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The time-dependent Green's function of the transverse vibration of a composite rectangular membrane

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Abstract: A new method for the approximate computation of the time-dependent Green's function for the equations of the transverse vibration of a multi stepped membrane is suggested. This method is based on generalization of the Fourier series expansion method and consists of the following steps. The first step is finding eigenvalues and an orthogonal set of eigenfunctions corresponding to an ordinary differential operator with boundary and matching conditions. The second step is a regularization (approximation) of the Dirac delta function in the form of the Fourier series with a finite number of terms, using the orthogonal set of eigenfunctions. The third step is an approximate computation of the Green's function in the form of the Fourier series with a finite number of terms relative to the orthogonal set of eigenfunctions. The computational experiment confirms the robustness of the method.

Keywords: Multi stepped membrane, equations of transverse vibration, Green's function, analytical method, simulation.

1 Introduction

In recent years, together with the improvements in technology and science, many researchers have attempted to analyze vibrations in composite structures like non-homogeneous membranes with continuously varying density or stepped density (see, for example [Spence and Horgan (1983); Masad (1996); Laura, Rossi and Gutierrez (1997); Laura, Bambill and Gutierrez (1997); Bambill, Gutierrez, Laura and Jederlinic (1997); Laura, Rossit and Malfa (1998); Laura, Rossit and Malfa (1999); Wang (1998); Buchanan and Jr (1999); Jabareen and Eisenberger (2001); Kang and Lee (2002); Kang (2004); Filipich and Rosales (2007)]).

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Most of investigations dealt with the natural frequencies and mode shapes for the non-homogeneous membranes of different shapes: Spence and Horgan [Spence and Horgan (1983)] presented the lower and upper bounds for the natural frequencies of vibration of a circular membrane with stepped radial density. Masad [Masad (1996)] used numerical integration and a perturbation method to obtain the natural frequencies of a non-homogeneous rectangular membrane with linearly varying density. Kang investigated the natural frequencies and mode shapes of the composite rectangular membranes with oblique and bent interfaces in his works [Kang and Lee (2002); Kang (2004)] respectively. Filipich and Rosales [Filipich and Rosales (2007)] found the natural frequencies and mode shapes of the composite membranes.

The Green's functions are a very important mathematical tool to solve vibration problems related to different fields such as electromagnetic, elastic, acoustic etc. and it is a basic tool to formulate a boundary integral equation of the considered problem, so it is usually used in boundary element method [Chew (1990); Tewary (1995); Berger and Tewary (2001); Tewary (2004); Pan and Yuan (2000a); Rashed (2004); Ting (2005); Nakamura and Tanuma (1997); Pan and Yuan (2000b); Yang and Tewary (2008); Gu, Young and Fan (2009); Chen, Ke and Liao (2009); Yakhno (2011); Yakhno and Yaslan (2011); Yakhno and Ozdek (2012)]. The Green's functions are preferable since they provide simplification for modelling waves and give powerful computational advantages for engineers to overcome calculational difficulties. The Green's functions have been used to solve problems of the wave propagation in composite elastic materials. Especially, they are applicable for two dimensional problems of composite structures. For example, Kukla used the Green's function to obtain a frequency equation and exact solutions for the problems of the plate systems [Kukla (1996); Kukla (1999); Kukla and Szewczyk (2007)].

The time-dependent Green's functions of the vibration of elastic materials are defined by the partial differential equations with piecewise constant coefficients and the Dirac delta function as inhomogeneous term. Methods for constructing the time dependent-Green's functions for the multi stepped membrane have not been developed so far.

The purpose of the present paper is an approximate computation of the time dependent Green's function for the equations of the transverse vibration of a multi stepped membrane with a piecewise constant varying density and tension. For the computation of the Green's function we suggest a new analytical method. This method has the following steps. The first step is determination of the eigenvalues and eigenfunctions of an ordinary differential equation with boundary and matching conditions. These eigenfunctions form an orthogonal set. Second step is an approximation (regularization) of the Dirac delta function in the form of Fourier series with a finite number of terms, using this orthogonal set of the eigenfunctions. The third step is an approximate computation of the Green's function in the form of the Fourier series with a finite number of terms relative to the orthogonal set of eigenfunctions.

The paper is organized as follows. The equations of the transverse vibration of a multi stepped membrane are stated in Section 2. Section 3 describes the steps of the approximate computation of the time-dependent Green's function. The computational experiment is given in Section 4. The conclusion and appendix with technical details are at the end of the paper.

2 The time-dependent Green's function of the transverse vibration of a composite multi stepped membrane

2.1 Equations for the Green's function

Let b_1, b_2, ρ_i, T_i, ℓ_i, $i = 1,2,\ldots,N$ be given real numbers such that $0 = \ell_0 < \ell_1 < \ell_2 < \ldots < \ell_N = b_2$, $\rho_i > 0$, $T_i > 0$ and $\rho(x_2)$, $T(x_2)$ be given functions of the form

$$\rho(x_2) = \{\rho_i,\ x_2 \in (\ell_{i-1},\ell_i),\ i = 1,2,\ldots,N\}, \tag{1}$$

$$T(x_2) = \{T_i,\ x_2 \in (\ell_{i-1},\ell_i),\ i = 1,2,\ldots,N\}. \tag{2}$$

Let us consider a multi layered membrane which is located in a rectangular domain $\mathscr{D}$ of the form

$$\mathscr{D} = \{\mathbf{x} = (x_1,x_2)|\ x_1 \in [0,b_1], x_2 \in [0,\ell_1) \cup \ldots \cup (\ell_{N-1},b_2]\}.$$

Here, $\rho(x_2)$ and $T(x_2)$ are the density and the tension of this membrane, respectively.

The Green's function of the transverse vibration on a multi layered membrane is a generalized function $G(x_1,x_2,t;\mathbf{x}^0)$ satisfying for $(x_1,x_2) \in (0,b_1) \times (0,\ell_1) \cup \ldots \cup (\ell_{N-1},b_2)$, $t \in \mathbf{R}$ the following equations

$$\rho(x_2)\frac{\partial^2 G}{\partial t^2} = \frac{\partial}{\partial x_2}\left(T(x_2)\frac{\partial G}{\partial x_2}\right) + T(x_2)\frac{\partial^2 G}{\partial x_1^2} + \delta(\mathbf{x}-\mathbf{x}^0)\delta(t), \tag{3}$$

$$G(x_1,x_2,t;\mathbf{x}^0)|_{t<0} = 0, \tag{4}$$

and the boundary and matching conditions

$$G(0,x_2,t;\mathbf{x}^0)=0, \quad G(b_1,x_2,t;\mathbf{x}^0)=0, \tag{5}$$

$$G(x_1,0,t;\mathbf{x}^0)=0, \quad G(x_1,b_2,t;\mathbf{x}^0)=0, \tag{6}$$

$$G(x_1,x_2,t;\mathbf{x}^0)\Big|_{x_2=\ell_i-0} = G(x_1,x_2,t;\mathbf{x}^0)\Big|_{x_2=\ell_i+0}, \tag{7}$$

$$T(x_2)\frac{\partial G}{\partial x_2}(x_1,x_2,t;\mathbf{x}^0)\Big|_{x_2=\ell_i-0} = T(x_2)\frac{\partial G}{\partial x_2}(x_1,x_2,t;\mathbf{x}^0)\Big|_{x_2=\ell_i+0}, \tag{8}$$

where $i=1,2,\ldots,N-1$; $\mathbf{x}^0=(x_1^0,x_2^0)\in(0,b_1)\times(0,\ell_1)\cup\ldots\cup(\ell_{N-1},b_2)$ is a fixed point; $\delta(\mathbf{x}-\mathbf{x}^0)=\delta(x_1-x_1^0,x_2-x_2^0)$ is the Dirac delta function concentrated at $\mathbf{x}^0$; $\delta(t)$ is the Dirac delta function concentrated at $t=0$.

Using the technique of the generalized functions (see, for example [Vladimirov (1971)]) the following remark holds.

Remark 1. Let $(x_1^0,x_2^0)\in(0,b_1)\times(0,\ell_1)\cup\ldots\cup(\ell_{N-1},b_2)$ be a fixed point, $\Theta(t)$ be the Heaviside function ($\Theta(t)=1$ for $t\geq 0$; $\Theta(t)=0$ for $t<0$) and $g(x_1,x_2,t;\mathbf{x}^0)$ be a generalized function satisfying for $(x_1,x_2)\in(0,b_1)\times(0,\ell_1)\cup\ldots\cup(\ell_{N-1},b_2)$, $t\in\mathbf{R}$ the following equation

$$\rho(x_2)\frac{\partial^2 g}{\partial t^2} = \frac{\partial}{\partial x_2}\left(T(x_2)\frac{\partial g}{\partial x_2}\right) + T(x_2)\frac{\partial^2 g}{\partial x_1^2}, \tag{9}$$

and conditions

$$g(x_1,x_2,0,\mathbf{x}^0)=0, \quad \frac{\partial g}{\partial t}(x_1,x_2,0,\mathbf{x}^0)=\frac{1}{\rho(x_2)}\delta(\mathbf{x}-\mathbf{x}^0), \tag{10}$$

$$g(0,x_2,t;\mathbf{x}^0)=0, \quad g(b_1,x_2,t;\mathbf{x}^0)=0, \tag{11}$$

$$g(x_1,0,t;\mathbf{x}^0)=0, \quad g(x_1,b_2,t;\mathbf{x}^0)=0, \tag{12}$$

$$g(x_1,x_2,t;\mathbf{x}^0)\Big|_{x_2=\ell_i-0} = g(x_1,x_2,t;\mathbf{x}^0)\Big|_{x_2=\ell_i+0}, \tag{13}$$

$$\frac{\partial g}{\partial x_2}(x_1,x_2,t;\mathbf{x}^0)\Big|_{x_2=\ell_i-0} = \beta_i\frac{\partial g}{\partial x_2}(x_1,x_2,t;\mathbf{x}^0)\Big|_{x_2=\ell_i+0}, \ \beta=\frac{T_{i+1}}{T_i}, \tag{14}$$

where $i=1,2,\ldots,N-1$. Then $G(x_1,x_2,t;\mathbf{x}^0)=\Theta(t)g(x_1,x_2,t;\mathbf{x}^0)$ is a generalized function satisfying $(3)-(8)$.

Therefore, to determine the Green's function $G(x_1,x_2,t;\mathbf{x}^0)$ it is sufficient to find a generalized function $g(x_1,x_2,t;\mathbf{x}^0)$ satisfying (9)-(14).

3 Approximate computation of a solution of (9)-(14)

3.1 The first step: Solving eigenvalue-eigenfunction problem in a multi layered rectangle

Let b_1, b_2 and ℓ_i, $i = 1, 2, \ldots, N$ be fixed positive numbers, $\rho(x_2)$, $T(x_2)$ be given functions of the form $(1), (2)$, respectively and let $d(x_2)$ be defined as a piecewise constant function of the form

$$d(x_2) = \{d_i = \rho_i / T_i,\ x_2 \in (\ell_{i-1}, \ell_i),\ i = 1, 2, \ldots, N\}\ . \tag{15}$$

Let us consider the following partial differential equation

$$\frac{\partial^2 V}{\partial x_1^2} + \frac{\partial^2 V}{\partial x_2^2} + \lambda d(x_2) V = 0, \quad x_1 \in (0, b_1),\ x_2 \in (0, \ell_1) \cup \ldots \cup (\ell_{N-1}, b_2), \tag{16}$$

subject to the boundary conditions

$$V(0, x_2) = 0, \quad V(b_1, x_2) = 0, \tag{17}$$

$$V(x_1, 0) = 0, \quad V(x_1, b_2) = 0, \tag{18}$$

and the matching conditions

$$V(x_1, \ell_i - 0) = V(x_1, \ell_i + 0), \tag{19}$$

$$\frac{\partial V}{\partial x_2}(x_1, \ell_i - 0) = \beta_i \frac{\partial V}{\partial x_2}(x_1, \ell_i + 0),\ \beta_i = \frac{T_{i+1}}{T_i},\ i = 1, 2, \ldots, N-1, \tag{20}$$

where λ is a parameter.

We note that a number λ for which there exists a nonzero function $V(x_1, x_2)$ satisfying $(16) - (20)$ is called an eigenvalue and a nonzero solution $V(x_1, x_2)$ of $(16) - (20)$ for this eigenvalue is called an eigenfunction.

Let

$$\xi_k = \left(\frac{k\pi}{b_1}\right)^2, \quad U_k(x_1) = \sqrt{\frac{2}{b_1}} \sin(\sqrt{\xi_k} x_1),\ k = 1, 2, \ldots \tag{21}$$

be eigenvalues and corresponding to them eigenfunctions of the following problem

$$U''(x_1) + \xi U(x_1) = 0, \quad x_1 \in (0, b_1), \tag{22}$$

$$U(0) = 0, \quad U(b_1) = 0. \tag{23}$$

For each fixed $k = 1, 2, \ldots$, a non-zero solution of $(16) - (20)$ can be found in the form

$$V(x_1, x_2) = U_k(x_1) Z(x_2). \tag{24}$$

Substituting (24) into (16) and using $(22),(23)$ we find

$$\frac{1}{d(x_2)}Z''(x_2)+\left(\lambda-\frac{\xi_k}{d(x_2)}\right)Z(x_2)=0,\quad x_2\in(0,\ell_1)\cup\ldots\cup(\ell_{N-1},b_2), \tag{25}$$

$$Z(0)=0,\quad Z(b_2)=0, \tag{26}$$

$$Z(\ell_i-0)=Z(\ell_i+0),\ Z'(\ell_i-0)=\beta_i Z'(\ell_i+0),\ \beta_i=\frac{T_{i+1}}{T_i}. \tag{27}$$

Therefore finding eigenvalues and eigenfunctions of $(16)-(20)$ is reducible to the construction of the eigenvalues and corresponding to them eigenfunctions of $(25)-(27)$.

Remark 2. The operator $L_k=\frac{1}{d(x_2)}(-\frac{d^2}{dx_2^2}+\xi_k)$ is positive-definite and symmetric for all positive ξ_k, $k=1,2,\ldots$ (see, for example , [Yakhno and Ozdek (2012)]). Using the general theory of symmetric and positive definite operators (see, for example [Vladimirov (1971)]) we find that all eigenvalues λ of $(25)-(27)$ are real and positive.

3.1.1 Construction of eigenvalues and eigenfunctions of $(25)-(27)$

For each $k=1,2,\ldots$, we find a general solution of the ordinary differential equation (25) in the form

$$Z(x_2)=A_i\cos(\sqrt{d_i\eta^i}x_2)+B_i\sin(\sqrt{d_i\eta^i}x_2),\ \ x_2\in(\ell_{i-1},\ell_i),\ i=1,2,\ldots,N,$$

with arbitrary constants A_i, B_i, where $\eta^i=\lambda-\frac{\xi_k}{d_i}$, $i=1,2,\ldots,N$. We need to determine the constants A_i and B_i to satisfy $(26),(27)$. Setting $B_1=1$ and using the boundary and matching conditions (26), (27) we find $A_1=0$ and the algebraic system

$$\mathbf{Q}(\lambda)\cdot\mathbf{S}=\mathbf{0}, \tag{28}$$

where $\mathbf{0}$ is zero column vector, $\mathbf{S}$ is a column vector with the components 1, A_2, $B_2,\ldots,A_N,B_N$ and Q is a block matrix of the form

$$\mathbf{Q}(\lambda)=\begin{bmatrix} \mathbf{P}_1(\lambda) & \mathbf{R}_1(\lambda) & 0 & 0 & 0 & 0 & 0\\ 0 & \mathbf{P_2}(\lambda) & \mathbf{R}_2(\lambda) & 0 & 0 & 0 & 0\\ 0 & 0 & \mathbf{P_3}(\lambda) & \mathbf{R}_3(\lambda) & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \mathbf{P}_{N-2}(\lambda) & \mathbf{R}_{N-2}(\lambda) & 0\\ 0 & 0 & 0 & 0 & 0 & \mathbf{P}_{N-1}(\lambda) & \mathbf{R}_{N-1}(\lambda)\\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{P}_N(\lambda) \end{bmatrix}, \tag{29}$$

where $\mathbf{P}_r(\lambda)$, $\mathbf{R}_j(\lambda)$ are the submatrices defined by

$$\mathbf{P}_1(\lambda) = \begin{bmatrix} \sin(\sqrt{d_1\eta_1}\ell_1) \\ \sqrt{d_1\eta_1}\cos(\sqrt{d_1\eta_1}\ell_1) \end{bmatrix}_{2\times 1};$$

$$\mathbf{P}_r(\lambda) = \begin{bmatrix} \cos(\sqrt{d_r\eta_r}\ell_r) & \sin(\sqrt{d_r\eta_r}\ell_r) \\ -\sqrt{d_r\eta_r}\sin(\sqrt{d_r\eta_r}\ell_r) & \sqrt{d_r\eta_r}\cos(\sqrt{d_r\eta_r}\ell_r) \end{bmatrix}_{2\times 2},$$

for $r = 2, \dots, N-1$;

$$\mathbf{P}_N(\lambda) = \begin{bmatrix} \cos(\sqrt{\eta_N d_N}\ell_N) & \sin(\sqrt{\eta_N d_N}\ell_N) \end{bmatrix}_{1\times 2};$$

$$\mathbf{R}_j(\lambda) = \begin{bmatrix} -\cos(\sqrt{d_{j+1}\eta_{j+1}}\ell_j) & -\sin(\sqrt{d_{j+1}\eta_{j+1}}\ell_j) \\ \beta_j\sqrt{d_{j+1}\eta_{j+1}}\sin(\sqrt{d_{j+1}\eta_{j+1}}\ell_j) & -\beta_j\sqrt{d_{j+1}\eta_{j+1}}\cos(\sqrt{d_{j+1}\eta_{j+1}}\ell_j) \end{bmatrix}_{2\times 2},$$

for $j = 1, \dots, N-1$.

Hence finding the nonzero functions $Z(x_2)$ satisfying (25) – (27) is reduced to determine the numbers $A_2, B_2, \dots, A_N, B_N$ satisfying (28). The equation (28) is the homogeneous linear algebraic system which is written in the matrix form. This system is consistent if and only if the determinant of the matrix $\mathbf{Q}(\lambda)$ is equal to zero. Moreover, the roots of $\det(\mathbf{Q}(\lambda)) = 0$ are eigenvalues of (25) – (27). The roots of $\det(\mathbf{Q}(\lambda)) = 0$ can be computed by MATLAB tools (see, Computational experiment).

Let us assume that roots λ_{km}, $k, m = 1, 2, \dots$ of $\det(\mathbf{Q}(\lambda)) = 0$ have been derived. Substituting $\lambda = \lambda_{km}$ and $A_1^{km} = 0$, $B_1^{km} = 1$ into (28), we find the following relations:

$$\mathbf{R}_1(\lambda_{km}) \begin{bmatrix} A_2^{km} \\ B_2^{km} \end{bmatrix} = -\mathbf{P}_1(\lambda_{km}), \tag{30}$$

$$\mathbf{R}_j(\lambda_{km}) \begin{bmatrix} A_{j+1}^{km} \\ B_{j+1}^{km} \end{bmatrix} = -\mathbf{P}_j(\lambda_{km}) \begin{bmatrix} A_j^{km} \\ B_j^{km} \end{bmatrix}, \; j = 2, \dots, N-1. \tag{31}$$

Notice that $\det \mathbf{R}_j(\lambda_{km}) = \beta_j\sqrt{\lambda_{km}d_{j+1}} \neq 0$ for all $j = 1, 2, \dots, N-1$. Then the values of A_2^{km}, B_2^{km}, $\dots$ A_N^{km}, B_N^{km} are found by the following recurrence relations:

$$\begin{bmatrix} A_2^{km} \\ B_2^{km} \end{bmatrix} = -\mathbf{R}_1^{-1}(\lambda_{km})\mathbf{P}_1(\lambda_{km}), \tag{32}$$

$$\begin{bmatrix} A_{j+1}^{km} \\ B_{j+1}^{km} \end{bmatrix} = -\mathbf{R}_j^{-1}(\lambda_{km})\mathbf{P}_j(\lambda_{km}) \begin{bmatrix} A_j^{km} \\ B_j^{km} \end{bmatrix}, \; j = 2, \dots, N-1. \tag{33}$$

Substituting the obtained values $A_2^{km}, B_2^{km}, \dots, A_N^{km}, B_N^{km}$ into (28), we find the following explicit formula for the eigenfunction $Z_{km}(x_2)$ of (25) – (27)

$$Z_{km}(x_2) = A_i^{km}\cos(\sqrt{\eta_{km}^i d_i x_2}) + B_i^{km}\sin(\sqrt{\eta_{km}^i d_i x_2}), \; x_2 \in (\ell_{i-1}, \ell_i), \tag{34}$$

where $i = 1, 2, \dots, N$; $k, m = 1, 2, \dots$.

Remark 3. Using (24) and (34), the eigenfunctions of (16) – (20) are constructed in the form

$$V_{km}(x_1,x_2) = U_k(x_1)Z_{km}(x_2),\;\; x_1 \in [0,b_1],\; x_2 \in [0,\ell_1)\cup\ldots\cup(\ell_{N-1},b_2], \tag{35}$$

where $U_k(x_1)$ has been defined by (21), $k,m = 1,2,\ldots$.

3.1.2 Orthogonality Property of Eigenfunctions

Let $\rho(x_2)$ be a function defined by (1), $V_{km}(x_1,x_2)$, $k,m = 1,2,\ldots$ be eigenfunctions of the form (35) and

$$X_{km}(x_1,x_2) = V_{km}(x_1,x_2)/\alpha_{km}, \quad k,m = 1,2,\ldots, \tag{36}$$

where $\alpha_{km}^2 = \int_0^{b_1}\int_0^{b_2}\rho(x_2)V_{km}^2(x_1,x_2)dx_1dx_2$. Then the set of the functions X_{km}, $k,m = 1,2\ldots$ and $X_{k'm'}$, $k',m' = 1,2\ldots$ is orthonormal, that is

$$\int_0^{b_1}\int_0^{b_2}\rho(x_2)X_{km}(x_1,x_2)X_{k'm'}(x_1,x_2)dx_2dx_1 = \begin{cases} 0, \text{ if } k \neq k' \text{ or } m \neq m\,,' \\ 1, \text{ if } k = k' \text{ and } m = m'\,. \end{cases} \tag{37}$$

The validity of the equation (37) can be found in Appendix A.

3.2 The second step: Approximation of the Dirac delta function

The Dirac delta function is very often used for modelling the point source in physics and engineering (see, [Vladimirov (1971); Yakhno (2008); Yakhno (2011); Yakhno and Ozdek (2012); Faydaoglu and Yakhno (2012)]). We note that the Dirac delta function does not have point-wise values and can not be drawn. For this reason, a regularization (approximation) of the Dirac delta function in the form of the classical function which has point-wise values, is usually used for drawing a graph of the Dirac delta function or for computations [Vladimirov (1971)].

Let M be a fixed natural number, then we consider a function $\delta_M(\mathbf{x},\mathbf{x^0})$ defined by the formula

$$\delta_M(\mathbf{x},\mathbf{x^0}) = \sum_{k=1}^{M}\sum_{m=1}^{M}\rho(x_2^0)X_{km}(x_1^0,x_2^0)X_{km}(x_1,x_2). \tag{38}$$

Using the formula (38), we have computed values of $\delta_M(\mathbf{x},\mathbf{x^0})$ for different M. The result of this computation for $(x_1^0,x_2^0) = (\frac{3}{2},\frac{7}{2})$, $(x_1,x_2) \in (0,5]\times[0,\frac{5}{2})\cup(\frac{5}{2},5]$ and the material properties

$$\rho_1 = 1400kg/m^3,\;\; \rho_2 = 2400kg/m^3,\; T_1 = 0.5\times10^2N/m,\;\; T_2 = 6\times10^2N/m.$$

is presented in Fig 1,2.

We note that the values of the Dirac delta function have been interpreted as zero for all points except (x_1^0,x_2^0). The value at (x_1^0,x_2^0) is $+\infty$. The result of the computation indicates that as we increase M, the values of $\delta_M(\mathbf{x},\mathbf{x^0})$, near (x_1^0,x_2^0), are increasing and values at

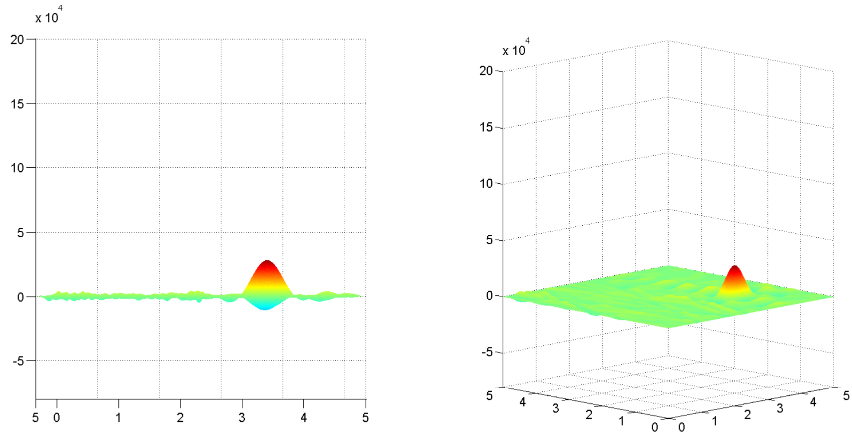

Figure 1: Screenshot of $\delta_M(\mathbf{x},\mathbf{x^0})$ for $M=15$; $x_1^0=\frac{3}{2}$; $x_2^0=\frac{7}{2}$.

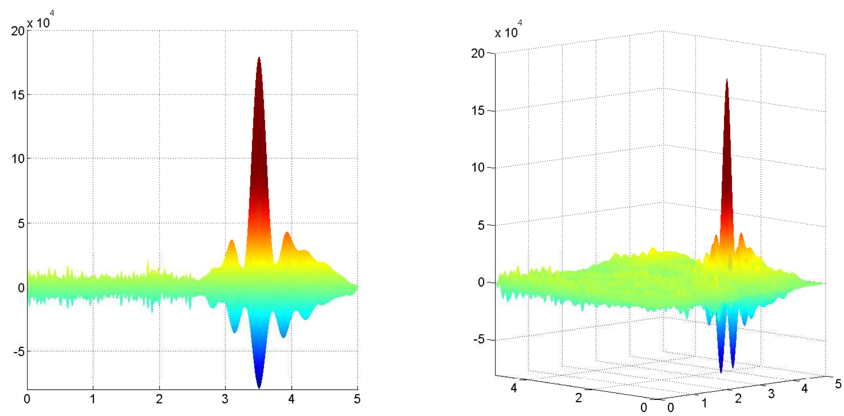

Figure 2: Screenshot of $\delta_M(\mathbf{x},\mathbf{x^0})$ for $M=35$; $x_1^0=\frac{3}{2}$; $x_2^0=\frac{7}{2}$.

other points (x_1,x_2) are vanishing. This means that $\delta_M(\mathbf{x},\mathbf{x^0})$ regularizes (approximates) the Dirac delta function $\delta(\mathbf{x}-\mathbf{x^0})$ and M is the parameter of the regularization.

Hence, $\delta_M(\mathbf{x},\mathbf{x^0})$, defined by the formula (38), is useful for an approximate computation of the Green's function.

Let us consider (9) – (14) when $\delta(\mathbf{x}-\mathbf{x}^0)$ is replaced by $\delta_M(\mathbf{x},\mathbf{x}^0)$. We obtain the following problem of finding $g_M(x_1,x_2,t;\mathbf{x}^0)$ satisfying

$$\rho(x_2)\frac{\partial^2 g_M}{\partial t^2}=\frac{\partial}{\partial x_2}\left(T(x_2)\frac{\partial g_M}{\partial x_2}\right)+T(x_2)\frac{\partial^2 g_M}{\partial x_2^1}, \tag{39}$$

$(x_1,x_2) \in (0,b_1)\times(0,\ell_1)\cup\ldots\cup(\ell_{N-1},b_2),$

$$g_M(x_1,x_2,+0;\mathbf{x}^0) = 0, \quad \frac{\partial g_M}{\partial t}(x_1,x_2,+0;\mathbf{x}^0) = \frac{1}{\rho(x_2^0)}\delta_M(\mathbf{x},\mathbf{x}^0), \tag{40}$$

$$g_M(0,x_2,t;\mathbf{x}^0) = 0, \quad g_M(b_1,x_2,t;x^0) = 0, \tag{41}$$

$$g_M(x_1,0,t;\mathbf{x}^0) = 0, \quad g_M(x_1,b_2,t;x^0) = 0, \tag{42}$$

$$g_M(x_1,\ell_i-0,t;\mathbf{x}^0) = g_M(x_1,\ell_i+0,t;\mathbf{x}^0), \tag{43}$$

$$\frac{\partial g_M}{\partial x_2}\Big|_{x_2=\ell_i-0} = \beta_i \frac{\partial g_M}{\partial x_2}\Big|_{x_2=\ell_i+0}, \ \beta_i = \frac{T_{i+1}}{T_i}, \ i = 1,2,\ldots,N-1, \tag{44}$$

where $\mathbf{x}^0 = (x_1^0,x_2^0) \in (0,b_1)\times(0,\ell_1)\cup\ldots\cup(\ell_{N-1},b_2)$.

3.3 The third step: Computation of a solution $g_M(x_1,x_2,t;\mathbf{x}^0)$ of $(39)-(44)$

We find a solution of $(39)-(44)$ in the form

$$g_M(x_1,x_2,t;\mathbf{x}^0) = \sum_{k=1}^{M}\sum_{m=1}^{M} H_{km}(t;\mathbf{x}^0)X_{km}(x_1,x_2), \tag{45}$$

where $H_{km}(t;\mathbf{x}^0)$ are unknown functions and $X_{km}(x_1,x_2)$ are eigenfunctions for all $k,m = 1,2,\ldots$ for $(x_1,x_2) \in [0,b_1)\times[0,\ell_1)\cup\ldots\cup(\ell_{N-1},b_2]$, $t > 0$. Substituting (45) into $(39)-(40)$ and using the orthogonality Property 3.1.2 we get for each $k,m = 1,2,\ldots,N$ the following ordinary differential equation with initial data

$$H''_{km}(t;\mathbf{x}^0) + \lambda_{km}H_{km}(t;\mathbf{x}^0) = 0, \tag{46}$$

$$H_{km}(0;\mathbf{x}^0) = 0, \quad H'_{km}(0;\mathbf{x}^0) = X_{km}(x_1^0,x_2^0). \tag{47}$$

The solution of $(46)-(47)$ is given by

$$H_{km}(t;\mathbf{x}^0) = \frac{X_{km}(x_1^0,x_2^0)}{\sqrt{\lambda_{km}}}\sin(\sqrt{\lambda_{km}}t),$$

for $k,m = 1,2,\ldots,M$. As a result of it, a solution of $(39)-(43)$ is found by

$$g_M(x_1,x_2,t;\mathbf{x}^0) = \sum_{k=1}^{M}\sum_{m=1}^{M} \frac{X_{km}(x_1^0,x_2^0)}{\sqrt{\lambda_{km}}}\sin(\sqrt{\lambda_{km}}t)X_{km}(x_1,x_2). \tag{48}$$

As a last step, using Remark 1 and above mentioned reasonings, we declare the function $G_M(x_1,x_2,t;\mathbf{x}^0) = \Theta(t)g_M(x_1,x_2,t;\mathbf{x}^0)$ as a regularized (approximate) Green's function of the transverse vibration of the multi-layered membrane. The number M is the parameter of the regularization.

4 Computational experiment

For the computational experiment, a two layered membrane, located in a rectangular domain

$\mathscr{D} = \{\mathbf{x} = (x_1, x_2) | \, x_1 \in [0,5], x_2 \in [0,(5/2)) \cup ((5/2),5]\}$

has been taken. The density and the tension of the first membrane are

$\rho_1 = 1400 kg/m^3, \; T_1 = 0.5 \times 10^2 N/m.$

The density and the tension of the second membrane are

$\rho_2 = 2400 kg/m^3, \; T_2 = 6 \times 10^2 N/m.$

Notice that in this example $N = 2$ and $d(x) = d_1$ for $x \in [0, \ell_1)$; $d(x) = d_2$ for $x \in (\ell_1, \ell_2]$.

The main goal of this experiment is the approximate computation of the Green's function for the transverse vibration of the two stepped membrane.

In the first step, we construct eigenvalues and corresponding to them eigenfunctions of (16) – (20) using the technique of Section 3.1.1. For this, we substitute $N = 2$ into the algebraic system (28), then the matrix $\mathbf{Q}(\lambda)$ takes the form

$$\mathbf{Q_k}(\lambda) = \begin{bmatrix} \sin(\sqrt{\eta_k^1 d_1}\ell_1) & -\cos(\sqrt{\eta_k^2 d_2}\ell_1) & -\sin(\sqrt{\eta_k^2 d_2}\ell_1) \\ \sqrt{\eta_k^1 d_1}\cos(\sqrt{\eta_k^1 d_1}\ell_1) & \beta\sqrt{\eta_k^2 d_2}\sin(\sqrt{\eta_k^2 d_2}\ell_1) & -\beta\sqrt{\eta_k^2 d_2}\cos(\sqrt{\eta_k^2 d_2}\ell_1) \\ 0 & \cos(\sqrt{\eta_k^2 d_2} b_2) & \sin(\sqrt{\eta_k^2 d_2} b_2) \end{bmatrix},$$

where $\eta_k^i = (\lambda - \frac{\xi_k}{d_i})$, $i = 1,2$; $k = 1,2,\ldots$.

For each $k = 1,2,\ldots,$ we have computed the roots $\lambda_{km}, \;\; m = 1,2,\ldots$ of the equation $\det(\mathbf{Q_k}(\lambda)) = 0$ using MATLAB. These roots are eigenvalues of (16) – (20).

After computation of eigenvalues $\lambda_{km},\;$ the set of eigenfunctions $V_{km}(x_1,x_2)$ is constructed by the following formulas

$$V_{km}(x_1,x_2) = U_k(x_1) Z_{km}(x_2), \;\; x_1 \in [0,b_1], \; x_2 \in [0,\ell_1) \cup (\ell_1,\ell_2], \tag{49}$$

where $U_k(x_1)$ are given by (21) and $Z_{km}(x_2)$ is defined by

$$Z_{km}(x_2) = \begin{cases} \sin(\sqrt{\lambda_{km} d_1 - \xi_k} x_2) & x_2 \in [0,r), \\ \sin(\sqrt{\lambda_{km} d_1 - \xi_k}\ell_1)\cos(\sqrt{\lambda_{km} d_2 - \xi_k}(x_2 - \ell_1)) & \\ + \frac{\sqrt{\lambda_{km} d_1 - \xi_k}}{\beta\sqrt{\lambda_{km} d_2 - \xi_k}} \cos(\sqrt{\lambda_{km} d_1 - \xi_k}\ell_1)\sin(\sqrt{\lambda_{km} d_2 - \xi_k}(x_2 - \ell_1)), & x_2 \in (r, b_2]. \end{cases}$$

Here the values of d_1, d_2, β are defined by $d_i = \rho_i / T_i$, $i = 1,2$; $\beta = \frac{T_2}{T_1}$.

The orthonormal set of eigenfunctions is defined by (36).

The second step is an approximation (regularization) of the Dirac delta function. For the computational experiment we take $\mathbf{x}^0 = (\frac{3}{2}, \frac{7}{2})$ and the parameter M of the regularization (approximation) has been chosen by the following natural logic. Using the formula (38), $\delta_M(\mathbf{x}, \mathbf{x}^0)$ has been computed for the different values of $M = 5, 10, \ldots, 35$. The results of the computation are presented in Fig. $1, 2$ for $M = 15$ and $M = 35$, respectively.

We note that the values of the Dirac delta function have been interpreted as zero for all points except (x_1^0, x_2^0). The value at (x_1^0, x_2^0) is $+\infty$. The result of the computation indicates that as we increase M, the values of $\delta_M(\mathbf{x}, \mathbf{x^0})$, when (x_1, x_2) is near (x_1^0, x_2^0), are increasing and values at other points (x_1, x_2) are vanishing. This means that $\delta_M(\mathbf{x}, \mathbf{x^0})$ regularizes (approximates) the Dirac delta function $\delta(\mathbf{x} - \mathbf{x^0})$ and M is the parameter of the regularization. We have observed that there is not much difference between the figures of regularization of Dirac delta function $S_M(\mathbf{x}, \mathbf{x}^0)$ for the values $M = 35$ and for the values of M greater than 35. So, we have chosen number $M = 35$ as an optimal value of the parameter of approximation.

In the third step, the problem $(39) - (44)$ has been solved for $M = 35$, $b_1 = 5$, $\ell_1 = \frac{5}{2}$, $b_2 = \ell_2 = 5$, $N = 2$. The solution $g_M(x_1, x_2, t; x^0)$ of $(39) - (44)$ has been computed by formula (48). The values of the function $g_M(x_1, x_2, t; x^0)$ $(M = 35)$ have been accepted as values of an approximate solution of $(9) - (14)$. This means

$$G_M(x, t; x^0) \quad = \quad \Theta(t) \sum_{k=1}^{M} \sum_{m=1}^{M} \frac{X_{km}(x_1^0, x_2^0)}{\sqrt{\lambda_{km}}} \sin(\sqrt{\lambda_{km}} t) X_{km}(x_1, x_2)$$

is an approximate Green's function for the equations of the transverse vibration $(3) - (8)$, respectively.

Using the formula for $G_M(x, t; x^0)$ the approximate computation of the Green's function has been made. The results of this computation are presented in Figs.3-7. The horizontal axis of the lower picture in Fig.3 is x_1-axis and the vertical one is the magnitude of $G_M(x, t; x^0)$. The upper picture in Fig.3 is the plan of the surface $z = G_M(x, t; x^0)$ (view from the top of $z-$axis). Here the horizontal axis is x_1-axis and vertical one is x_2-axis. The plan of the surface $z = G_M(x, t; x^0)$ for the different time t is presented in Figs.4-7, where the vertical axis is x_2-axis and horizontal one is x_1-axis. The different colors correspond to different values of $G_M(x, t; x^0)$. The scale of the values of $G_M(x, t; x^0)$ and corresponding to them colors is marked out in Fig.3. The graphs of $G_M(x, t; x^0)$ at time $t = 0.1$ (Fig.3) are similar to graphs of the initial excitation at $t = 0$, which is modeled by the function $\delta_M(\mathbf{x}, \mathbf{x}^0)$ (Fig.2). We note that from physical point of view $G_M(x, t; x^0)$ describes the wave propagation in the considered membrane (the vibration of the membrane) arising from the pulse source modeled by $\delta_M(\mathbf{x}, \mathbf{x}^0) \cdot \delta(t)$. Figs.4-7 demonstrate the process of this wave propagation. For example, we see the wave front at $t = 1$ in the form of the circle in Fig.4. The further propagation of the wave, arising from the pulse source modeled by $\delta_M(\mathbf{x}, \mathbf{x}^0) \cdot \delta(t)$ $(M = 35)$, is presented in Figs.5-7. The wave front crossed the line of the interface of two materials at time $t = 3$ (Fig.5). We see the fronts of the transmitted wave (indicated by arrow 2) and the reflected wave (indicated by arrow 1) in Fig.5. In Fig.6, it is clearly seen that the reflected and the transmitted waves are moving in the opposite directions. Moreover, the front of the transmitted wave has touched the boundaries of the

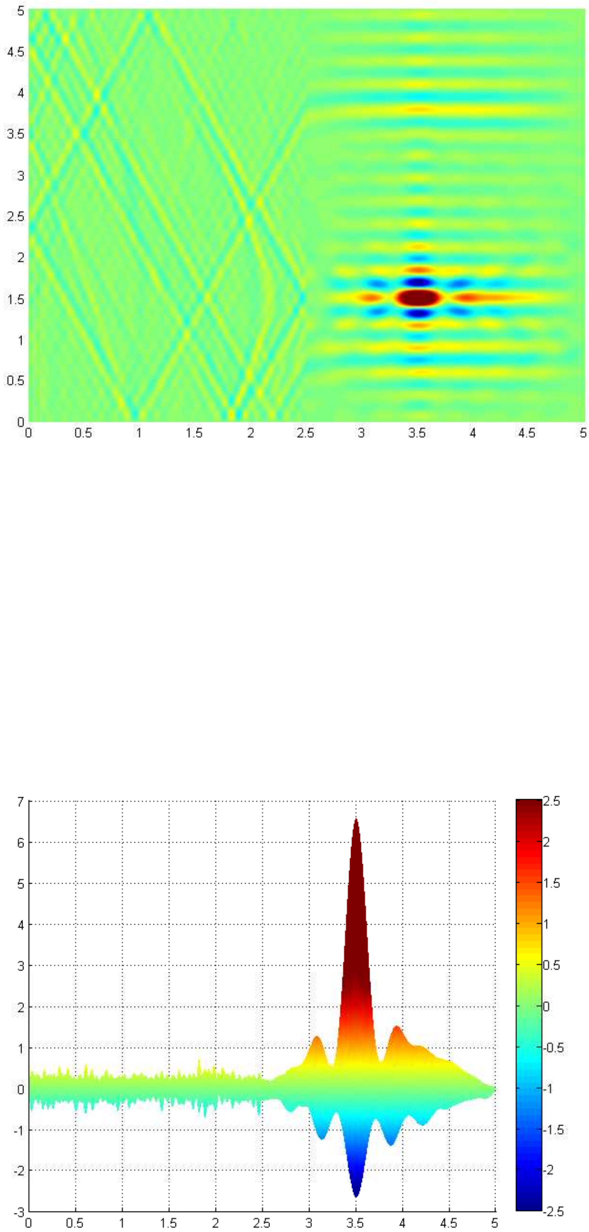

Figure 3: Screenshots of the Green's function $G(x_1, x_2, x_1^0, x_2^0, t)$ for $t = 0.1$; $x_1^0 = \frac{3}{2}$; $x_2^0 = \frac{5}{2}$.

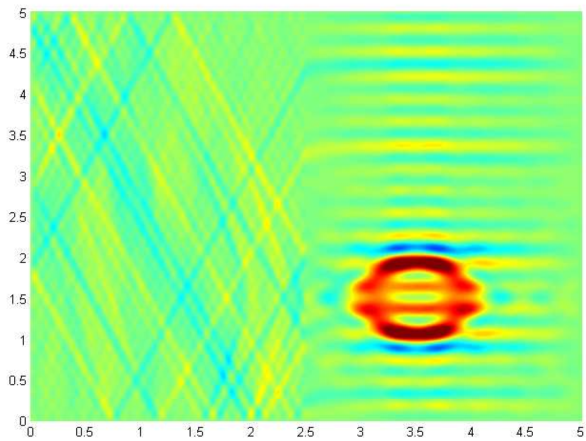

Figure 4: Screenshots of the Green's function $G(x_1, x_2, x_1^0, x_2^0, t)$ for $t = 1$.

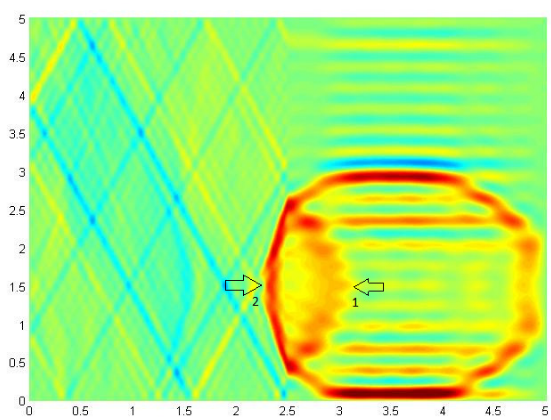

Figure 5: Screenshots of the Green's function $G(x_1, x_2, x_1^0, x_2^0, t)$ for $t = 3$.

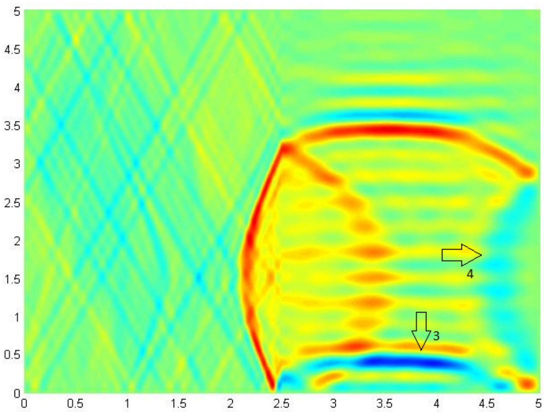

Figure 6: Screenshots of the Green's function $G(x_1,x_2,x_1^0,x_2^0,t)$ for $t=4$.

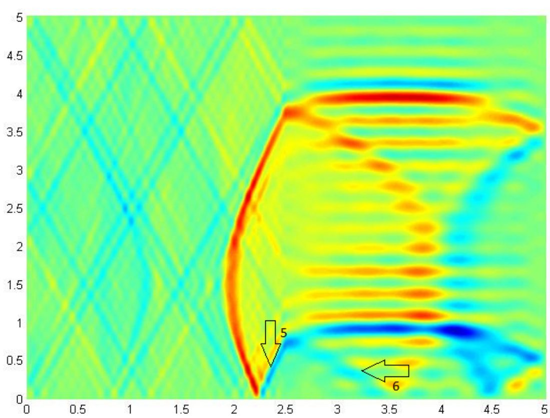

Figure 7: Screenshots of the Green's function $G(x_1,x_2,x_1^0,x_2^0,t)$ for $t=5$.

membrane at right side ($x_1 = 5$) and the bottom ($x_2 = 0$). As a result, the reflected wave fronts have arisen. In Fig.6 the fronts of reflected waves from the boundaries $x_1 = 5$ and $x_2 = 0$ are marked by arrows 4 and 3 respectively. The reflected wave from the boundary $x_2 = 0$ has gone through the line of the interface of two materials at time $t = 5$. In Fig.7, the arrow 5 indicates the transmitted wave front while the arrow 6 indicates the reflected wave front.

5 Conclusion

The method of the approximate computation of the time-dependent Green's function for the equations of the transverse vibration of a multi stepped membrane is suggested. This method is based on determination of the eigenvalues and orthogonal set of the eigenfunctions; regularization of the Dirac delta function in the form of the Fourier series with a finite number of terms; expansion of the unknown Green's function in the form of Fourier series with unknown coefficients and computation of a finite number of unknown Fourier coefficients. Computational experiment confirms the robustness of the method for the approximate computation of the Dirac delta function and Green's function. Using visualization of the computed values of the Green's function we can observe in details the wave propagation in the composite rectangular membrane, in particular the movement of the wave fronts.

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Appendix A:

Let $x_1 \in [0,b_1]$ and $x_2 \in [\ell_0,\ell_1) \cup (\ell_1,\ell_2) \cup \ldots \cup (\ell_{N-1},\ell_N]$ be variables ($\ell_0 = 0$, $\ell_N = b_2$); $\rho(x_2)$, $T(x_2)$, $d(x_2)$ be functions defined by (1), (2), (15); λ_{km}, $\lambda_{k'm'}$ be eigenvalues and $V_{km}(x_1,x_2), V_{k'm'}(x_1,x_2)$ be corresponding to them eigenfunctions of **EEP** (16) – (20) for $k,m = 1,2,\ldots;\ \ k',m' = 1,2,\ldots,$ respectively. Then multiplying the partial differential equation (16) by $V_{km}(x_1,x_2)$, $V_{k'm'}(x_1,x_2)$, respectively, we find

$$\frac{1}{d(x_2)}\frac{\partial^2 V_{km}}{\partial x_1^2}V_{k'm'} + \frac{1}{d(x_2)}\frac{\partial^2 V_{km}}{\partial x_2^2}V_{k'm'} + \lambda_{km}V_{km}V_{k'm'} = 0,$$

$$\frac{1}{d(x_2)}\frac{\partial^2 V_{k'm'}}{\partial x_1^2}V_{km} + \frac{1}{d(x_2)}\frac{\partial^2 V_{k'm'}}{\partial x_2^2}V_{km} + \lambda_{k'm'}V_{k'm'}V_{k'm'} = 0,$$

Subtracting gives

$$\frac{1}{d(x_2)}\left[\frac{\partial}{\partial x_1}(\frac{\partial V_{km}}{\partial x_1}V_{k'm'}-\frac{\partial V_{k'm'}}{\partial x_1}V_{km})+\frac{\partial}{\partial x_2}(\frac{\partial V_{km}}{\partial x_2}V_{k'm'}-\frac{\partial V_{k'm'}}{\partial x_2}V_{km})\right]$$
$$+(\lambda_{km}-\lambda_{k'm'})V_{km}V_{k'm'}=0.$$

Integrating the obtained relation with respect to x_1 from 0 to b_1 and with respect to x_2 from ℓ_{i-1} to ℓ_i we get

$$\int_{\ell_{i-1}}^{\ell_i}\frac{1}{d_i}\left(\frac{\partial V_{km}}{\partial x_1}V_{k'm'}-\frac{\partial V_{k'm'}}{\partial x_1}V_{km}\right)\Big|_0^{b_1}dx_2+\int_0^{b_1}\frac{1}{d_i}\left(\frac{\partial V_{km}}{\partial x_2}V_{k'm'}-\frac{\partial V_{k'm'}}{\partial x_2}V_{km}\right)\Big|_{\ell_{i-1}}^{\ell_i}dx_1$$
$$+\int_0^{b_1}\int_{\ell_{i-1}}^{\ell_i}(\lambda_{km}-\lambda_{k'm'})V_{km}V_{k'm'}dx_2dx_1=0.$$

Multiplying the last relation by ρ_i and then summing with respect to i from $i=1$ to $i=N$, we find

$$(\lambda_{km}-\lambda_{k'm'})\int_0^{b_1}\left[\int_0^{\ell_1}\rho_1 V_{km}V_{k'm'}dx_2+\ldots+\int_{\ell_{N-1}}^{b_2}\rho_{N-1}V_{km}V_{k'm'}dx_2\right]dx_1=0.$$

Here we have used the boundary and matching conditions $(17)-(20)$.

As a result, we have

$$\int_0^{b_1}\int_0^{b_2}\rho(x_2)V_{km}(x_1,x_2)V_{k'm'}(x_1,x_2)dx=\begin{cases}0, \text{ if } k\neq k' \text{ or } m\neq m';\\ \alpha_{km}, \text{ if } k=k' \text{ and } m=m,\end{cases}$$

where $\alpha_{km}=\int_0^{b_1}\int_0^{b_2}\rho(x_2)V_{km}^2(x_1,x_2)dx_2dx_1$.