\delta_{2h}^{(1)})^2}Y_1(\delta_{2h}^{(1)}) + \frac{(\zeta'^{(1)}_2)^3}{2\delta_{2h}^{(1)}}(Y_2(\delta_{2h}^{(1)})-Y_0(\delta_{2h}^{(1)})) + \beta_2^{(1)}\zeta'^{(1)}_2 Y_1(\delta_{2h}^{(1)})) & if\,(\zeta_2^{(1)})^2>0 \end{cases}$$

$$\alpha_{23} = \begin{cases} \frac{\omega'^{(1)}_{1313}}{\lambda_1^{(1)}}\zeta'^{(1)}_3 I_1(\delta_{3h}^{(1)}) + \frac{\omega'^{(1)}_{1331}}{\lambda_3^{(1)}}(\beta_1^{(1)}((\zeta'^{(1)}_3)^3(3I_1(\delta_{3h}^{(1)})+I_3(\delta_{3h}^{(1)}))/4- \\ -\frac{\omega'^{(1)}_{1313}}{\lambda_1^{(1)}}\zeta'^{(1)}_3 J_1(\delta_{3h}^{(1)}) + \frac{\omega'^{(1)}_{1331}}{\lambda_3^{(1)}}(\beta_1^{(1)}((\zeta'^{(1)}_3)^3(3J_1(\delta_{3h}^{(1)})-J_3(\delta_{3h}^{(1)}))/4+ \end{cases}$$

$$\begin{cases} \frac{(\zeta'^{(1)}_3)^3}{(\delta_{3h}^{(1)})^2}I_1(\delta_{3h}^{(1)}) + \frac{(\zeta'^{(1)}_3)^3}{2\delta_{3h}^{(1)}}(I_2(\delta_{3h}^{(1)})+I_0(\delta_{3h}^{(1)})) - \beta_2^{(1)}\zeta'^{(1)}_3 I_1(\delta_{3h}^{(1)})) & if\,(\zeta_3^{(1)})^2<0 \\ \frac{(\zeta'^{(1)}_3)^3}{(\delta_{3h}^{(1)})^2}J_1(\delta_{3h}^{(1)}) + \frac{(\zeta'^{(1)}_3)^3}{2\delta_{3h}^{(1)}}(J_2(\delta_{3h}^{(1)})-J_0(\delta_{3h}^{(1)})) + \beta_2^{(1)}\zeta'^{(1)}_3 J_1(\delta_{3h}^{(1)})) & if\,(\zeta_3^{(1)})^2>0 \end{cases}$$

$$\alpha_{24} = \begin{cases} -\frac{\omega'^{(1)}_{1313}}{\lambda_1^{(1)}}\zeta'^{(1)}_3 K_1(\delta_{3h}^{(1)}) + \frac{\omega'^{(1)}_{1331}}{\lambda_3^{(1)}}(\beta_1^{(1)}((\zeta'^{(1)}_3)^3(-3K_1(\delta_{3h}^{(1)})-K_3(\delta_{3h}^{(1)}))/4+ \\ -\frac{\omega'^{(1)}_{1313}}{\lambda_1^{(1)}}\zeta'^{(1)}_3 Y_1(\delta_{3h}^{(1)}) + \frac{\omega'^{(1)}_{1331}}{\lambda_3^{(1)}}(\beta_1^{(1)}((\zeta'^{(1)}_3)^3(3Y_1(\delta_{3h}^{(1)})-Y_3(\delta_{3h}^{(1)}))/4+ \end{cases}$$

$$\begin{cases} \frac{(\zeta'^{(1)}_3)^3}{(\delta_{3h}^{(1)})^2}K_1(\delta_{3h}^{(1)}) + \frac{(\zeta'^{(1)}_3)^3}{2\delta_{3h}^{(1)}}(K_2(\delta_{3h}^{(1)})+K_0(\delta_{3h}^{(1)})) + \beta_2^{(1)}\zeta'^{(1)}_3 K_1(\delta_{3h}^{(1)})) & if\,(\zeta_3^{(1)})^2<0 \\ \frac{(\zeta'^{(1)}_3)^3}{(\delta_{3h}^{(1)})^2}Y_1(\delta_{3h}^{(1)}) + \frac{(\zeta'^{(1)}_3)^3}{2\delta_{3h}^{(1)}}(Y_2(\delta_{3h}^{(1)})-Y_0(\delta_{3h}^{(1)})) + \beta_2^{(1)}\zeta'^{(1)}_3 Y_1(\delta_{3h}^{(1)})) & if\,(\zeta_3^{(1)})^2>0 \end{cases}$$

$\alpha_{15} = \alpha_{16} = \alpha_{17} = \alpha_{18} = \alpha_{25} = \alpha_{26} = \alpha_{27} = \alpha_{28} = 0,$

$$\alpha_{31}=\begin{cases}\frac{\omega'^{(1)}_{1111}}{2\lambda_3^{(1)}}(-(\zeta'^{(1)}_2)^2(I_2(\delta_2^{(1)})+I_0(\delta_2^{(1)}))-\frac{\omega'^{(1)}_{1122}}{\lambda_3^{(1)}}\frac{(\zeta'^{(1)}_2)^2}{\delta_2^{(1)}}I_1(\delta_2^{(1)})+\frac{\omega'^{(1)}_{1133}}{\lambda_1^{(1)}}(\beta_1^{(1)}\times\\ \frac{\omega'^{(1)}_{1111}}{2\lambda_3^{(1)}}(-(\zeta'^{(1)}_2)^2(J_2(\delta_2^{(1)})-J_0(\delta_2^{(1)}))+\frac{\omega'^{(1)}_{1122}}{\lambda_3^{(1)}}\frac{(\zeta'^{(1)}_2)^2}{\delta_2^{(1)}}J_1(\delta_2^{(1)})+\frac{\omega'^{(1)}_{1133}}{\lambda_1^{(1)}}(\beta_1^{(1)}\times\end{cases}$$

$$\begin{cases}(\frac{(\zeta_2^{(1)})^2}{2}(I_2(\delta_2^{(1)})+I_0(\delta_2^{(1)}))+\frac{(\zeta_2^{(1)})^2}{\delta_2^{(1)}}I_1(\delta_2^{(1)})-\beta_2^{(1)}I_0(\delta_2^{(1)})) & if\,(\zeta_2^{(1)})^2<0\\ (\frac{(\zeta_2^{(1)})^2}{2}(J_2(\delta_2^{(1)})-J_0(\delta_2^{(1)}))-\frac{(\zeta_2^{(1)})^2}{\delta_2^{(1)}}J_1(\delta_2^{(1)})-\beta_2^{(1)}J_0(\delta_2^{(1)})) & if\,(\zeta_2^{(1)})^2>0\end{cases}$$

$$\alpha_{32}=\begin{cases}\frac{\omega'^{(1)}_{1111}}{2\lambda_3^{(1)}}(-(\zeta'^{(1)}_2)^2(K_2(\delta_2^{(1)})+K_0(\delta_2^{(1)}))+\frac{\omega'^{(1)}_{1122}}{\lambda_3^{(1)}}\frac{(\zeta'^{(1)}_2)^2}{\delta_2^{(1)}}K_1(\delta_2^{(1)})+\frac{\omega'^{(1)}_{1133}}{\lambda_1^{(1)}}(\beta_1^{(1)}\times\\ \frac{\omega'^{(1)}_{1111}}{2\lambda_3^{(1)}}(-(\zeta'^{(1)}_2)^2(Y_2(\delta_2^{(1)})-Y_0(\delta_2^{(1)}))+\frac{\omega'^{(1)}_{1122}}{\lambda_3^{(1)}}\frac{(\zeta'^{(1)}_2)^2}{\delta_2^{(1)}}Y_1(\delta_2^{(1)})+\frac{\omega'^{(1)}_{1133}}{\lambda_1^{(1)}}(\beta_1^{(1)}\times\end{cases}$$

$$\begin{cases}(\frac{(\zeta_2^{(1)})^2}{2}(K_2(\delta_2^{(1)})+K_0(\delta_2^{(1)}))-\frac{(\zeta_2^{(1)})^2}{\delta_2^{(1)}}K_1(\delta_2^{(1)})-\beta_2^{(1)}K_0(\delta_2^{(1)})) & if\,(\zeta_2^{(1)})^2<0\\ (\frac{(\zeta_2^{(1)})^2}{2}(Y_2(\delta_2^{(1)})-Y_0(\delta_2^{(1)}))-\frac{(\zeta_2^{(1)})^2}{\delta_2^{(1)}}Y_1(\delta_2^{(1)})-\beta_2^{(1)}Y_0(\delta_2^{(1)})) & if\,(\zeta_2^{(1)})^2>0\end{cases}$$

$$\alpha_{33}=\begin{cases}\frac{\omega'^{(1)}_{1111}}{2\lambda_3^{(1)}}(-(\zeta'^{(1)}_3)^2(I_2(\delta_3^{(1)})+I_0(\delta_3^{(1)}))-\frac{\omega'^{(1)}_{1122}}{\lambda_3^{(1)}}\frac{(\zeta'^{(1)}_3)^2}{\delta_3^{(1)}}I_1(\delta_3^{(1)})+\frac{\omega'^{(1)}_{1133}}{\lambda_1^{(1)}}(\beta_1^{(1)}\times\\ \frac{\omega'^{(1)}_{1111}}{2\lambda_3^{(1)}}(-(\zeta'^{(1)}_3)^2(J_2(\delta_3^{(1)})-J_0(\delta_3^{(1)}))+\frac{\omega'^{(1)}_{1122}}{\lambda_3^{(1)}}\frac{(\zeta'^{(1)}_3)^2}{\delta_3^{(1)}}J_1(\delta_3^{(1)})+\frac{\omega'^{(1)}_{1133}}{\lambda_1^{(1)}}(\beta_1^{(1)}\times\end{cases}$$

$$\begin{cases}(\frac{(\zeta_3^{(1)})^2}{2}(I_2(\delta_3^{(1)})+I_0(\delta_3^{(1)}))+\frac{(\zeta_3^{(1)})^2}{\delta_3^{(1)}}I_1(\delta_3^{(1)})-\beta_2^{(1)}I_0(\delta_3^{(1)})) & if\,(\zeta_3^{(1)})^2<0\\ (\frac{(\zeta_3^{(1)})^2}{2}(J_2(\delta_3^{(1)})-J_0(\delta_3^{(1)}))-\frac{(\zeta_3^{(1)})^2}{\delta_3^{(1)}}J_1(\delta_3^{(1)})-\beta_2^{(1)}J_0(\delta_3^{(1)})) & if\,(\zeta_3^{(1)})^2>0\end{cases}$$

$$\alpha_{34}=\begin{cases}\frac{\omega'^{(1)}_{1111}}{2\lambda_3^{(1)}}(-(\zeta'^{(1)}_3)^2(K_2(\delta_3^{(1)})+K_0(\delta_3^{(1)}))+\frac{\omega'^{(1)}_{1122}}{\lambda_3^{(1)}}\frac{(\zeta'^{(1)}_3)^2}{\delta_3^{(1)}}K_1(\delta_3^{(1)})+\frac{\omega'^{(1)}_{1133}}{\lambda_1^{(1)}}(\beta_1^{(1)}\times\\ \frac{\omega'^{(1)}_{1111}}{2\lambda_3^{(1)}}(-(\zeta'^{(1)}_3)^2(Y_2(\delta_3^{(1)})-Y_0(\delta_3^{(1)}))+\frac{\omega'^{(1)}_{1122}}{\lambda_3^{(1)}}\frac{(\zeta'^{(1)}_3)^2}{\delta_3^{(1)}}Y_1(\delta_3^{(1)})+\frac{\omega'^{(1)}_{1133}}{\lambda_1^{(1)}}(\beta_1^{(1)}\times\end{cases}$$

$$\begin{cases}(\frac{(\zeta_3^{(1)})^2}{2}(K_2(\delta_3^{(1)})+K_0(\delta_3^{(1)}))-\frac{(\zeta_3^{(1)})^2}{\delta_3^{(1)}}K_1(\delta_3^{(1)})-\beta_2^{(1)}K_0(\delta_3^{(1)})) & if\,(\zeta_3^{(1)})^2<0\\ (\frac{(\zeta_3^{(1)})^2}{2}(Y_2(\delta_3^{(1)})-Y_0(\delta_3^{(1)}))-\frac{(\zeta_3^{(1)})^2}{\delta_3^{(1)}}Y_1(\delta_3^{(1)})-\beta_2^{(1)}Y_0(\delta_3^{(1)})) & if\,(\zeta_3^{(1)})^2>0\end{cases}$$

$$\alpha_{35}=-\begin{cases}\frac{\omega'^{(2)}_{1111}}{2\lambda_3^{(2)}}(-(\zeta'^{(2)}_2)^2(I_2(\delta_2^{(2)})+I_0(\delta_2^{(2)}))-\frac{\omega'^{(2)}_{1122}}{\lambda_3^{(2)}}\frac{(\zeta'^{(2)}_2)^2}{\delta_2^{(2)}}I_1(\delta_2^{(2)})+\frac{\omega'^{(2)}_{1133}}{\lambda_1^{(1)}}(\beta_1^{(2)}\times\\ \frac{\omega'^{(2)}_{1111}}{2\lambda_3^{(2)}}(-(\zeta'^{(2)}_2)^2(J_2(\delta_2^{(2)})-J_0(\delta_2^{(2)}))+\frac{\omega'^{(2)}_{1122}}{\lambda_3^{(2)}}\frac{(\zeta'^{(2)}_2)^2}{\delta_2^{(2)}}J_1(\delta_2^{(2)})+\frac{\omega'^{(2)}_{1133}}{\lambda_1^{(1)}}(\beta_1^{(2)}\times\end{cases}$$

$$\begin{cases}(\frac{(\zeta_2^{(2)})^2}{2}(I_2(\delta_2^{(2)})+I_0(\delta_2^{(2)}))+\frac{(\zeta_2^{(2)})^2}{\delta_2^{(2)}}I_1(\delta_2^{(2)})-\beta_2^{(2)}I_0(\delta_2^{(2)})) & if\,(\zeta_2^{(1)})^2<0\\ (\frac{(\zeta_2^{(2)})^2}{2}(J_2(\delta_2^{(2)})-J_0(\delta_2^{(2)}))-\frac{(\zeta_2^{(2)})^2}{\delta_2^{(2)}}J_1(\delta_2^{(2)})-\beta_2^{(2)}J_0(\delta_2^{(2)})) & if\,(\zeta_2^{(1)})^2>0\end{cases}$$

$$\alpha_{36}=-\begin{cases}\frac{\omega'^{(2)}_{1111}}{2\lambda_3^{(2)}}(-(\zeta'^{(2)}_2)^2(K_2(\delta_2^{(2)})+K_0(\delta_2^{(2)}))+\frac{\omega'^{(2)}_{1122}}{\lambda_3^{(1)}}\frac{(\zeta'^{(2)}_2)^2}{\delta_2^{(2)}}K_1(\delta_2^{(2)})+\frac{\omega'^{(2)}_{1133}}{\lambda_1^{(2)}}(\beta_1^{(2)}\times\\ \frac{\omega'^{(2)}_{1111}}{2\lambda_3^{(2)}}(-(\zeta'^{(2)}_2)^2(Y_2(\delta_2^{(2)})-Y_0(\delta_2^{(2)}))+\frac{\omega'^{(2)}_{1122}}{\lambda_3^{(2)}}\frac{(\zeta'^{(2)}_2)^2}{\delta_2^{(2)}}Y_1(\delta_2^{(2)})+\frac{\omega'^{(2)}_{1133}}{\lambda_1^{(2)}}(\beta_1^{(2)}\times\end{cases}$$

$$\begin{cases}(\frac{(\zeta_2^{(2)})^2}{2}(K_2(\delta_2^{(2)})+K_0(\delta_2^{(2)}))-\frac{(\zeta_2^{(2)})^2}{\delta_2^{(2)}}K_1(\delta_2^{(2)})-\beta_2^{(2)}K_0(\delta_2^{(2)})) & if\,(\zeta_2^{(1)})^2<0\\ (\frac{(\zeta_2^{(2)})^2}{2}(Y_2(\delta_2^{(2)})-Y_0(\delta_2^{(2)}))-\frac{(\zeta_2^{(2)})^2}{\delta_2^{(2)}}Y_1(\delta_2^{(2)})-\beta_2^{(2)}Y_0(\delta_2^{(2)})) & if\,(\zeta_2^{(1)})^2>0\end{cases}$$

$$\alpha_{37}=-\begin{cases}\frac{\omega'^{(2)}_{1111}}{2\lambda_3^{(2)}}(-(\zeta'^{(2)}_3)^2(I_2(\delta_3^{(2)})+I_0(\delta_3^{(2)}))-\frac{\omega'^{(2)}_{1122}}{\lambda_3^{(2)}}\frac{(\zeta'^{(2)}_3)^2}{\delta_3^{(2)}}I_1(\delta_3^{(2)})+\frac{\omega'^{(2)}_{1133}}{\lambda_1^{(2)}}(\beta_1^{(2)}\times\\ \frac{\omega'^{(2)}_{1111}}{2\lambda_3^{(2)}}(-(\zeta'^{(2)}_3)^2(J_2(\delta_3^{(2)})-J_0(\delta_3^{(2)}))+\frac{\omega'^{(2)}_{1122}}{\lambda_3^{(2)}}\frac{(\zeta'^{(2)}_3)^2}{\delta_3^{(2)}}J_1(\delta_3^{(2)})+\frac{\omega'^{(2)}_{1133}}{\lambda_1^{(2)}}(\beta_1^{(2)}\times\end{cases}$$

$$\begin{cases}(\frac{(\zeta_3^{(2)})^2}{2}(I_2(\delta_3^{(2)})+I_0(\delta_3^{(2)}))+\frac{(\zeta_3^{(2)})^2}{\delta_3^{(2)}}I_1(\delta_3^{(2)})-\beta_2^{(2)}I_0(\delta_3^{(2)})) & if\,(\zeta_3^{(1)})^2<0\\ (\frac{(\zeta_3^{(2)})^2}{2}(J_2(\delta_3^{(2)})-J_0(\delta_3^{(2)}))-\frac{(\zeta_3^{(2)})^2}{\delta_3^{(2)}}J_1(\delta_3^{(2)})-\beta_2^{(2)}J_0(\delta_3^{(2)})) & if\,(\zeta_3^{(1)})^2>0\end{cases}$$

$$\alpha_{38}=-\begin{cases}\frac{\omega'^{(2)}_{1111}}{2\lambda_3^{(2)}}(-(\zeta'^{(2)}_3)^2(K_2(\delta_3^{(2)})+K_0(\delta_3^{(2)}))+\frac{\omega'^{(2)}_{1122}}{\lambda_3^{(2)}}\frac{(\zeta'^{(2)}_3)^2}{\delta_3^{(2)}}K_1(\delta_3^{(2)})+\frac{\omega'^{(2)}_{1133}}{\lambda_1^{(2)}}(\beta_1^{(2)}\times\\ \frac{\omega'^{(2)}_{1111}}{2\lambda_3^{(2)}}(-(\zeta'^{(2)}_3)^2(Y_2(\delta_3^{(2)})-Y_0(\delta_3^{(2)}))+\frac{\omega'^{(2)}_{1122}}{\lambda_3^{(2)}}\frac{(\zeta'^{(2)}_3)^2}{\delta_3^{(2)}}Y_1(\delta_3^{(2)})+\frac{\omega'^{(2)}_{1133}}{\lambda_1^{(2)}}(\beta_1^{(2)}\times\end{cases}$$

$$\begin{cases}(\frac{(\zeta_3^{(2)})^2}{2}(K_2(\delta_3^{(2)})+K_0(\delta_3^{(2)}))-\frac{(\zeta_3^{(2)})^2}{\delta_3^{(2)}}K_1(\delta_3^{(2)})-\beta_2^{(2)}K_0(\delta_3^{(2)})) & if\,(\zeta_3^{(1)})^2<0\\ (\frac{(\zeta_3^{(2)})^2}{2}(Y_2(\delta_3^{(2)})-Y_0(\delta_3^{(2)}))-\frac{(\zeta_3^{(2)})^2}{\delta_3^{(2)}}Y_1(\delta_3^{(2)})-\beta_2^{(2)}Y_0(\delta_3^{(2)})) & if\,(\zeta_3^{(1)})^2>0\end{cases}$$

$$\alpha_{41}=\begin{cases}\frac{\omega'^{(1)}_{1313}}{\lambda_1^{(1)}}\zeta'^{(1)}_2I_1(\delta_2^{(1)})+\frac{\omega'^{(1)}_{1331}}{\lambda_3^{(1)}}(\beta_1^{(1)}((\zeta'^{(1)}_2)^3(3I_1(\delta_2^{(1)})+I_3(\delta_2^{(1)}))/4-\\ -\frac{\omega'^{(1)}_{1313}}{\lambda_1^{(1)}}\zeta'^{(1)}_2J_1(\delta_2^{(1)})+\frac{\omega'^{(1)}_{1331}}{\lambda_3^{(1)}}(\beta_1^{(1)}((\zeta'^{(1)}_2)^3(3J_1(\delta_2^{(1)})-J_3(\delta_2^{(1)}))/4+\end{cases}$$

$$\begin{cases}\frac{(\zeta'^{(1)}_2)^3}{(\delta_2^{(1)})^2}I_1(\delta_2^{(1)})+\frac{(\zeta'^{(1)}_2)^3}{2\delta_2^{(1)}}(I_2(\delta_2^{(1)})+I_0(\delta_2^{(1)}))-\beta_2^{(1)}\zeta'^{(1)}_2I_1(\delta_2^{(1)})) & if\,(\zeta_2^{(1)})^2<0\\ \frac{(\zeta'^{(1)}_2)^3}{(\delta_2^{(1)})^2}J_1(\delta_2^{(1)})+\frac{(\zeta'^{(1)}_2)^3}{2\delta_2^{(1)}}(J_2(\delta_2^{(1)})-J_0(\delta_2^{(1)}))+\beta_2^{(1)}\zeta'^{(1)}_2J_1(\delta_2^{(1)})) & if\,(\zeta_2^{(1)})^2>0\end{cases}$$

$$\alpha_{42}=\begin{cases}-\frac{\omega'^{(1)}_{1313}}{\lambda_1^{(1)}}\zeta'^{(1)}_2K_1(\delta_2^{(1)})+\frac{\omega'^{(1)}_{1331}}{\lambda_3^{(1)}}(\beta_1^{(1)}((\zeta'^{(1)}_2)^3(-3K_1(\delta_2^{(1)})-K_3(\delta_2^{(1)}))/4+\\ -\frac{\omega'^{(1)}_{1313}}{\lambda_1^{(1)}}\zeta'^{(1)}_2Y_1(\delta_2^{(1)})+\frac{\omega'^{(1)}_{1331}}{\lambda_3^{(1)}}(\beta_1^{(1)}((\zeta'^{(1)}_2)^3(3Y_1(\delta_2^{(1)})-Y_3(\delta_2^{(1)}))/4+\end{cases}$$

$$\begin{cases}\frac{(\zeta'^{(1)}_2)^3}{(\delta_2^{(1)})^2}K_1(\delta_2^{(1)})+\frac{(\zeta'^{(1)}_2)^3}{2\delta_2^{(1)}}(K_2(\delta_2^{(1)})+K_0(\delta_2^{(1)}))+\beta_2^{(1)}\zeta'^{(1)}_2K_1(\delta_2^{(1)})) & if\,(\zeta_2^{(1)})^2<0\\ \frac{(\zeta'^{(1)}_2)^3}{(\delta_2^{(1)})^2}Y_1(\delta_2^{(1)})+\frac{(\zeta'^{(1)}_2)^3}{2\delta_2^{(1)}}(Y_2(\delta_2^{(1)})-Y_0(\delta_2^{(1)}))+\beta_2^{(1)}\zeta'^{(1)}_2Y_1(\delta_2^{(1)})) & if\,(\zeta_2^{(1)})^2>0\end{cases}$$

$$\alpha_{43}=\begin{cases}\frac{\omega'^{(1)}_{1313}}{\lambda_1^{(1)}}\zeta'^{(1)}_3 I_1(\delta_3^{(1)})+\frac{\omega'^{(1)}_{1331}}{\lambda_3^{(1)}}(\beta_1^{(1)}((\zeta'^{(1)}_3)^3(3I_1(\delta_3^{(1)})+I_3(\delta_3^{(1)}))/4-\\ -\frac{\omega'^{(1)}_{1313}}{\lambda_1^{(1)}}\zeta'^{(1)}_3 J_1(\delta_3^{(1)})+\frac{\omega'^{(1)}_{1331}}{\lambda_3^{(1)}}(\beta_1^{(1)}((\zeta'^{(1)}_3)^3(3J_1(\delta_3^{(1)})-J_3(\delta_3^{(1)}))/4+\end{cases}$$

$$\begin{cases}\frac{(\zeta'^{(1)}_3)^3}{(\delta_3^{(1)})^2}I_1(\delta_3^{(1)})+\frac{(\zeta'^{(1)}_3)^3}{2\delta_3^{(1)}}(I_2(\delta_3^{(1)})+I_0(\delta_3^{(1)}))-\beta_2^{(1)}\zeta'^{(1)}_3 I_1(\delta_3^{(1)})) & if\,(\zeta_3^{(1)})^2<0\\ \frac{(\zeta'^{(1)}_3)^3}{(\delta_3^{(1)})^2}J_1(\delta_3^{(1)})+\frac{(\zeta'^{(1)}_3)^3}{2\delta_3^{(1)}}(J_2(\delta_3^{(1)})-J_0(\delta_3^{(1)}))+\beta_2^{(1)}\zeta'^{(1)}_3 J_1(\delta_3^{(1)})) & if\,(\zeta_3^{(1)})^2>0\end{cases},$$

$$\alpha_{44}=\begin{cases}-\frac{\omega'^{(1)}_{1313}}{\lambda_1^{(1)}}\zeta'^{(1)}_3 K_1(\delta_3^{(1)})+\frac{\omega'^{(1)}_{1331}}{\lambda_3^{(1)}}(\beta_1^{(1)}((\zeta'^{(1)}_3)^3(-3K_1(\delta_3^{(1)})-K_3(\delta_3^{(1)}))/4+\\ -\frac{\omega'^{(1)}_{1313}}{\lambda_1^{(1)}}\zeta'^{(1)}_3 Y_1(\delta_3^{(1)})+\frac{\omega'^{(1)}_{1331}}{\lambda_3^{(1)}}(\beta_1^{(1)}((\zeta'^{(1)}_3)^3(3Y_1(\delta_3^{(1)})-Y_3(\delta_3^{(1)}))/4+\end{cases}$$

$$\begin{cases}\frac{(\zeta'^{(1)}_3)^3}{(\delta_3^{(1)})^2}K_1(\delta_3^{(1)})+\frac{(\zeta'^{(1)}_3)^3}{2\delta_3^{(1)}}(K_2(\delta_3^{(1)})+K_0(\delta_3^{(1)}))+\beta_2^{(1)}\zeta'^{(1)}_3 K_1(\delta_3^{(1)})) & if\,(\zeta_3^{(1)})^2<0\\ \frac{(\zeta'^{(1)}_3)^3}{(\delta_3^{(1)})^2}Y_1(\delta_3^{(1)})+\frac{(\zeta'^{(1)}_3)^3}{2\delta_3^{(1)}}(Y_2(\delta_3^{(1)})-Y_0(\delta_3^{(1)}))+\beta_2^{(1)}\zeta'^{(1)}_3 Y_1(\delta_3^{(1)})) & if\,(\zeta_3^{(1)})^2>0\end{cases},$$

$$\alpha_{45}=-\begin{cases}\frac{\omega'^{(2)}_{1313}}{\lambda_1^{(2)}}\zeta'^{(2)}_2 I_1(\delta_2^{2)})+\frac{\omega'^{(2)}_{1331}}{\lambda_3^{(1)}}(\beta_1^{(2)}((\zeta'^{(2)}_2)^3(3I_1(\delta_2^{(2)})+I_3(\delta_2^{(2)}))/4-\\ -\frac{\omega'^{(2)}_{1313}}{\lambda_1^{(2)}}\zeta'^{(2)}_2 J_1(\delta_2^{(2)})+\frac{\omega'^{(2)}_{1331}}{\lambda_3^{(2)}}(\beta_1^{(2)}((\zeta'^{(2)}_2)^3(3J_1(\delta_2^{(2)})-J_3(\delta_2^{(2)}))/4+\end{cases}$$

$$\begin{cases}\frac{(\zeta'^{(2)}_2)^3}{(\delta_2^{(2)})^2}I_1(\delta_2^{(2)})+\frac{(\zeta'^{(2)}_2)^3}{2\delta_2^{(2)}}(I_2(\delta_2^{(2)})+I_0(\delta_2^{(2)}))-\beta_2^{(2)}\zeta'^{(2)}_2 I_1(\delta_2^{(2)})) & if\,(\zeta_2^{(1)})^2<0\\ \frac{(\zeta'^{(2)}_2)^3}{(\delta_2^{(2)})^2}J_1(\delta_2^{(2)})+\frac{(\zeta'^{(2)}_2)^3}{2\delta_2^{(2)}}(J_2(\delta_2^{(2)})-J_0(\delta_2^{(2)}))+\beta_2^{(2)}\zeta'^{(2)}_2 J_1(\delta_2^{(2)})) & if\,(\zeta_2^{(1)})^2>0\end{cases},$$

$$\alpha_{46}=-\begin{cases}-\frac{\omega'^{(2)}_{1313}}{\lambda_1^{(2)}}\zeta'^{(2)}_2 K_1(\delta_2^{(2)})+\frac{\omega'^{(2)}_{1331}}{\lambda_3^{(2)}}(\beta_1^{(2)}((\zeta'^{(2)}_2)^3(-3K_1(\delta_2^{(2)})-K_3(\delta_2^{(2)}))/4+\\ -\frac{\omega'^{(2)}_{1313}}{\lambda_1^{(2)}}\zeta'^{(2)}_2 Y_1(\delta_2^{(2)})+\frac{\omega'^{(2)}_{1331}}{\lambda_3^{(2)}}(\beta_1^{(2)}((\zeta'^{(2)}_2)^3(3Y_1(\delta_2^{(2)})-Y_3(\delta_2^{(2)}))/4+\end{cases}$$

$$\begin{cases}\frac{(\zeta'^{(2)}_2)^3}{(\delta_2^{(2)})^2}K_1(\delta_2^{(2)})+\frac{(\zeta'^{(2)}_2)^3}{2\delta_2^{(2)}}(K_2(\delta_2^{(2)})+K_0(\delta_2^{(2)}))+\beta_2^{(2)}\zeta'^{(2)}_2 K_1(\delta_2^{(2)})) & if\,(\zeta_2^{(1)})^2<0\\ \frac{(\zeta'^{(2)}_2)^3}{(\delta_2^{(2)})^2}Y_1(\delta_2^{(2)})+\frac{(\zeta'^{(2)}_2)^3}{2\delta_2^{(2)}}(Y_2(\delta_2^{(2)})-Y_0(\delta_2^{(2)}))+\beta_2^{(2)}\zeta'^{(2)}_2 Y_1(\delta_2^{(2)})) & if\,(\zeta_2^{(1)})^2>0\end{cases},$$

$$\alpha_{47}=-\begin{cases}\frac{\omega'^{(2)}_{1313}}{\lambda_1^{(2)}}\zeta'^{(2)}_3 I_1(\delta_3^{(2)})+\frac{\omega'^{(2)}_{1331}}{\lambda_3^{(2)}}(\beta_1^{(2)}((\zeta'^{(2)}_3)^3(3I_1(\delta_3^{(2)})+I_3(\delta_3^{(2)}))/4-\\ -\frac{\omega'^{(2)}_{1313}}{\lambda_1^{(2)}}\zeta'^{(2)}_3 J_1(\delta_3^{(2)})+\frac{\omega'^{(2)}_{1331}}{\lambda_3^{(2)}}(\beta_1^{(2)}((\zeta'^{(2)}_3)^3(3J_1(\delta_3^{(2)})-J_3(\delta_3^{(2)}))/4+\end{cases}$$

$$\begin{cases}\frac{(\zeta'^{(2)}_3)^3}{(\delta_3^{(2)})^2}I_1(\delta_3^{(2)})+\frac{(\zeta'^{(2)}_3)^3}{2\delta_3^{(2)}}(I_2(\delta_3^{(2)})+I_0(\delta_3^{(2)}))-\beta_2^{(2)}\zeta'^{(2)}_3 I_1(\delta_3^{(2)})) & if\,(\zeta_3^{(1)})^2<0\\ \frac{(\zeta'^{(2)}_3)^3}{(\delta_3^{(2)})^2}J_1(\delta_3^{(2)})+\frac{(\zeta'^{(2)}_3)^3}{2\delta_3^{(2)}}(J_2(\delta_3^{(2)})-J_0(\delta_3^{(2)}))+\beta_2^{(2)}\zeta'^{(2)}_3 J_1(\delta_3^{(2)})) & if\,(\zeta_3^{(1)})^2>0\end{cases},$$

$$\alpha_{48} = -\begin{cases} -\frac{\omega'^{(2)}_{1313}}{\lambda_1^{(2)}}\zeta'^{(2)}_3 K_1(\delta_3^{(2)}) + \frac{\omega'^{(2)}_{1331}}{\lambda_3^{(2)}}(\beta_1^{(2)}((\zeta'^{(2)}_3)^3(-3K_1(\delta_3^{(2)}) - K_3(\delta_3^{(2)}))/4+ \\ -\frac{\omega'^{(2)}_{1313}}{\lambda_1^{(2)}}\zeta'^{(2)}_3 Y_1(\delta_3^{(2)}) + \frac{\omega'^{(2)}_{1331}}{\lambda_3^{(2)}}(\beta_1^{(2)}((\zeta'^{(2)}_3)^3(3Y_1(\delta_3^{(2)}) - Y_3(\delta_3^{(2)}))/4+ \end{cases}$$

$$\begin{cases} \frac{(\zeta'^{(2)}_3)^3}{(\delta_3^{(2)})^2}K_1(\delta_3^{(2)}) + \frac{(\zeta'^{(2)}_3)^3}{2\delta_3^{(2)}}(K_2(\delta_3^{(2)}) + K_0(\delta_3^{(2)})) + \beta_2^{(2)}\zeta'^{(2)}_3 K_1(\delta_3^{(2)})) & if\,(\zeta_3^{(1)})^2 < 0 \\ \frac{(\zeta'^{(2)}_3)^3}{(\delta_3^{(2)})^2}Y_1(\delta_3^{(2)}) + \frac{(\zeta'^{(2)}_3)^3}{2\delta_3^{(2)}}(Y_2(\delta_3^{(2)}) - Y_0(\delta_3^{(2)})) + \beta_2^{(2)}\zeta'^{(2)}_3 Y_1(\delta_3^{(2)})) & if\,(\zeta_3^{(1)})^2 > 0 \end{cases},$$

$$\alpha_{51} = \begin{cases} -\zeta_2^{(1)} I_1(\delta_2^{(1)}) \\ \zeta_2^{(1)} J_1(\delta_2^{(1)}) \end{cases}, \begin{matrix} if\,(\zeta_2^{(1)})^2 < 0 \\ if\,(\zeta_2^{(1)})^2 > 0 \end{matrix}, \quad \alpha_{52} = \begin{cases} \zeta_2^{(1)} K_1(\delta_2^{(1)}) \\ \zeta_2^{(1)} J_1(\delta_2^{(1)}) \end{cases}, \begin{matrix} if\,(\zeta_2^{(1)})^2 < 0 \\ if\,(\zeta_2^{(1)})^2 > 0 \end{matrix},$$

$$\alpha_{53} = \begin{cases} -\zeta_3^{(1)} I_1(\delta_3^{(1)}) \\ \zeta_3^{(1)} J_1(\delta_3^{(1)}) \end{cases}, \begin{matrix} if\,(\zeta_3^{(1)})^2 < 0 \\ if\,(\zeta_3^{(1)})^2 > 0 \end{matrix}, \quad \alpha_{54} = \begin{cases} \zeta_3^{(1)} K_1(\delta_3^{(1)}) \\ \zeta_3^{(1)} J_1(\delta_3^{(1)}) \end{cases}, \begin{matrix} if\,(\zeta_3^{(1)})^2 < 0 \\ if\,(\zeta_3^{(1)})^2 > 0 \end{matrix},$$

$$\alpha_{55} = -\begin{cases} -\zeta_2^{(2)} I_1(\delta_2^{(2)}) \\ \zeta_2^{(2)} J_1(\delta_2^{(2)}) \end{cases}, \begin{matrix} if\,(\zeta_2^{(1)})^2 < 0 \\ if\,(\zeta_2^{(1)})^2 > 0 \end{matrix}, \quad \alpha_{56} = \begin{cases} \zeta_2^{(2)} K_1(\delta_2^{(2)}) \\ \zeta_2^{(2)} J_1(\delta_2^{(2)}) \end{cases}, \begin{matrix} if\,(\zeta_2^{(1)})^2 < 0 \\ if\,(\zeta_2^{(1)})^2 > 0 \end{matrix},$$

$$\alpha_{57} = \begin{cases} -\zeta_3^{(2)} I_1(\delta_3^{(2)}) \\ \zeta_3^{(2)} J_1(\delta_3^{(2)}) \end{cases}, \begin{matrix} if\,(\zeta_3^{(1)})^2 < 0 \\ if\,(\zeta_3^{(1)})^2 > 0 \end{matrix}, \quad \alpha_{58} = \begin{cases} \zeta_3^{(2)} K_1(\delta_3^{(2)}) \\ \zeta_3^{(2)} J_1(\delta_3^{(2)}) \end{cases}, \begin{matrix} if\,(\zeta_3^{(1)})^2 < 0 \\ if\,(\zeta_3^{(1)})^2 > 0 \end{matrix},$$

$$\alpha_{61} = \begin{cases} (\beta_1^{(1)}(\zeta_2^{(1)})^2 - \beta_2^{(1)}) I_0(\delta_2^{(1)}) \\ (-\beta_1^{(1)}(\zeta_2^{(1)})^2 - \beta_2^{(1)}) J_0(\delta_2^{(1)}) \end{cases}, \begin{matrix} if\,(\zeta_2^{(1)})^2 < 0 \\ if\,(\zeta_2^{(1)})^2 > 0 \end{matrix},$$

$$\alpha_{62} = \begin{cases} (\beta_1^{(1)}(\zeta_2^{(1)})^2 - \beta_2^{(1)}) K_0(\delta_2^{(1)}) \\ (-\beta_1^{(1)}(\zeta_2^{(1)})^2 - \beta_2^{(1)}) Y_0(\delta_2^{(1)}) \end{cases}, \begin{matrix} if\,(\zeta_2^{(1)})^2 < 0 \\ if\,(\zeta_2^{(1)})^2 > 0 \end{matrix},$$

$$\alpha_{63} = \begin{cases} (\beta_1^{(1)}(\zeta_3^{(1)})^2 - \beta_2^{(1)}) I_0(\delta_3^{(1)}) \\ (-\beta_1^{(1)}(\zeta_3^{(1)})^2 - \beta_2^{(1)}) J_0(\delta_3^{(1)}) \end{cases}, \begin{matrix} if\,(\zeta_3^{(1)})^2 < 0 \\ if\,(\zeta_3^{(1)})^2 > 0 \end{matrix}$$

$$\alpha_{64} = \begin{cases} (\beta_1^{(1)}(\zeta_3^{(1)})^2 - \beta_2^{(1)}) K_0(\delta_3^{(1)}) \\ (-\beta_1^{(1)}(\zeta_3^{(1)})^2 - \beta_2^{(1)}) Y_0(\delta_3^{(1)}) \end{cases}, \begin{matrix} if\,(\zeta_3^{(1)})^2 < 0 \\ if\,(\zeta_3^{(1)})^2 > 0 \end{matrix},$$

$$\alpha_{65} = -\begin{cases} (\beta_1^{(2)}(\zeta_2^{(2)})^2 - \beta_2^{(2)}) I_0(\delta_2^{(2)}) \\ (-\beta_1^{(2)}(\zeta_2^{(2)})^2 - \beta_2^{(2)}) J_0(\delta_2^{(2)}) \end{cases}, \begin{matrix} if\,(\zeta_2^{(1)})^2 < 0 \\ if\,(\zeta_2^{(1)})^2 > 0 \end{matrix},$$

$$\alpha_{66} = -\begin{cases} (\beta_1^{(2)}(\zeta_2^{(2)})^2 - \beta_2^{(2)}) K_0(\delta_2^{(2)}) \\ (-\beta_1^{(2)}(\zeta_2^{(2)})^2 - \beta_2^{(2)}) Y_0(\delta_2^{(2)}) \end{cases}, \begin{matrix} if\,(\zeta_2^{(1)})^2 < 0 \\ if\,(\zeta_2^{(1)})^2 > 0 \end{matrix},$$

$$\alpha_{67} = -\begin{cases} (\beta_1^{(2)}(\zeta_3^{(2)})^2 - \beta_2^{(2)}) I_0(\delta_3^{(2)}) \\ (-\beta_1^{(2)}(\zeta_3^{(2)})^2 - \beta_2^{(2)}) J_0(\delta_3^{(2)}) \end{cases}, \begin{matrix} if\,(\zeta_3^{(1)})^2 < 0 \\ if\,(\zeta_3^{(1)})^2 > 0 \end{matrix},$$

$$\alpha_{68} = -\begin{cases} (\beta_1^{(2)}(\zeta_3^{(2)})^2 - \beta_2^{(2)})K_0(\delta_3^{(2)}) & , \quad if\,(\zeta_3^{(1)})^2 < 0 \\ (-\beta_1^{(2)}(\zeta_3^{(2)})^2 - \beta_2^{(2)})Y_0(\delta_3^{(2)}) & \quad if\,(\zeta_3^{(1)})^2 > 0 \end{cases},$$

$$\alpha_{75} = \begin{cases} \frac{\omega'^{(2)}_{1111}}{2\lambda_3^{(2)}}(-(\zeta'^{(2)}_2)^2(I_2(\delta_{2h}^{(2)}) + I_0(\delta_{2h}^{(2)})) - \frac{\omega'^{(2)}_{1122}}{\lambda_3^{(2)}}\frac{(\zeta'^{(2)}_2)^2}{\delta_{2h}^{(2)}}I_1(\delta_{2h}^{(2)}) + \frac{\omega'^{(2)}_{1133}}{\lambda_1^{(1)}}(\beta_1^{(2)}\times \\ \frac{\omega'^{(2)}_{1111}}{2\lambda_3^{(2)}}(-(\zeta'^{(2)}_2)^2(J_2(\delta_{2h}^{(2)}) - J_0(\delta_{2h}^{(2)})) + \frac{\omega'^{(2)}_{1122}}{\lambda_3^{(2)}}\frac{(\zeta'^{(2)}_2)^2}{\delta_{2h}^{(2)}}J_1(\delta_{2h}^{(2)}) + \frac{\omega'^{(2)}_{1133}}{\lambda_1^{(1)}}(\beta_1^{(2)}\times \end{cases}$$

$$\begin{cases} (\frac{(\zeta_2^{(2)})^2}{2}(I_2(\delta_{2h}^{(2)}) + I_0(\delta_{2h}^{(2)})) + \frac{(\zeta_2^{(2)})^2}{\delta_{2h}^{(1)}}I_1(\delta_{2h}^{(2)}) - \beta_2^{(2)}I_0(\delta_{2h}^{(2)})) & , \quad if\,(\zeta_2^{(1)})^2 < 0 \\ (\frac{(\zeta_2^{(2)})^2}{2}(J_2(\delta_{2h}^{(2)}) - J_0(\delta_{2h}^{(2)})) - \frac{(\zeta_2^{(2)})^2}{\delta_{2h}^{(2)}}J_1(\delta_{2h}^{(2)}) - \beta_2^{(2)}J_0(\delta_{2h}^{(2)})) & \quad if\,(\zeta_2^{(1)})^2 > 0 \end{cases},$$

$$\alpha_{76} = \begin{cases} \frac{\omega'^{(2)}_{1111}}{2\lambda_3^{(2)}}(-(\zeta'^{(2)}_2)^2(K_2(\delta_{2h}^{(2)}) + K_0(\delta_{2h}^{(2)})) + \frac{\omega'^{(2)}_{1122}}{\lambda_3^{(2)}}\frac{(\zeta'^{(2)}_2)^2}{\delta_{2h}^{(2)}}K_1(\delta_{2h}^{(2)}) + \frac{\omega'^{(2)}_{1133}}{\lambda_1^{(2)}}(\beta_1^{(2)}\times \\ \frac{\omega'^{(2)}_{1111}}{2\lambda_3^{(2)}}(-(\zeta'^{(2)}_2)^2(Y_2(\delta_{2h}^{(2)}) - Y_0(\delta_{2h}^{(2)})) + \frac{\omega'^{(2)}_{1122}}{\lambda_3^{(2)}}\frac{(\zeta'^{(2)}_2)^2}{\delta_{2h}^{(2)}}Y_1(\delta_{2h}^{(2)}) + \frac{\omega'^{(2)}_{1133}}{\lambda_1^{(2)}}(\beta_1^{(2)}\times \end{cases}$$

$$\begin{cases} (\frac{(\zeta_2^{(2)})^2}{2}(K_2(\delta_{2h}^{(2)}) + K_0(\delta_{2h}^{(2)})) - \frac{(\zeta_2^{(2)})^2}{\delta_{2h}^{(2)}}K_1(\delta_{2h}^{(2)}) - \beta_2^{(2)}K_0(\delta_{2h}^{(2)})) & if\,(\zeta_2^{(1)})^2 < 0 \\ (\frac{(\zeta_2^{(2)})^2}{2}(Y_2(\delta_{2h}^{(2)}) - Y_0(\delta_{2h}^{(2)})) - \frac{(\zeta_2^{(2)})^2}{\delta_{2h}^{(2)}}Y_1(\delta_{2h}^{(2)}) - \beta_2^{(2)}Y_0(\delta_{2h}^{(2)})) & if\,(\zeta_2^{(1)})^2 > 0 \end{cases},$$

$$\alpha_{77} = \begin{cases} \frac{\omega'^{(2)}_{1111}}{2\lambda_3^{(2)}}(-(\zeta'^{(2)}_3)^2(I_2(\delta_{3h}^{(2)}) + I_0(\delta_{3h}^{(2)})) - \frac{\omega'^{(2)}_{1122}}{\lambda_3^{(2)}}\frac{(\zeta'^{(2)}_3)^2}{\delta_{3h}^{(2)}}I_1(\delta_{3h}^{(2)}) + \frac{\omega'^{(2)}_{1133}}{\lambda_1^{(2)}}(\beta_1^{(2)}\times \\ \frac{\omega'^{(2)}_{1111}}{2\lambda_3^{(2)}}(-(\zeta'^{(2)}_3)^2(J_2(\delta_{3h}^{(2)}) - J_0(\delta_{3h}^{(2)})) + \frac{\omega'^{(2)}_{1122}}{\lambda_3^{(2)}}\frac{(\zeta'^{(2)}_3)^2}{\delta_{3h}^{(2)}}J_1(\delta_{3h}^{(2)}) + \frac{\omega'^{(2)}_{1133}}{\lambda_1^{(2)}}(\beta_1^{(2)}\times \end{cases}$$

$$\begin{cases} (\frac{(\zeta_3^{(2)})^2}{2}(I_2(\delta_{3h}^{(2)}) + I_0(\delta_{3h}^{(2)})) + \frac{(\zeta_3^{(2)})^2}{\delta_{3h}^{(2)}}I_1(\delta_{3h}^{(2)}) - \beta_2^{(2)}I_0(\delta_{3h}^{(2)})) & if\,(\zeta_3^{(1)})^2 < 0 \\ (\frac{(\zeta_3^{(2)})^2}{2}(J_2(\delta_{3h}^{(2)}) - J_0(\delta_{3h}^{(2)})) - \frac{(\zeta_3^{(2)})^2}{\delta_{3h}^{(2)}}J_1(\delta_{3h}^{(2)}) - \beta_2^{(2)}J_0(\delta_{3h}^{(2)})) & if\,(\zeta_3^{(1)})^2 > 0 \end{cases},$$

$$\alpha_{78} = \begin{cases} \frac{\omega'^{(2)}_{1111}}{2\lambda_3^{(2)}}(-(\zeta'^{(2)}_3)^2(K_2(\delta_{3h}^{(2)}) + K_0(\delta_{3h}^{(2)})) + \frac{\omega'^{(2)}_{1122}}{\lambda_3^{(2)}}\frac{(\zeta'^{(2)}_3)^2}{\delta_{3h}^{(2)}}K_1(\delta_{3h}^{(2)}) + \frac{\omega'^{(2)}_{1133}}{\lambda_1^{(2)}}(\beta_1^{(2)}\times \\ \frac{\omega'^{(2)}_{1111}}{2\lambda_3^{(2)}}(-(\zeta'^{(2)}_3)^2(Y_2(\delta_{3h}^{(2)}) - Y_0(\delta_{3h}^{(2)})) + \frac{\omega'^{(2)}_{1122}}{\lambda_3^{(2)}}\frac{(\zeta'^{(2)}_3)^2}{\delta_{3h}^{(2)}}Y_1(\delta_{3h}^{(2)}) + \frac{\omega'^{(2)}_{1133}}{\lambda_1^{(2)}}(\beta_1^{(2)}\times \end{cases}$$

$$\begin{cases} (\frac{(\zeta_3^{(2)})^2}{2}(K_2(\delta_{3h}^{(2)}) + K_0(\delta_{3h}^{(2)})) - \frac{(\zeta_3^{(2)})^2}{\delta_{3h}^{(2)}}K_1(\delta_{3h}^{(2)}) - \beta_2^{(2)}K_0(\delta_{3h}^{(2)})) & if\,(\zeta_3^{(1)})^2 < 0 \\ (\frac{(\zeta_3^{(2)})^2}{2}(Y_2(\delta_{3h}^{(2)}) - Y_0(\delta_{3h}^{(2)})) - \frac{(\zeta_3^{(2)})^2}{\delta_{3h}^{(2)}}Y_1(\delta_{3h}^{(2)}) - \beta_2^{(2)}Y_0(\delta_{3h}^{(2)})) & if\,(\zeta_3^{(1)})^2 > 0 \end{cases},$$

$$\alpha_{85} = \begin{cases} \frac{\omega'^{(2)}_{1313}}{\lambda_1^{(2)}}\zeta'^{(2)}_2 I_1(\delta_{2h}^{(2)}) + \frac{\omega'^{(2)}_{1331}}{\lambda_3^{(2)}}(\beta_1^{(2)}((\zeta'^{(2)}_2)^3(3I_1(\delta_{2h}^{(2)}) + I_3(\delta_{2h}^{(2)}))/4- \\ -\frac{\omega'^{(2)}_{1313}}{\lambda_1^{(2)}}\zeta'^{(2)}_2 J_1(\delta_{2h}^{(2)}) + \frac{\omega'^{(2)}_{1331}}{\lambda_3^{(2)}}(\beta_1^{(2)}((\zeta'^{(2)}_2)^3(3J_1(\delta_{2h}^{(2)}) - J_3(\delta_{2h}^{(2)}))/4+ \end{cases}$$

$$\begin{cases} \frac{(\zeta'^{(2)}_2)^3}{(\delta^{(2)}_{2h})^2} I_1(\delta^{(2)}_{2h}) + \frac{(\zeta'^{(2)}_2)^3}{2\delta^{(2)}_{2h}} (I_2(\delta^{(2)}_{2h}) + I_0(\delta^{(2)}_{2h})) - \beta^{(2)}_2 \zeta'^{(2)}_2 I_1(\delta^{(2)}_{2h})) & if\,(\zeta^{(1)}_2)^2 < 0 \\ \frac{(\zeta'^{(2)}_2)^3}{(\delta^{(2)}_{2h})^2} J_1(\delta^{(2)}_{2h}) + \frac{(\zeta'^{(2)}_2)^3}{2\delta^{(2)}_{2h}} (J_2(\delta^{(2)}_{2h}) - J_0(\delta^{(2)}_{2h})) + \beta^{(2)}_2 \zeta'^{(2)}_2 J_1(\delta^{(2)}_{2h})) & if\,(\zeta^{(1)}_2)^2 > 0 \end{cases},$$

$$\alpha_{86} = \begin{cases} -\frac{\omega'^{(2)}_{1313}}{\lambda^{(2)}_1} \zeta'^{(2)}_2 K_1(\delta^{(2)}_{2h}) + \frac{\omega'^{(2)}_{1331}}{\lambda^{(2)}_3} (\beta^{(2)}_1 ((\zeta'^{(2)}_2)^3 (-3K_1(\delta^{(2)}_{2h}) - K_3(\delta^{(2)}_{2h}))/4 + \\ -\frac{\omega'^{(2)}_{1313}}{\lambda^{(2)}_1} \zeta'^{(2)}_2 Y_1(\delta^{(2)}_{2h}) + \frac{\omega'^{(2)}_{1331}}{\lambda^{(2)}_3} (\beta^{(2)}_1 ((\zeta'^{(2)}_2)^3 (3Y_1(\delta^{2)}_{2h}) - Y_3(\delta^{(2)}_{2h}))/4 + \end{cases}$$

$$\begin{cases} \frac{(\zeta'^{(2)}_2)^3}{(\delta^{(2)}_{2h})^2} K_1(\delta^{(2)}_{2h}) + \frac{(\zeta'^{(2)}_2)^3}{2\delta^{(2)}_{2h}} (K_2(\delta^{(2)}_{2h}) + K_0(\delta^{(2)}_{2h})) + \beta^{(2)}_2 \zeta'^{(2)}_2 K_1(\delta^{(2)}_{2h})) & if\,(\zeta^{(1)}_2)^2 < 0 \\ \frac{(\zeta'^{(2)}_2)^3}{(\delta^{(2)}_{2h})^2} Y_1(\delta^{(2)}_{2h}) + \frac{(\zeta'^{(2)}_2)^3}{2\delta^{(2)}_{2h}} (Y_2(\delta^{(2)}_{2h}) - Y_0(\delta^{(2)}_{2h})) + \beta^{(2)}_2 \zeta'^{(2)}_2 Y_1(\delta^{(2)}_{2h})) & if\,(\zeta^{(1)}_2)^2 > 0 \end{cases},$$

$$\alpha_{87} = \begin{cases} \frac{\omega'^{(2)}_{1313}}{\lambda^{(2)}_1} \zeta'^{(2)}_3 I_1(\delta^{(2)}_{3h}) + \frac{\omega'^{(2)}_{1331}}{\lambda^{(2)}_3} (\beta^{(2)}_1 ((\zeta'^{(2)}_3)^3 (3I_1(\delta^{(2)}_{3h}) + I_3(\delta^{(2)}_{3h}))/4 - \\ -\frac{\omega'^{(2)}_{1313}}{\lambda^{(2)}_1} \zeta'^{(2)}_3 J_1(\delta^{(2)}_{3h}) + \frac{\omega'^{(2)}_{1331}}{\lambda^{(2)}_3} (\beta^{(2)}_1 ((\zeta'^{(2)}_3)^3 (3J_1(\delta^{(2)}_{3h}) - J_3(\delta^{(2)}_{3h}))/4 + \end{cases}$$

$$\begin{cases} \frac{(\zeta'^{(2)}_3)^3}{(\delta^{(2)}_{3h})^2} I_1(\delta^{(2)}_{3h}) + \frac{(\zeta'^{(2)}_3)^3}{2\delta^{(2)}_{3h}} (I_2(\delta^{(2)}_{3h}) + I_0(\delta^{(2)}_{3h})) - \beta^{(2)}_2 \zeta'^{(2)}_3 I_1(\delta^{(2)}_{3h})) & if\,(\zeta^{(1)}_3)^2 < 0 \\ \frac{(\zeta'^{(2)}_3)^3}{(\delta^{(2)}_{3h})^2} J_1(\delta^{(2)}_{3h}) + \frac{(\zeta'^{(2)}_3)^3}{2\delta^{(2)}_{3h}} (J_2(\delta^{(2)}_{3h}) - J_0(\delta^{(2)}_{3h})) + \beta^{(2)}_2 \zeta'^{(2)}_3 J_1(\delta^{(2)}_{3h})) & if\,(\zeta^{(1)}_3)^2 > 0 \end{cases},$$

$$\alpha_{88} = \begin{cases} -\frac{\omega'^{(2)}_{1313}}{\lambda^{(2)}_1} \zeta'^{(2)}_3 K_1(\delta^{(2)}_{3h}) + \frac{\omega'^{(2)}_{1331}}{\lambda^{(2)}_3} (\beta^{(2)}_1 ((\zeta'^{(2)}_3)^3 (-3K_1(\delta^{(2)}_{3h}) - K_3(\delta^{(2)}_{3h}))/4 + \\ -\frac{\omega'^{(2)}_{1313}}{\lambda^{(2)}_1} \zeta'^{(2)}_3 Y_1(\delta^{(2)}_{3h}) + \frac{\omega'^{(2)}_{1331}}{\lambda^{(2)}_3} (\beta^{(2)}_1 ((\zeta'^{(2)}_3)^3 (3Y_1(\delta^{(2)}_{3h}) - Y_3(\delta^{(2)}_{3h}))/4 + \end{cases}$$

$$\begin{cases} \frac{(\zeta'^{(2)}_3)^3}{(\delta^{(2)}_{3h})^2} K_1(\delta^{(2)}_{3h}) + \frac{(\zeta'^{(2)}_3)^3}{2\delta^{(2)}_{3h}} (K_2(\delta^{(2)}_{3h}) + K_0(\delta^{(2)}_{3h})) + \beta^{(2)}_2 \zeta'^{(2)}_3 K_1(\delta^{(2)}_{3h})) & if\,(\zeta^{(1)}_3)^2 < 0 \\ \frac{(\zeta'^{(2)}_3)^3}{(\delta^{(2)}_{3h})^2} Y_1(\delta^{(2)}_{3h}) + \frac{(\zeta'^{(2)}_3)^3}{2\delta^{(2)}_{3h}} (Y_2(\delta^{(2)}_{3h}) - Y_0(\delta^{(2)}_{3h})) + \beta^{(2)}_2 \zeta'^{(2)}_3 Y_1(\delta^{(2)}_{3h})) & if\,(\zeta^{(1)}_3)^2 > 0 \end{cases},$$

$$\alpha_{71} = \alpha_{72} = \alpha_{73} = \alpha_{74} = \alpha_{81} = \alpha_{82} = \alpha_{83} = \alpha_{84} = 0, \tag{A2}$$

where $J_n(x)$ and $Y_n(x)$ are Bessel functions of the first and second kind of the $n-th$ order; $I_0(x)$ and $K_0(x)$ are Bessel functions of a purely imaginary argument in the $n-$th order and Macdonald function in the $n-$th order, respectively.

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