 CMC, vol.36, no.2, pp.177-201, 2013

Heat Conduction Analysis of Nonhomogeneous Functionally Graded Three-Layer Media

Chien-Ching Ma[1,2], **Yi-Tzu Chen**[2]

Abstract: Functionally graded material (FGM) is a particulate composite with continuously changing its thermal and mechanical properties in order to raise the bonding strength in the discrete composite made from different phases of material constituents. Furthermore, FGM is a potent tool to create an intermediate layer in metal–ceramic composites to avoid the properties discontinuities and reduce, thereby, the residual stresses. For the nonhomogeneous problem, the mathematical derivation is much complicated than the homogeneous case since the material properties vary with coordinate. To analyze the problem, the Fourier transform is applied and the general solution in transform domain is obtained. The inverse Fourier transform is performed to get the results in physical domain for temperature and heat fluxes. Numerical results for the full-field distributions of temperature and heat fluxes with different functionally graded parameters are presented. The continuous characteristics of the temperature and heat flux along the interface are emphasized and some interesting phenomena are presented in this study. The results show that all the fields (temperature and heat fluxes) are continuous at the interface if the conductivities are continuous at the interface. Moreover, the first derivatives of temperature and heat flux q_y are continuous at the interface.

Keywords: Functionally graded material; Full-field solution; Fourier transform; Heat flux; Interface; Three-layered medium

1 Introduction

In recent decades, in order to have high temperature resistance and strong mechanical properties simultaneously, composite materials are wildly used. The aircraft and aerospace industry and the computer circuit industry are interested in the possibility of materials that can withstand very high thermal gradients. This is normally

[1] Corresponding author. Tel.: +886-2-23659996; fax:+ 886-2-23631755. E-mail address: ccma@ntu.edu.tw

[2] Department of Mechanical Engineering, National Taiwan University, Taipei, Taiwan 10617, R.O.C.

achieved by using a ceramic layer connected with a metallic layer. In metal-ceramic composite, ceramics can suffer high-temperature environment, but it mismatches with the metal supplying high-toughness. Since homogeneous material properties are constant and discontinuous at the interface, cracks are often generated at the bonded interface. In order to avoid the discontinuity in material properties at the interface, the functionally graded materials (FGMs) are made in such a way that the material properties are continuous functions of the thickness coordinate, and have the best properties of both materials at the interface. Functionally graded materials have also drawn considerable attention in engine combustion chamber or nuclear fusion reaction container to reduce the stress concentration or debonding at the interface. Therefore, FGM has been widely used in the junction and can significantly eliminate thermal residual stresses caused at the interface.

We must point out, however, that the dependences of material properties on the spatial coordinate involves the essential difficulty into mathematical treatment of the corresponding problems of mechanics for a nonhomogeneous solid. Such difficulty consists in the need to solve the partial differential equations with variable coefficients. This makes it impossible, except for a few particular cases, to solve the problems analytically. Most researchers analyzed the composition of FGM with three types, power-law, polynomial, and exponential functions, which are widely used due to the reason that these functions provide convenient process of theoretical investigation. In the literature for FGM with power-law function, Jabbari et al. (2002) provided an analytical solution for steady-state thermal stresses in a hollow thick cylinder made of power-law FGM. Jin and Paulino (2001) presented asymptotic analysis of a power-law FGM strip containing an edge crack under transient thermal loading condition. The second type of FGM for polynomial function were studied by Chiu and Erdogan (1999), Abu-Alshaikh and Köklüce (2006). Chiu and Erdogan (1999) assumed that the stiffness and density of the FGM slab vary continuously with arbitrary polynomial function, and stress wave with a rectangular pressure pulse in nickel-zirconia, and aluminum-silicon media with either free-free or fixed-free boundary conditions was analyzed in detail. Ma et al. (2012) analyzed the transient response in a functionally graded material (FGM) slab by Laplace transform technique and the numerical Laplace inversion (Durbin's formula (1974)) was used to calculate the dynamic behavior of the FGM slab. In addition, the FGM slab is approximated as a multilayered medium with homogeneous material in each layer, and the transient responses of FGM formulation and multilayered solution are discussed in detail. For the third type of FGM, i.e., exponential variation of material constants in Cartesian coordinates, it was widely used by many authors. Transient heat transfer in FGMs with an exponential spatial variation of material constants has been examined by Noda and Jin (1994). Erdogan and

Wu (1995) investigated the thermal stress problem of FGM with an exponential-form for an embedded or a surface crack. Jin and Batra (1996) analyzed thermal stresses and the stress intensity factor in an edge-cracked strip of an FGM subjected to sudden cooling at the crack surface. They assumed that the shear modulus decreased hyperbolically from the surface and the thermal conductivity varied exponentially. Ma and Lee (2009a, 2009b) and Lee and Ma (2010) derived analytical full-field solutions for two-dimensional problem of bimaterials and layered half-plane for functionally graded magnetoelectroelastic materials. Tokovyy and Ma (2008, 2009) provided an analytical approach to solve plane thermoelasticity problems for inhomogeneous functionally graded hollow cylinders and half-planes. The material properties were assumed to be arbitrary functions. Chen et al. (2004) investigated the free vibration of an arbitrarily thick orthotropic piezoelectric hollow cylinder with a functionally graded property along the thickness direction and ?lled with a non-viscous compressible fluid medium. Wu et al. (2008) presented an overview of various three-dimensional analytical approaches for the analysis of multilayered and functionally graded (FG) piezoelectric plates and shells. Dondero et al. (2011) proposed a numerical methodology for the design of random micro-heterogeneous materials with functionally graded effective thermal conductivities. This methodology was applied for the design of foam-like microstructures consisting of random distributions of circular insulated holes. Dong and Atluri (2012) developed T-Trefftz Voronoi Cell Finite Elements for micromechanical modeling of composite and porous materials. This class of elements is very useful for micromechanical modeling of composite and porous materials.

Because of the mathematical difficulties, the analytical solution for the functionally graded media subjected to a heat source has not yet been obtained for the FGMs with multilayered media. It is necessary to provide an effective method to understand the behaviors of the nonhomogeneous materials. The general methodology presented in this study could be useful to the analysis and design of layered composites of nonhomogeneous materials. This paper presents the theoretical results of two-dimensional problem for functionally graded three-layer media. From the Fourier transform method, the full-field solutions of temperature and heat flux are obtained in explicit forms. Numerical calculations based on the analytical solutions are performed and are discussed in detail. For the computational result, the full-field distributions of temperature and heat fluxes subjected to one or two heat sources are presented with different functionally graded factors. Three different cases of functionally graded factors are used to investigate the interesting phenomenon for the field quantities near the interface. It is noted that the temperature and heat flux fields along the interface for nonhomogeneous functionally graded materials are continuous if the conductivities are identical at the interface.

Furthermore, it is also proved that the contour curves for the temperature T and heat flux $q_y$at the interface have the same slopes (first derivative).

2 Governing Equation for Heat Conduction Problems and General Solutions

In this study, the two-dimensional steady-state heat conduction problem of functionally graded materials is considered. Assuming that the thermal properties depend on the y-axis, the governing equation in the absence of heat source is given by

$$\frac{\partial}{\partial x}\left(k(y)\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k(y)\frac{\partial T}{\partial y}\right)=0 \tag{1}$$

where $k,\ T$ are conductivity and temperature, respectively. From Fourier's law of conduction, the relationships between temperature and heat fluxes are

$$q_x=-k\frac{\partial T}{\partial x},\ q_y=-k\frac{\partial T}{\partial y}. \tag{2}$$

We assume that k varies along the $y$axis with an exponential form, i.e., $k=k(y)=\alpha\, e^{\beta y}$, where $\alpha$is a positive constant and β is the functionally graded factor which represents the degree of the material gradient in the y direction. Base on the assumption, the governing equation for functionally graded materials can be rewritten as

$$\frac{\partial^2 T}{\partial x^2}+\beta\frac{\partial T}{\partial y}+\frac{\partial^2 T}{\partial y^2}=0 \tag{3}$$

The governing equation presented in Eq. (3) is a second order PDE. We apply Fourier transform of the spatial coordinate x and the Fourier transform pairs of temperature $T(x,y)$ are defined as

$$\tilde{T}(\omega,y)=\int_{-\infty}^{\infty}T(x,y)e^{-i\omega x}dx \quad T(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{T}(\omega,y)e^{i\omega x}d\omega \tag{4}$$

where ω is the transform parameter and $i=\sqrt{-1}$. Then Eq. (3) becomes a second-order linear ordinary differential equation,

$$\frac{\partial^2\tilde{T}}{\partial y^2}+\beta\frac{\partial\tilde{T}}{\partial y}-\omega^2\tilde{T}=0 \tag{5}$$

Assume $\tilde{T}=e^{sy}$, and substitute into Eq. (5), then we have the characteristic equation

$$s^2+\beta s-\omega^2=0, \tag{6}$$

with roots

$$s_1 = \frac{-\beta + Q}{2}, \ s_2 = \frac{-\beta - Q}{2},$$

where

$$Q = \sqrt{\left(\frac{\beta}{2}\right)^2 + \omega^2}. \tag{7}$$

The general solution of temperature in the transform domain is represented as

$$\tilde{T} = c\, e^{s_1 y} + d\, e^{s_2 y} \tag{8}$$

where $c$and d are undetermined coefficients and can be obtained from boundary conditions. The general solutions of heat fluxes in the Fourier transform domain are expressed as

$$\begin{aligned} \tilde{q}_y(\omega, y) &= -k(y)\tfrac{\partial \tilde{T}}{\partial y} = -\alpha e^{\beta y}(s_1 c e^{s_1 y} + s_2 d e^{s_2 y}), \\ \tilde{q}_x(\omega, y) &= -k(y) i\omega \tilde{T} = -\alpha e^{\beta y} i\omega (c e^{s_1 y} + d e^{s_2 y}). \end{aligned} \tag{9}$$

The inverse Fourier transform is performed to get the solution in physical domain as follows:

$$q_x(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{q}_x(\omega, y) e^{i\omega x} d\omega, \ q_y(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{q}_y(\omega, y) e^{i\omega x} d\omega. \tag{10}$$

The heat conduction equation for functionally graded material is different from the classical heat conduction equation, so we can expect that the solution will be more complicated than the classical one.

3 The Applied Heat Source in the Middle Layer $(0 \le y_0 \le h_2)$

Consider a two-dimensional problem of functionally graded three-layer media subjected to a heat source q_0 at $(x_0,\ y_0)$ in material 2 as indicated in Figure 1. Generally, it is assumed that the thicknesses of the three layers are different, i.e., the thicknesses are $h_1,\ h_2,\ h_3$, and the conductivities are exponentially varying along y axis, i.e.,

$$k^{(j)} = \alpha^{(j)} e^{\beta^{(j)} y}, \quad j = 1,\ 2, 3. \tag{11}$$

The superscript j is employed to label material$j$, j=1, 2, 3. The governing equation for heat conduction problem in functionally graded materials is

$$\frac{\partial^2 T^{(j)}}{\partial x^2} + \beta^{(j)}\frac{\partial T^{(j)}}{\partial y} + \frac{\partial^2 T^{(j)}}{\partial y^2} = 0. \tag{12}$$

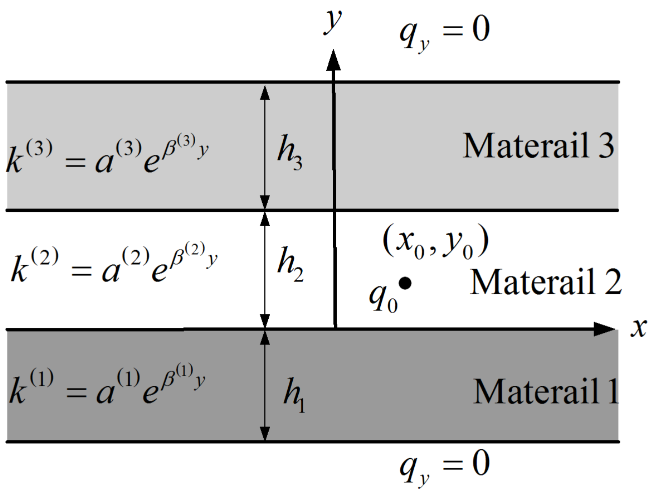

Figure 1: Configuration and coordinate system of functionally graded three-layer media subjected to a heat source in material 2.

The boundary conditions are

$$q_y^{(1)}\Big|_{y=-h_1} = 0, \quad q_y^{(3)}\Big|_{y=h_2+h_3} = 0. \tag{13}$$

And the continuity conditions at the interfaces are

$$\begin{aligned} T^{(2-)}\Big|_{y=0} &= T^{(1)}\Big|_{y=0}, \quad q_y^{(2-)}\Big|_{y=0} = q_y^{(1)}\Big|_{y=0}, \\ T^{(3)}\Big|_{y=h_2} &= T^{(2+)}\Big|_{y=h_2}, \quad q_y^{(3)}\Big|_{y=h_2} = q_y^{(2+)}\Big|_{y=h_2}. \end{aligned} \tag{14}$$

Near the heat source, the jump conditions are

$$T^{(2+)}\Big|_{y\to y_0^+} = T^{(2-)}\Big|_{y\to y_0^-}, \quad q_y^{(2+)}\Big|_{y\to y_0^+} - q_y^{(2-)}\Big|_{y\to y_0^-} = q_0\delta(x-x_0). \tag{15}$$

Applying the Fourier transformation on Eq. (12), we have

$$\frac{\partial^2 \tilde{T}^{(j)}}{\partial y^2} + \beta^{(j)} \frac{\partial \tilde{T}^{(j)}}{\partial y} - \omega^2 \tilde{T}^{(j)} = 0, \quad j = 1,\ 2, 3 \tag{16}$$

$$
\begin{aligned}
&\tilde{q}_y^{(1)}\Big|_{y=-h_1}=0, \quad \tilde{q}_y^{(3)}\Big|_{y=h_2+h_3}=0, \quad \tilde{T}^{(2-)}\Big|_{y=0}=\tilde{T}^{(1)}\Big|_{y=0}, \\
&\tilde{q}_y^{(2-)}\Big|_{y=0}=\tilde{q}_y^{(1)}\Big|_{y=0}, \quad \tilde{T}^{(3)}\Big|_{y=h_2}=\tilde{T}^{(2+)}\Big|_{y=h_2}, \quad \tilde{q}_y^{(3)}\Big|_{y=h_2}=\tilde{q}_y^{(2+)}\Big|_{y=h_2}, \\
&\tilde{T}^{(2+)}\Big|_{y\to y_0^+}=\tilde{T}^{(2-)}\Big|_{y\to y_0^-}, \quad \tilde{q}_y^{(2+)}\Big|_{y\to y_0^+}-\tilde{q}_y^{(2-)}\Big|_{y\to y_0^-}=q_0e^{-i\omega x_0}.
\end{aligned}
\tag{17}
$$

The general solution of Eq. (16) is

$$
\tilde{T}^{(j)}=c^{(j)}e^{s_1^{(j)}y}+d^{(j)}e^{s_2^{(j)}y}, \tag{18}
$$

where $c^{(j)}$ and $d^{(j)}$ are undetermined coefficients and can be obtained from the conditions indicated in Eq. (17) . Substituting Eq. (18) into Eq. (17), we can rewrite the conditions as following form:

$$
\begin{bmatrix}
1 & 1 & -1 & -1 & 0 & 0l & 0 & 0 \\
s_1^{(1)} & s_2^{(1)} & \frac{-\alpha^{(2)}s_1^{(2)}}{\alpha^{(1)}} & \frac{-\alpha^{(2)}s_2^{(2)}}{\alpha^{(1)}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & e^{s_1^{(2)}h_2} & e^{s_2^{(2)}h_2} & -e^{s_1^{(3)}h_2} & -e^{s_2^{(3)}h_2} \\
0 & 0 & 0 & 0 & s_1^{(2)}e^{s_1^{(2)}h_2} & s_2^{(2)}e^{s_2^{(2)}h_2} & A & B \\
0 & 0 & e^{s_1^{(2)}y_0} & e^{s_2^{(2)}y_0} & -e^{s_1^{(2)}y_0} & -e^{s_2^{(2)}y_0} & 0 & 0 \\
0 & 0 & s_1^{(2)}e^{s_1^{(2)}y_0} & s_2^{(2)}e^{s_2^{(2)}y_0} & -s_1^{(2)}e^{s_1^{(2)}y_0} & -s_2^{(2)}e^{s_2^{(2)}y_0} & 0 & 0 \\
s_1^{(1)}e^{-s_1^{(1)}h_1} & s_2^{(1)}e^{-s_2^{(1)}h_1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & s_1^{(3)}e^{s_1^{(3)}(h_2+h_3)} & s_2^{(3)}e^{s_2^{(3)}(h_2+h_3)}
\end{bmatrix}
\begin{bmatrix}
c^{(1)} \\ d^{(1)} \\ c^{(2-)} \\ d^{(2-)} \\ c^{(2+)} \\ d^{(2+)} \\ c^{(3)} \\ d^{(3)}
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{q_0e^{-i\omega x_0}}{\alpha^{(2)}e^{\beta^{(2)}y_0}} \\ 0 \\ 0
\end{bmatrix}
\tag{19}
$$

where

$$
A=\frac{-\alpha^{(3)}e^{h_2(\beta^{(3)}-\beta^{(2)})}s_1^{(3)}e^{s_1^{(3)}h_2}}{\alpha^{(2)}}, \quad B=\frac{-\alpha^{(3)}e^{h_2(\beta^{(3)}-\beta^{(2)})}s_2^{(3)}e^{s_2^{(3)}h_2}}{\alpha^{(2)}}.
$$

Thus the unknowns can be determined as follows:

$$
c^{(1)}=-\frac{s_2^{(1)}}{s_1^{(1)}}e^{(s_1^{(1)}-s_2^{(1)})h_1}d^{(1)}, \quad d^{(1)}=\frac{-e^{h_2s_2^{(3)}}s_1^{(1)}L(y_0,\omega)}{s_1^{(3)}P(y_0,\omega)}d^{(3)},
$$

$$
c^{(2-)}=\frac{-\Omega_1s_1^{(1)}s_2^{(1)}\alpha^{(1)}-\Omega_2s_2^{(2)}\alpha^{(2)}}{s_1^{(1)}(s_1^{(2)}-s_2^{(2)})\alpha^{(2)}}d^{(1)}, \quad d^{(2-)}=\frac{\Omega_1s_1^{(1)}s_2^{(1)}\alpha^{(1)}+\Omega_2s_1^{(2)}\alpha^{(2)}}{s_1^{(1)}(s_1^{(2)}-s_2^{(2)})\alpha^{(2)}}d^{(1)},
$$

$$
c^{(2+)}=\frac{e^{-h_2(s_1^{(2)}-s_2^{(3)})}\left[-s_2^{(2)}\alpha^{(2)}\Omega_4+e^{h_2(-\beta^{(2)}+\beta^{(3)})}s_1^{(3)}s_2^{(3)}\alpha^{(3)}\Omega_3\right]}{s_1^{(3)}\alpha^{(2)}\left(s_1^{(2)}-s_2^{(2)}\right)}d^{(3)},
$$

$$d^{(2+)} = \frac{e^{-h_2(s_2^{(2)}-s_2^{(3)})}\left[s_1^{(2)}\alpha^{(2)}\Omega_4 - e^{h_2(-\beta^{(2)}+\beta^{(3)})}s_1^{(3)}s_2^{(3)}\alpha^{(3)}\Omega_3\right]}{s_1^{(3)}\alpha^{(2)}\left(s_1^{(2)}-s_2^{(2)}\right)}d^{(3)},$$

$$c^{(3)} = -\frac{s_2^{(3)}}{s_1^{(3)}}e^{(s_2^{(3)}-s_1^{(3)})(h_2+h_3)}d^{(3)}, \quad d^{(3)} = \frac{-e^{h_2(s_1^{(3)}-\beta^{(3)})-i\omega x_0}q_0 s_1^{(3)}P(y_0)}{e^{-2\beta^{(3)}h_2}M}, \tag{20}$$

where

$$\Omega_1 = -1+e^{h_1(s_1^{(1)}-s_2^{(1)})},\ \ \Omega_2 = s_1^{(1)} - e^{h_1(s_1^{(1)}-s_2^{(1)})}s_2^{(1)},$$
$$\Omega_3 = 1-e^{h_3(-s_1^{(3)}+s_2^{(3)})},\ \ \Omega_4 = s_1^{(3)} - e^{h_3(-s_1^{(3)}+s_2^{(3)})}s_2^{(3)},$$

$$\begin{aligned} M = {} & e^{h_2 s_2^{(2)}}\left(s_1^{(2)}\alpha^{(2)}\Omega_2 + s_1^{(1)}s_2^{(1)}\alpha^{(1)}\Omega_1\right)\left(e^{h_2\beta^{(2)}}s_2^{(2)}\alpha^{(2)}\Omega_4 - e^{h_2\beta^{(3)}}s_1^{(3)}s_2^{(3)}\alpha^{(3)}\Omega_3\right) \\ & - e^{h_2 s_1^{(2)}}\left(s_2^{(2)}\alpha^{(2)}\Omega_2 + s_1^{(1)}s_2^{(1)}\alpha^{(1)}\Omega_1\right)\left(e^{h_2\beta^{(2)}}s_1^{(2)}\alpha^{(2)}\Omega_4 - e^{h_2\beta^{(3)}}s_1^{(3)}s_2^{(3)}\alpha^{(3)}\Omega_3\right), \end{aligned}$$

and we define functions $L(\xi,\omega)$ and$P(\xi,\omega)$ as follows:

$$\begin{aligned} L(\xi,\omega) = {} & e^{h_2\beta^{(3)}}(e^{h_2 s_2^{(2)}+s_1^{(2)}\xi} - e^{h_2 s_1^{(2)}+s_2^{(2)}\xi})s_1^{(3)}s_2^{(3)}\alpha^{(3)}\Omega_3 + e^{h_2\beta^{(2)}}(e^{h_2 s_1^{(2)}+s_2^{(2)}\xi}s_1^{(2)} \\ & - e^{h_2 s_2^{(2)}+s_1^{(2)}\xi}s_2^{(2)})\alpha^{(2)}\Omega_4, \end{aligned}$$

$$P(\xi,\omega) = \left(e^{s_1^{(2)}\xi} - e^{s_2^{(2)}\xi}\right)s_1^{(1)}s_2^{(1)}\alpha^{(1)}\Omega_1 + \left(e^{s_1^{(2)}\xi}s_2^{(2)} - e^{s_2^{(2)}\xi}s_1^{(2)}\right)\alpha^{(2)}\Omega_2.$$

We substitute Eq. (20) into Eq. (18) and perform the inverse Fourier transform, Green's functions of the temperature and the heat fluxes in the functionally graded three-layer media are presented as follows:

$$T^{(1)} = \frac{q_0}{\pi}\int_0^\infty \frac{(e^{s_2^{(1)}y}s_1^{(1)} - e^{h_1(s_1^{(1)}-s_2^{(1)})+s_1^{(1)}y}s_2^{(1)})L(y_0,\omega)}{M}\cos[\omega(x-x_0)]d\omega, \tag{21}$$

$$q_y^{(1)} = \frac{q_0\alpha^{(1)}}{\pi}\int_0^\infty \frac{e^{-h_1 s_2^{(1)}+\beta^{(1)}y}L(y_0,\omega)(e^{s_2^{(1)}(y+h_1)} - e^{s_1^{(1)}(y+h_1)})\omega^2}{M}\cos[\omega(x-x_0)]d\omega, \tag{22}$$

$$q_x^{(1)} = \frac{\alpha^{(1)}q_0}{\pi}\int_0^\infty \frac{e^{\beta^{(1)}y}(e^{s_2^{(1)}y}s_1^{(1)} - e^{h_1(s_1^{(1)}-s_2^{(1)})+s_1^{(1)}y}s_2^{(1)})L(y_0,\omega)\omega}{M}\sin[\omega(x-x_0)]d\omega, \tag{23}$$

$$T^{(2-)} = \frac{-q_0}{2\pi\alpha^{(2)}} \int_0^\infty \frac{L(y_0,\omega)P(y,\omega)}{Q^{(2)}M} \cos[\omega(x-x_0)]d\omega, \tag{24}$$

$$q_y^{(2-)} = \frac{q_0}{2\pi} \int_0^\infty \left\{ \frac{e^{\beta^{(2)}y}L(y_0,\omega)}{Q^{(2)}M} \left.\frac{\partial P(\xi,\omega)}{\partial \xi}\right|_{\xi=y} \cos[\omega(x-x_0)] \right\} d\omega, \tag{25}$$

$$q_x^{(2-)} = \frac{-q_0}{2\pi} \int_0^\infty \frac{e^{\beta^{(2)}y}L(y_0,\omega)P(y,\omega)\omega}{Q^{(2)}M} \sin[\omega(x-x_0)]d\omega, \tag{26}$$

$$T^{(2+)} = \frac{-q_0}{2\pi\alpha^{(2)}} \int_0^\infty \frac{P(y_0,\omega)\ L(y,\omega)}{Q^{(2)}M} \cos[\omega(x-x_0)]d\omega, \tag{27}$$

$$q_y^{(2+)} = \frac{q_0}{2\pi} \int_0^\infty \left\{ \frac{e^{\beta^{(2)}y}P(y_0,\omega)}{Q^{(2)}M} \left.\frac{\partial L(\xi,\omega)}{\partial \xi}\right|_{\xi=y} \cos[\omega(x-x_0)] \right\} d\omega, \tag{28}$$

$$q_x^{(2+)} = \frac{-q_0}{2\pi} \int_0^\infty \frac{e^{\beta^{(2)}y}P(y_0,\omega)L(y,\omega)\omega}{Q^{(2)}M} \sin[\omega(x-x_0)]d\omega, \tag{29}$$

$$T^{(3)} = \frac{-q_0}{\pi} \int_0^\infty \frac{P(y_0,\omega)}{M} \left(e^{s_2^{(3)}(y-h_2)} s_1^{(3)} - e^{h_3 s_2^{(3)} + s_1^{(3)}(y-h_2-h_3)} s_2^{(3)} \right) \cos[\omega(x-x_0)]d\omega, \tag{30}$$

$$q_y^{(3)} = \frac{\alpha^{(3)}q_0}{\pi} \int_0^\infty \frac{e^{\beta^{(3)}y}\omega^2 P(y_0,\omega)}{M} \left(e^{h_3 s_2^{(3)} + s_1^{(3)}(y-h_2-h_3)} - e^{s_2^{(3)}(y-h_2)} \right) \cos[\omega(x-x_0)]d\omega, \tag{31}$$

$$q_x^{(3)} = \frac{-\alpha^{(3)}q_0}{\pi} \int_0^\infty \left\{ \frac{e^{\beta^{(3)}y}P(y_0,\omega)\omega}{M} \left(e^{s_2^{(3)}(y-h_2)} s_1^{(3)} - e^{h_3 s_2^{(3)} + s_1^{(3)}(y-h_2-h_3)} s_2^{(3)} \right) \sin[\omega(x-x_0)] \right\} d\omega. \tag{32}$$

4 The Applied Heat Source is in the Bottom Layer $(-h_1 \le y_0 \le 0)$ or on the Boundary $(y_0 = -h_1)$

Here we consider the three-layer media subjected to a heat source at material 1 as indicated in Figure 2. The boundary conditions and continuity conditions at the interface are the same as the previous case. Jump condition and continuity condition near the heat source are

$$\left.\tilde{T}^{(1+)}\right|_{y\to y_0^+} = \left.\tilde{T}^{(1-)}\right|_{y\to y_0^-}, \quad \left.\tilde{q}_y^{(1+)}\right|_{y\to y_0^+} - \left.\tilde{q}_y^{(1-)}\right|_{y\to y_0^-} = q_0 e^{-i\omega x_0}. \tag{33}$$

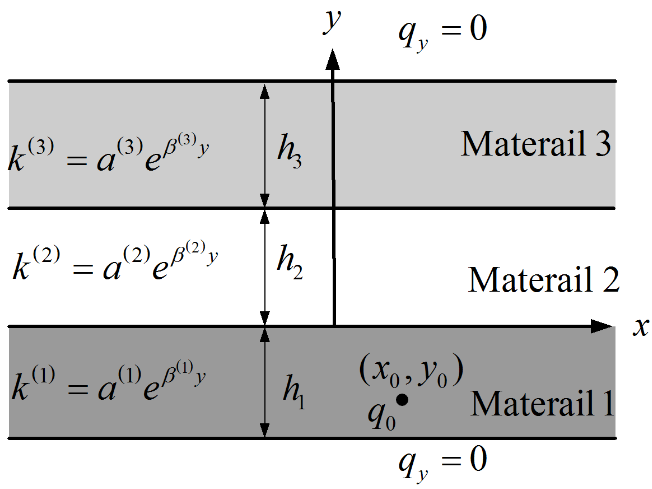

Figure 2: Configuration and coordinate system of functionally graded three-layer media subjected to a heat source in material 1.

From Eq. (17), Eq.(18) and Eq. (33), the undetermined coefficients can be obtained as follows:

$$c^{(1-)} = -\frac{s_2^{(1)}}{s_1^{(1)}} e^{h_1\left(s_1^{(1)} - s_2^{(1)}\right)} d^{(1-)},$$

$$d^{(1-)} = \frac{-e^{-i\omega x_0} q_0 s_1^{(1)}}{2\alpha^{(1)} Q^{(2)}} \times \frac{\left[\alpha^{(1)}\left(e^{s_2^{(1)} y_0} s_1^{(1)} - e^{s_1^{(1)} y_0} s_2^{(1)}\right) L(0,\omega) + \alpha^{(2)}\left(e^{s_1^{(1)} y_0} - e^{s_2^{(1)} y_0}\right) \left.\frac{\partial L(\xi,\omega)}{\partial \xi}\right|_{\xi=0}\right]}{\left(\alpha^{(1)} \Omega_1 \omega^2 L(0,\omega) + \alpha^{(2)} \Omega_2 \left.\frac{\partial L(\xi,\omega)}{\partial \xi}\right|_{\xi=0}\right)},$$

$$c^{(1+)} = -\frac{s_2^{(1)}}{s_1^{(1)}} e^{h_1\left(s_1^{(1)} - s_2^{(1)}\right)} d^{(1-)} - \frac{e^{s_2^{(1)} y_0 - i\omega x_0} q_0}{\left(s_1^{(1)} - s_2^{(1)}\right) \alpha^{(1)}},$$

$$d^{(1+)} = d^{(1-)} + \frac{e^{s_1^{(1)} y_0 - i\omega x_0} q_0}{\left(s_1^{(1)} - s_2^{(1)}\right) \alpha^{(1)}},$$

$$c^{(2)} = \frac{e^{-h_2\left(s_1^{(2)} - s_2^{(3)}\right)} \left[e^{h_2\left(-\beta^{(2)} + \beta^{(3)}\right)} s_1^{(3)} s_2^{(3)} \alpha^{(3)} \Omega_3 - s_2^{(2)} \alpha^{(2)} \Omega_4\right]}{\left(s_1^{(2)} - s_2^{(2)}\right) s_1^{(3)} \alpha^{(2)}} d^{(3)},$$

$$d^{(2)} = \frac{e^{h_2\left(s_2^{(3)} - s_2^{(2)}\right)} \left[e^{h_2\left(\beta^{(3)} - \beta^{(2)}\right)} s_1^{(3)} s_2^{(3)} \alpha^{(3)} \Omega_3 - s_1^{(2)} \alpha^{(2)} \Omega_4\right]}{\left(s_2^{(2)} - s_1^{(2)}\right) s_1^{(3)} \alpha^{(2)}} d^{(3)},$$

$$c^{(3)} = -\frac{s_2^{(3)}}{s_1^{(3)}} e^{(h_2+h_3)(s_2^{(3)}-s_1^{(3)})} d^{(3)},$$

$$d^{(3)} = \frac{e^{-h_2 s_2^{(3)}} Q^{(2)} s_1^{(3)} \alpha^{(2)}}{L(0,\omega)} \left[\frac{e^{-i\omega x_0} q_0 \left(e^{s_1^{(1)} y_0} - s^{s_2^{(1)} y_0} \right)}{\alpha^{(1)} Q^{(1)}} + \frac{2\Omega_2 d^{(1-)}}{s_1^{(1)}} \right]. \tag{34}$$

Thus the full-field solutions of functionally graded three-layer media subjected to a heat source in material 1 can be presented as follows:

$$T^{(1-)} = \frac{-q_0}{2\alpha^{(1)}\pi} \times \int_0^\infty \left\{ \begin{array}{l} \left[\alpha^{(1)} \left(e^{s_2^{(1)} y_0} s_1^{(1)} - e^{s_1^{(1)} y_0} s_2^{(1)} \right) L(0,\omega) + \alpha^{(2)} \left(e^{s_1^{(1)} y_0} - e^{s_2^{(1)} y_0} \right) \left. \frac{\partial L(\xi,\omega)}{\partial \xi} \right|_{\xi=0} \right] \\ \times \left(e^{s_2^{(1)} y} s_1^{(1)} - e^{s_1^{(1)} y + 2h_1 Q^{(1)}} s_2^{(1)} \right) \cos[\omega(x-x_0)] \\ \times \left[Q^{(1)} \left(-\Omega_1 \alpha^{(1)} \omega^2 L(0,\omega) + \alpha^{(2)} \Omega_2 \left. \frac{\partial L(\xi,\omega)}{\partial \xi} \right|_{\xi=0} \right) \right]^{-1} \end{array} \right\} d\omega, \tag{35}$$

$$q_y^{(1-)} = \frac{q_0}{2\pi} e^{\beta^{(1)} y} \times \int_0^\infty \left\{ \begin{array}{l} \left[\alpha^{(1)} \left(e^{s_2^{(1)} y_0} s_1^{(1)} - e^{s_1^{(1)} y_0} s_2^{(1)} \right) L(0,\omega) + \alpha^{(2)} \left(e^{s_1^{(1)} y_0} - e^{s_2^{(1)} y_0} \right) \left. \frac{\partial L(\xi,\omega)}{\partial \xi} \right|_{\xi=0} \right] \\ \times \left(e^{s_2^{(1)} y} - e^{s_1^{(1)} y + 2h_1 Q^{(1)}} \right) \omega^2 \cos[\omega(x-x_0)] \\ \times \left[Q^{(1)} \left(-\Omega_1 \alpha^{(1)} \omega^2 L(0,\omega) + \alpha^{(2)} \Omega_2 \left. \frac{\partial L(\xi,\omega)}{\partial \xi} \right|_{\xi=0} \right) \right]^{-1} \end{array} \right\} d\omega, \tag{36}$$

$$q_x^{(1-)} = \frac{-q_0}{2\pi} e^{\beta^{(1)} y} \times \int_0^\infty \left\{ \begin{array}{l} \left[\alpha^{(1)} \left(e^{s_2^{(1)} y_0} s_1^{(1)} - e^{s_1^{(1)} y_0} s_2^{(1)} \right) L(0,\omega) + \alpha^{(2)} \left(e^{s_1^{(1)} y_0} - e^{s_2^{(1)} y_0} \right) \left. \frac{\partial L(\xi,\omega)}{\partial \xi} \right|_{\xi=0} \right] \\ \times \left(e^{s_2^{(1)} y} s_1^{(1)} - e^{s_1^{(1)} y + 2h_1 Q^{(1)}} s_2^{(1)} \right) \omega \sin[\omega(x-x_0)] \\ \times \left[Q^{(1)} \left(-\Omega_1 \alpha^{(1)} \omega^2 L(0,\omega) + \alpha^{(2)} \Omega_2 \left. \frac{\partial L(\xi,\omega)}{\partial \xi} \right|_{\xi=0} \right) \right]^{-1} \end{array} \right\} d\omega, \tag{37}$$

$$T^{(1+)} = \frac{-q_0}{2\alpha^{(1)}\pi} \times \int_0^\infty \left\{ \begin{array}{l} \left[\alpha^{(1)} \left(e^{s_1^{(1)} y} s_2^{(1)} - e^{s_2^{(1)} y} s_1^{(1)} \right) L(0,\omega) + \alpha^{(2)} \left(e^{s_2^{(1)} y} - e^{s_1^{(1)} y} \right) \left. \frac{\partial L(\xi,\omega)}{\partial \xi} \right|_{\xi=0} \right] \\ \times \left(e^{s_1^{(1)} y_0 + 2h_1 Q^{(1)}} s_2^{(1)} - e^{s_2^{(1)} y_0} s_1^{(1)} \right) \cos[\omega(x-x_0)] \\ \times \left[Q^{(1)} \left(-\Omega_1 \alpha^{(1)} \omega^2 L(0,\omega) + \alpha^{(2)} \Omega_2 \left. \frac{\partial L(\xi,\omega)}{\partial \xi} \right|_{\xi=0} \right) \right]^{-1} \end{array} \right\} d\omega, \tag{38}$$

$$q_y^{(1+)} = \frac{-q_0}{2\pi} e^{\beta^{(1)} y} \times \int_0^\infty \left\{ \begin{array}{l} \left[\alpha^{(1)} \left(e^{s_2^{(1)} y} - e^{s_1^{(1)} y} \right) \omega^2 L(0,\omega) + \alpha^{(2)} \left(e^{s_1^{(1)} y} s_1^{(1)} - e^{s_2^{(1)} y} s_2^{(1)} \right) \left. \frac{\partial L(\xi,\omega)}{\partial \xi} \right|_{\xi=0} \right] \\ \times \left(e^{s_1^{(1)} y_0 + 2h_1 Q^{(1)}} s_2^{(1)} - e^{s_2^{(1)} y_0} s_1^{(1)} \right) \cos[\omega(x-x_0)] \\ \times \left[Q^{(1)} \left(-\Omega_1 \alpha^{(1)} \omega^2 L(0,\omega) + \alpha^{(2)} \Omega_2 \left. \frac{\partial L(\xi,\omega)}{\partial \xi} \right|_{\xi=0} \right) \right]^{-1} \end{array} \right\} d\omega,$$

(39)

$$q_x^{(1+)} = \frac{-q_0}{2\pi} e^{\beta^{(1)}y} \times \int_0^\infty \left\{ \begin{array}{l} \left[\alpha^{(1)} \left(e^{s_1^{(1)}y} s_2^{(1)} - e^{s_2^{(1)}y} s_1^{(1)} \right) L(0,\omega) + \alpha^{(2)} \left(e^{s_2^{(1)}y} - e^{s_1^{(1)}y} \right) \left. \frac{\partial L(\xi,\omega)}{\partial \xi} \right|_{\xi=0} \right] \\ \times \left(e^{s_1^{(1)}y_0 + 2h_1 Q^{(1)}} s_2^{(1)} - e^{s_2^{(1)}y_0} s_1^{(1)} \right) \omega \sin[\omega(x-x_0)] \\ \times \left[Q^{(1)} \left(-\Omega_1 \alpha^{(1)} \omega^2 L(0,\omega) + \alpha^{(2)} \Omega_2 \left. \frac{\partial L(\xi,\omega)}{\partial \xi} \right|_{\xi=0} \right) \right]^{-1} \end{array} \right\} d\omega, \tag{40}$$

$$T^{(2)} = \frac{q_0}{\pi} \int_0^\infty \frac{\left(e^{s_1^{(1)}y_0 + 2h_1 Q^{(1)}} s_2^{(1)} - e^{s_2^{(1)}y_0} s_1^{(1)} \right) L(y,\omega)}{-\alpha^{(1)} \Omega_1 \omega^2 L(0,\omega) + \alpha^{(2)} \Omega_2 \left. \frac{\partial L(\xi,\omega)}{\partial \xi} \right|_{\xi=0}} \cos[\omega(x-x_0)] d\omega, \tag{41}$$

$$q_y^{(2)} = \frac{-\alpha^{(2)} q_0 e^{\beta^{(2)}y}}{\pi} \int_0^\infty \frac{\left(e^{s_1^{(1)}y_0 + 2h_1 Q^{(1)}} s_2^{(1)} - e^{s_2^{(1)}y_0} s_1^{(1)} \right) \left. \frac{\partial L(\xi,\omega)}{\xi} \right|_{\xi=y}}{-\alpha^{(1)} \Omega_1 \omega^2 L(0,\omega) + \alpha^{(2)} \Omega_2 \left. \frac{\partial L(\xi,\omega)}{\partial \xi} \right|_{\xi=0}} \cos[\omega(x-x_0)] d\omega, \tag{42}$$

$$q_x^{(2)} = \frac{q_0 \alpha^{(2)} e^{\beta^{(2)}y}}{\pi} \int_0^\infty \frac{\left(e^{s_1^{(1)}y_0 + 2h_1 Q^{(1)}} s_2^{(1)} - e^{s_2^{(1)}y_0} s_1^{(1)} \right) \omega L(y,\omega)}{-\alpha^{(1)} \Omega_1 \omega^2 L(0,\omega) + \alpha^{(2)} \Omega_2 \left. \frac{\partial L(\xi,\omega)}{\partial \xi} \right|_{\xi=0}} \sin[\omega(x-x_0)] d\omega, \tag{43}$$

$$T^{(3)} = \frac{2q_0 \alpha^{(2)}}{\pi} \int_0^\infty \left\{ \begin{array}{l} Q^{(2)} \left(e^{2h_1 Q^{(1)} + s_1^{(1)}y_0} s_2^{(1)} - e^{s_2^{(1)}y_0} s_1^{(1)} \right) \cos[\omega(x-x_0)] \\ \times \left(e^{s_2^{(3)}(y-h_2)} s_1^{(3)} - e^{-2h_3 Q^{(3)} + s_1^{(3)}(y-h_2)} s_2^{(3)} \right) \\ \times \left(-\Omega_1 \alpha^{(1)} \omega^2 L(0,\omega) + \Omega_2 \alpha^{(2)} \left. \frac{\partial L(\xi,\omega)}{\partial \xi} \right|_{\xi=0} \right)^{-1} \end{array} \right\} d\omega, \tag{44}$$

$$q_y^{(3)} = \frac{-2q_0 \alpha^{(2)} \alpha^{(3)}}{\pi} \int_0^\infty \left\{ \begin{array}{l} Q^{(2)} \left(e^{2h_1 Q^{(1)} + s_1^{(1)}y_0} s_2^{(1)} - e^{s_2^{(1)}y_0} s_1^{(1)} \right) \cos[\omega(x-x_0)] \\ \times \left(e^{s_2^{(3)}(y-h_2)} - e^{-2h_3 Q^{(3)} + s_1^{(3)}(y-h_2)} \right) \omega^2 e^{\beta^{(3)}y} \\ \times \left(-\Omega_1 \alpha^{(1)} \omega^2 L(0,\omega) + \Omega_2 \alpha^{(2)} \left. \frac{\partial L(\xi,\omega)}{\partial \xi} \right|_{\xi=0} \right)^{-1} \end{array} \right\} d\omega, \tag{45}$$

$$q_x^{(3)} = \frac{2q_0 \alpha^{(2)} \alpha^{(3)}}{\pi} \int_0^\infty \left\{ \begin{array}{l} Q^{(2)} \left(e^{2h_1 Q^{(1)} + s_1^{(1)}y_0} s_2^{(1)} - e^{s_2^{(1)}y_0} s_1^{(1)} \right) \sin[\omega(x-x_0)] \\ \times \left(e^{s_2^{(3)}(y-h_2)} s_1^{(3)} - e^{-2h_3 Q^{(3)} + s_1^{(3)}(y-h_2)} s_2^{(3)} \right) \omega e^{\beta^{(3)}y} \\ \times \left(-\Omega_1 \alpha^{(1)} \omega^2 L(0,\omega) + \Omega_2 \alpha^{(2)} \left. \frac{\partial L(\xi,\omega)}{\partial \xi} \right|_{\xi=0} \right)^{-1} \end{array} \right\} d\omega. \tag{46}$$

The last case we consider is a nonhomogeneous three-layer media subjected to a heat source on the boundary$y = -h_1$ as indicated in Figure 3. The solutions in this case can be obtained from Eqs. (35) - (46) by setting$y_0 = -h_1$.

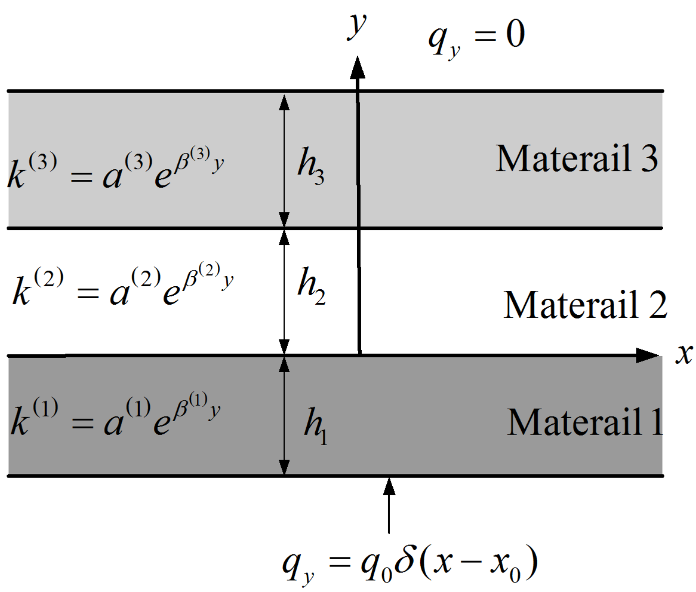

Figure 3: Configuration and coordinate system of functionally graded three-layer media subjected to a heat source at boundary $y = -h_1$

5 The Characteristics at the Interface for Continuous Conductivities in the Nonhomogeneous Three-Layer Media

Functionally graded materials are used to connect two dissimilar materials, and reduce discontinuous jump at the interface. In the previous section, the full-field solutions of nonhomogeneous three-layer media subjected to a heat source are presented. For special case of nonhomogeneous three-layer media with continuous conductivities at the interfaces, there are some phenomena we will discuss here in detail.

For case (A), the conductivities are continuous at the interfaces (i.e., $k^{(1)}(0) = k^{(2)}(0)$ and $k^{(2)}(h_2) = k^{(3)}(h_2)$). From Eqs. (21) - (32), it is interesting to find out that the temperature and heat fluxes are all continuous at the interface even for the heat flux q_x. In this case, the parameter M and functions $P(\xi, \omega)$ and $L(\xi, \omega)$ are degenerated to

$$M = -e^{-h_2 s_2^{(2)}}\left(\alpha^{(1)}\right)^2\left(s_2^{(2)}\Omega_2 + s_1^{(1)}s_2^{(1)}\Omega_1\right)\left(s_1^{(2)}\Omega_4 - s_1^{(3)}s_2^{(3)}\Omega_3\right)$$
$$+e^{-h_2 s_1^{(2)}}\left(\alpha^{(1)}\right)^2\left(s_1^{(2)}\Omega_2 + s_1^{(1)}s_2^{(1)}\Omega_1\right)\left(s_2^{(2)}\Omega_4 - s_1^{(3)}s_2^{(3)}\Omega_3\right),$$

$$L(\xi,\omega) = \alpha^{(1)}e^{h_2\beta^{(2)}}\left[(e^{h_2 s_2^{(2)}+s_1^{(2)}\xi} - e^{h_2 s_1^{(2)}+s_2^{(2)}\xi})s_1^{(3)}s_2^{(3)}\Omega_3 + (e^{h_2 s_1^{(2)}+s_2^{(2)}\xi}s_1^{(2)} - e^{h_2 s_2^{(2)}+s_1^{(2)}\xi}s_2^{(2)})\Omega_4\right],$$

$$P(\xi,\omega) = \alpha^{(1)}\left[\left(e^{s_1^{(2)}\xi} - e^{s_2^{(2)}\xi}\right)s_1^{(1)}s_2^{(1)}\Omega_1 + \left(-e^{s_2^{(2)}\xi}s_1^{(2)} + e^{s_1^{(2)}\xi}s_2^{(2)}\right)\Omega_2\right].$$

From Eqs. (21) – (32), the heat fluxes at the interfaces are

$$q_x^{(1)}(x,0) = q_x^{(2-)}(x,0) = \frac{\alpha^{(1)}q_0}{\pi}\int_0^\infty \frac{\Omega_2 L(y_0,\omega)\omega}{M}\sin[\omega(x-x_0)]d\omega, \tag{47}$$

$$q_y^{(1)}(x,0) = q_y^{(2-)}(x,0) = \frac{-q_0\alpha^{(1)}}{\pi}\int_0^\infty \frac{L(y_0,\omega)\Omega_1\omega^2}{M}\cos[\omega(x-x_0)]d\omega, \tag{48}$$

$$q_x^{(2+)}(x,h_2) = q_x^{(3)}(x,h_2) = \frac{-\alpha^{(2)}e^{\beta^{(2)}h_2}q_0}{\pi}\int_0^\infty \left\{\frac{P(y_0,\omega)\omega\Omega_4}{M}\sin[\omega(x-x_0)]\right\}d\omega, \tag{49}$$

$$q_y^{2+)}(x,h_2) = q_y^{(3)}(x,h_2) = \frac{-\alpha^{(2)}e^{\beta^{(2)}h_2}q_0}{\pi}\int_0^\infty \left\{\frac{P(y_0,\omega)\omega^2\Omega_3}{M}\cos[\omega(x-x_0)]\right\}d\omega. \tag{50}$$

Furthermore, the first derivatives of temperature T and heat flux q_y are also continuous at the interface, and the interesting results are presented as follows:

$$\frac{q_{x,x}^{(1)}(x,0)}{q_{x,y}^{(1)}(x,0)} \neq \frac{q_{x,x}^{(2-)}(x,0)}{q_{x,y}^{(2-)}(x,0)}, \quad \frac{q_{y,x}^{(1)}(x,0)}{q_{y,y}^{(1)}(x,0)} = \frac{q_{y,x}^{(2-)}(x,0)}{q_{y,y}^{(2-)}(x,0)}, \tag{51}$$

$$\frac{q_{x,x}^{(2+)}(x,h_2)}{q_{x,y}^{(2+)}(x,h_2)} \neq \frac{q_{x,x}^{(3)}(x,h_2)}{q_{x,y}^{(3)}(x,h_2)}, \quad \frac{q_{y,x}^{(2+)}(x,h_2)}{q_{y,y}^{(2+)}(x,h_2)} = \frac{q_{y,x}^{(3)}(x,h_2)}{q_{y,y}^{(3)}(x,h_2)}. \tag{52}$$

where

$$q_{y,x}^{(1)}(x,0) = q_{y,x}^{(2-)}(x,0) = \frac{q_0\alpha^{(1)}}{\pi}\int_0^\infty \frac{L(y_0,\omega)\Omega_1\omega^3}{M}\sin[\omega(x-x_0)]d\omega, \tag{53}$$

$$q_{y,y}^{(1)}(x,0) = q_{y,y}^{(2-)}(x,0) = \frac{-q_0\alpha^{(1)}}{\pi}\int_0^\infty \frac{L(y_0,\omega)\Omega_2\omega^2}{M}\cos[\omega(x-x_0)]d\omega, \tag{54}$$

$$q_{y,x}^{(2+)}(x,h_2) = q_{y,x}^{(3)}(x,h_2) = \frac{\alpha^{(2)}e^{\beta^{(2)}h_2}q_0}{\pi}\int_0^\infty \left\{\frac{P(y_0,\omega)\omega^3\Omega_3}{M}\sin[\omega(x-x_0)]\right\}d\omega, \tag{55}$$

$$q_{y,y}^{(2+)}(x,h_2) = q_{y,y}^{(3)}(x,h_2) = \frac{\alpha^{(2)}e^{\beta^{(2)}h_2}q_0}{\pi}\int_0^\infty \left\{\frac{\omega^2 P(y_0,\omega)\Omega_4}{M}\cos[\omega(x-x_0)]\right\}d\omega. \tag{56}$$

So it can be concluded that if the conductivities are continuous at the interfaces, not only the temperature and heat fluxes are continuous at the interfaces, but also the first derivatives of temperature T and heat flux q_y are continuous.

Case (B), we investigate the case that a functionally graded layer sandwiched between two homogeneous layers, and the conductivities are continuous at the interfaces, i.e., $\beta^{(1)} = \beta^{(3)} = 0, \beta^{(2)} \neq 0, \alpha^{(1)} = \alpha^{(2)}$, and $k^{(2)}(h_2) = k^{(3)}(h_2)$. In this case, $s_1^{(1)} = \omega$, $s_2^{(1)} = -\omega$, $s_1^{(3)} = \omega$, $s_2^{(3)} = -\omega$ and the simplify functions are indicated as follows:

$$\Omega_1 = -1 + e^{2h_1\omega}, \quad \Omega_2 = \omega(1+e^{2h_1\omega}), \quad \Omega_3 = 1 - e^{-2h_3\omega}, \quad \Omega_4 = \omega(1+e^{-2h_3\omega}),$$

$$\begin{aligned} M =& e^{h_2 s_2^{(2)}}\alpha^{(3)}\left(s_1^{(2)}\alpha^{(2)}\Omega_2 - \omega^2\alpha^{(1)}\Omega_1\right)\left(s_2^{(2)}\Omega_4 + \omega^2\Omega_3\right) \\ &- e^{h_2 s_1^{(2)}}\alpha^{(3)}\left(s_2^{(2)}\alpha^{(2)}\Omega_2 - \omega^2\alpha^{(1)}\Omega_1\right)\left(s_1^{(2)}\Omega_4 + \omega^2\Omega_3\right), \end{aligned}$$

$$\begin{aligned} L(\xi,\omega) =& (e^{h_2 s_1^{(2)} + s_2^{(2)}\xi} - e^{h_2 s_2^{(2)} + s_1^{(2)}\xi})\omega^2\alpha^{(3)}\Omega_3 \\ &+ (e^{h_2 s_1^{(2)} + s_2^{(2)}\xi}s_1^{(2)} - e^{h_2 s_2^{(2)} + s_1^{(2)}\xi}s_2^{(2)})\alpha^{(3)}\Omega_4, \end{aligned}$$

$$P(\xi,\omega) = \left(e^{s_2^{(2)}\xi} - e^{s_1^{(2)}\xi}\right)\omega^2\alpha^{(1)}\Omega_1 + \left(e^{s_1^{(2)}\xi}s_2^{(2)} - e^{s_2^{(2)}\xi}s_1^{(2)}\right)\alpha^{(2)}\Omega_2.$$

From Eqs. (21) – (32), the solutions are presented as follows:

$$T^{(1)} = \frac{q_0}{\pi}\int_0^\infty \frac{(e^{-\omega y} + e^{\omega(2h_1+y)})\omega L(y_0,\omega)}{M}\cos[\omega(x-x_0)]d\omega, \tag{57}$$

$$q_y^{(1)} = \frac{q_0\alpha^{(1)}}{\pi}\int_0^\infty \frac{e^{h_1\omega}L(y_0,\omega)(e^{-\omega(y+h_1)} - e^{\omega(y+h_1)})\omega^2}{M}\cos[\omega(x-x_0)]d\omega, \tag{58}$$

$$q_x^{(1)} = \frac{\alpha^{(1)} q_0}{\pi} \int_0^\infty \frac{(e^{-\omega y} + e^{\omega(2h_1+y)}) L(y_0,\omega)\omega^2}{M} \sin[\omega(x-x_0)] d\omega, \tag{59}$$

$$T^{(2-)} = \frac{-q_0}{2\pi\alpha^{(2)}} \int_0^\infty \frac{L(y_0,\omega)P(y,\omega)}{Q^{(2)}M} \cos[\omega(x-x_0)] d\omega, \tag{60}$$

$$q_y^{(2-)} = \frac{q_0}{2\pi} \int_0^\infty \left\{ \frac{e^{\beta^{(2)}y} L(y_0,\omega)}{Q^{(2)}M} \left. \frac{\partial P(\xi,\omega)}{\partial \xi} \right|_{\xi=y} \cos[\omega(x-x_0)] \right\} d\omega, \tag{61}$$

$$q_x^{(2-)} = \frac{-q_0}{2\pi} \int_0^\infty \frac{e^{\beta^{(2)}y} L(y_0,\omega)P(y,\omega)\omega}{Q^{(2)}M} \sin[\omega(x-x_0)] d\omega, \tag{62}$$

$$T^{(2+)} = \frac{-q_0}{2\pi\alpha^{(2)}} \int_0^\infty \frac{P(y_0,\omega)\, L(y,\omega)}{Q^{(2)}M} \cos[\omega(x-x_0)] d\omega, \tag{63}$$

$$q_y^{(2+)} = \frac{q_0}{2\pi} \int_0^\infty \left\{ \frac{e^{\beta^{(2)}y} P(y_0,\omega)}{Q^{(2)}M} \left. \frac{\partial L(\xi,\omega)}{\partial \xi} \right|_{\xi=y} \cos[\omega(x-x_0)] \right\} d\omega, \tag{64}$$

$$q_x^{(2+)} = \frac{-q_0}{2\pi} \int_0^\infty \frac{e^{\beta^{(2)}y} P(y_0,\omega)L(y,\omega)\omega}{Q^{(2)}M} \sin[\omega(x-x_0)] d\omega, \tag{65}$$

$$T^{(3)} = \frac{-q_0}{\pi} \int_0^\infty \left\{ \frac{P(y_0,\omega)}{M} \omega \cos[\omega(x-x_0)] \left(e^{-\omega(y-h_2)} + e^{-2h_3\omega+\omega(y-h_2)} \right) \right\} d\omega, \tag{66}$$

$$q_y^{(3)} = \frac{\alpha^{(3)} q_0}{\pi} \int_0^\infty \left\{ \frac{e^{-h_3\omega} P(y_0,\omega)\omega^2}{M} \left(e^{\omega(y-h_2-h_3)} - e^{-\omega(y-h_2-h_3)} \right) \cos[\omega(x-x_0)] \right\} d\omega, \tag{67}$$

$$q_x^{(3)} = \frac{-\alpha^{(3)} q_0}{\pi} \int_0^\infty \left\{ \frac{e^{-h_3\omega} P(y_0,\omega)\omega^2}{M} \left(e^{-\omega(y-h_2-h_3)} + e^{\omega(y-h_2-h_3)} \right) \sin[\omega(x-x_0)] \right\} d\omega. \tag{68}$$

6 Numerical Results and Discussions

In this section, numerical calculations of the full-field distributions are constructed by using the solutions presented in previous sections. Contour plot is used to demonstrate the full-filed distributions of temperature and heat fluxes. In full-field distribution contours, short dash lines and solid lines are used to indicate negative and positive values, respectively.

Figure 4 - Figure 6 show the full-field contours of normalized temperature and heat fluxes for the functionally graded factors $\alpha^{(1)} = 5,\ \alpha^{(2)} = 2,\ \alpha^{(3)} = 1,\ \beta^{(1)} = 2,\ \beta^{(2)} = 1,\ \beta^{(3)} = 3$, and subjected to a heat source at $(x_0, y_0) = (0, 0.5h_2)$, $(0, -0.5\ h_2)$, and $(0, -h_1)$, respectively. The thicknesses of the three layers are set to be different, and $h_1 = 1.2,\ h_2 = 1,\ h_3 = 0.8$. It is noted that the temperature and heat flux q_y are continuous at the interface due to the continuity conditions while heat flux q_x is discontinuous at the interface. For discontinuous conductivities at the interface, we can see that the slopes of all the full-field distributions are discontinuous at the interface.

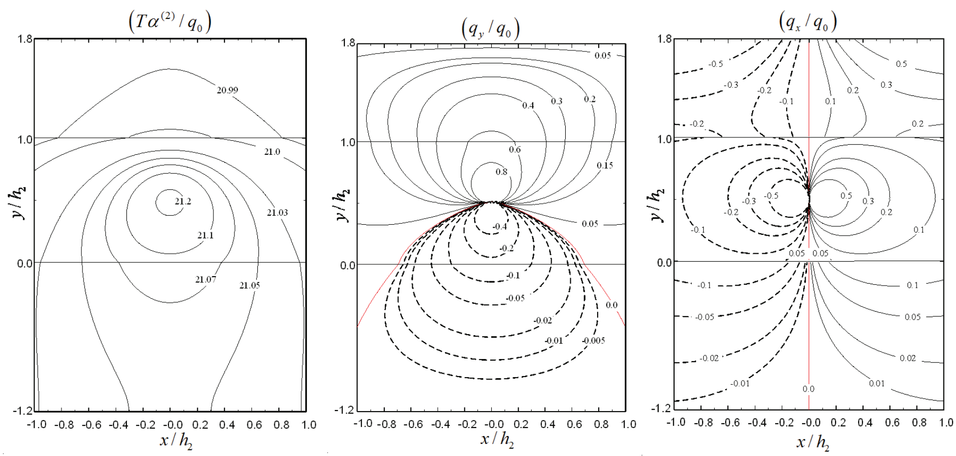

Figure 4: Full-field distribution of temperature, heat flux q_y and heat flux q_x for a three-layer media subjected to a heat source q_0 at $(0, 0.5h_2)$ with $\alpha^{(1)} = 5,\ \alpha^{(2)} = 2,\ \alpha^{(3)} = 1,\ \beta^{(1)} = 2,\ \beta^{(2)} = 1,\ \beta^{(3)} = 3$ and $h_1 = 1.2,\ h_2 = 1,\ h_3 = 0.8$.

Next, the continuity characteristics at the interfaces are presented in Figure 7 - Figure 9 with continuous conductivities at the interfaces. In Figure 7 - Figure 9, the functionally graded factors are$\alpha^{(1)} = 1,\ \alpha^{(2)} = 1,\ \alpha^{(3)} = 3,\ \beta^{(1)} = -2,\ \beta^{(2)} = \ln 3 + 2,\ \beta^{(3)} = 2$. Since the conductivities are continuous at the interfaces, we can see that the temperature and heat fluxes are all continuous at the interfaces,

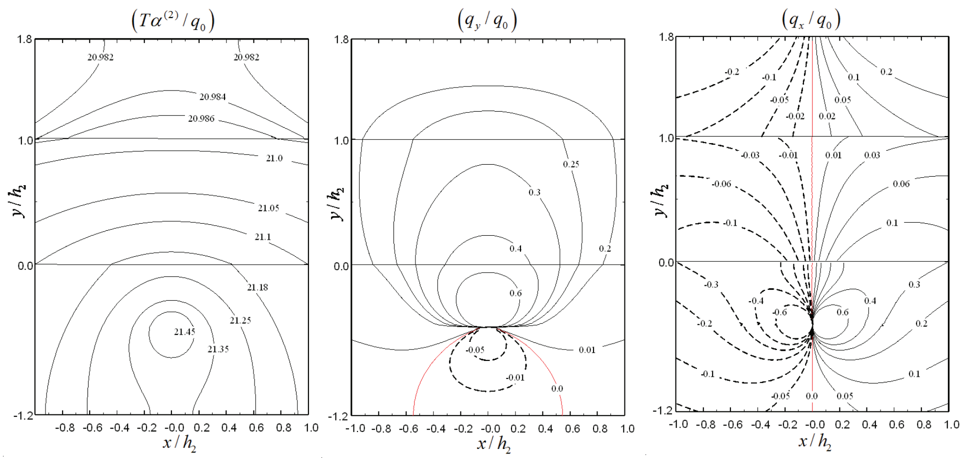

Figure 5: Full-field distribution of temperature, heat flux q_y and heat flux q_x for a three-layer media subjected to a heat source q_0 at $(0,-0.5h_2)$ with $\alpha^{(1)}=5,\ \alpha^{(2)}=2,\ \alpha^{(3)}=1,\ \beta^{(1)}=2,\ \beta^{(2)}=1,\ \beta^{(3)}=3$ and $h_1=1.2,\ h_2=1,\ h_3=0.8$.

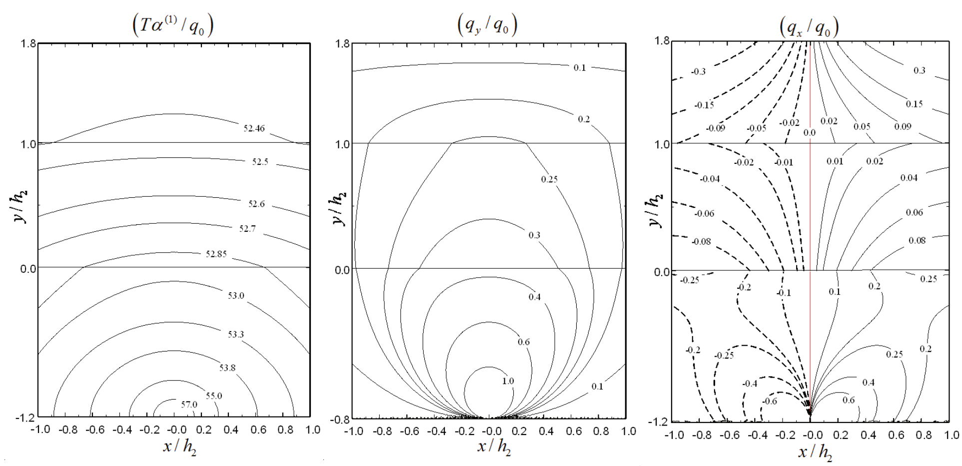

Figure 6: Full-field distribution of temperature , heat flux q_y and heat flux q_x for a three-layer media subjected to a heat source q_0 at $(0,-h_1)$ with $\alpha^{(1)}=5,\ \alpha^{(2)}=2,\ \alpha^{(3)}=1,\ \beta^{(1)}=2,\ \beta^{(2)}=1,\ \beta^{(3)}=3$ and $h_1=1.2,\ h_2=1,\ h_3=0.8$.

and moreover, the slopes of the contour plots of temperature and heat flux q_y are also continuous at the interfaces. Figure 10 - Figure 12 show the results for a functionally graded layer sandwiched between two homogeneous layers, and the factors are$\alpha^{(1)} = 1,\ \alpha^{(2)} = 1,\ \alpha^{(3)} = e,\ \beta^{(1)} = 0,\ \beta^{(2)} = 1,\ \beta^{(3)} = 0$. Because conductivities are continuous at the interfaces, the continuity characteristics are also found in this case.

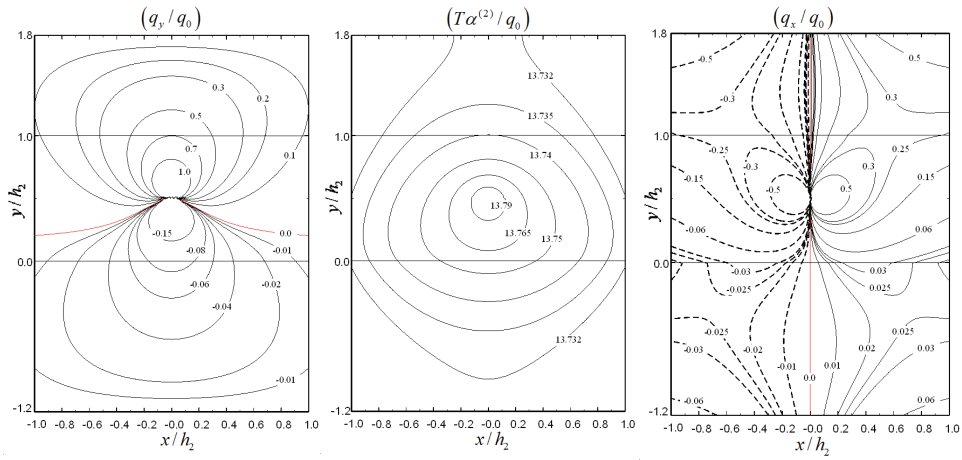

Figure 7: Full-field distribution of temperature, heat flux q_y and heat flux q_x for a three-layer media subjected to a heat source q_0 at $(0, 0.5h_2)$ with $\alpha^{(1)} = 1,\ \alpha^{(2)} = 1,\ \alpha^{(3)} = 3,\ \beta^{(1)} = -2,\ \beta^{(2)} = \ln 3 + 2,\ \beta^{(3)} = 2$ and $h_1 = 1.2,\ h_2 = 1,\ h_3 = 0.8$.

From the concept of Green's function, if a structure is subjected to more than one heat source, the solution can be constructed by superposing solutions for the structure subjected to one point heat source. Figure 13 - Figure 14 show that there are two heat sources in this structure and the full-field distributions are constructed by superposing the solutions in previous sections. Figure 13 show the case that the conductivities are discontinuous at the interfaces, and Figure 14 show the results that the conductivities are continuous at the interfaces.

In this section, from the contour plots, we can see the continuity characteristics at the interfaces. If the conductivities are continuous at the interfaces, all the fields are continuous at the interface, including the heat flux q_x. Moreover, the slopes of the contour plots for the temperature and heat flux q_y are continuous at the interfaces.

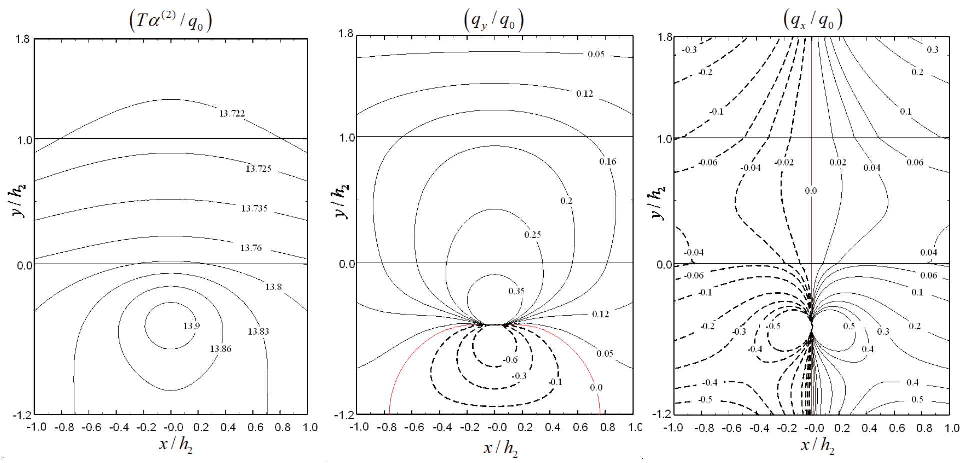

Figure 8: Full-field distribution of temperature, heat flux q_y and heat flux q_x for a three-layer media subjected to a heat source q_0 at $(0, -0.5h_2)$ with $\alpha^{(1)} = 1$, $\alpha^{(2)} = 1$, $\alpha^{(3)} = 3$, $\beta^{(1)} = -2$, $\beta^{(2)} = \ln 3 + 2$, $\beta^{(3)} = 2$ and $h_1 = 1.2$, $h_2 = 1$, $h_3 = 0.8$.

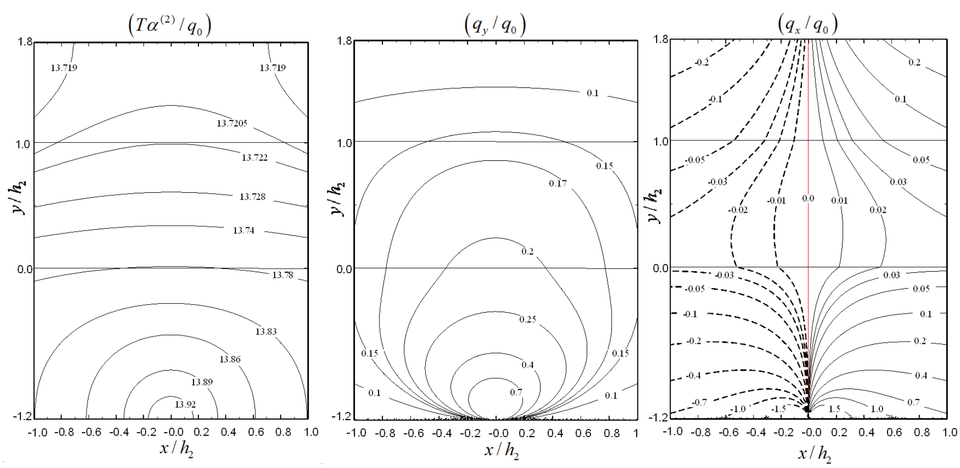

Figure 9: Full-field distribution of temperature heat flux q_y and heat flux q_x for a three-layer media subjected to a heat source q_0 at $(0, -h_1)$ with $\alpha^{(1)} = 1$, $\alpha^{(2)} = 1$, $\alpha^{(3)} = 3$, $\beta^{(1)} = -2$, $\beta^{(2)} = \ln 3 + 2$, $\beta^{(3)} = 2$ and $h_1 = 1.2$, $h_2 = 1$, $h_3 = 0.8$.

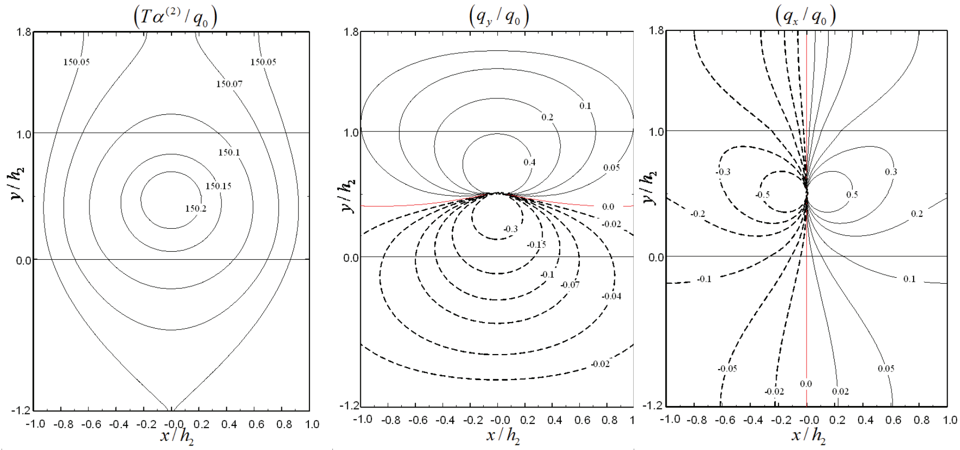

Figure 10: Full-field distribution of temperature, heat flux q_y and heat flux q_x for a three-layer media subjected to a heat source q_0 at $(0, 0.5h_2)$ with $\alpha^{(1)} = 1,\ \alpha^{(2)} = 1,\ \alpha^{(3)} = e,\ \beta^{(1)} = 0,\ \beta^{(2)} = 1,\ \beta^{(3)} = 0$ and $h_1 = 1.2,\ h_2 = 1,\ h_3 = 0.8$.

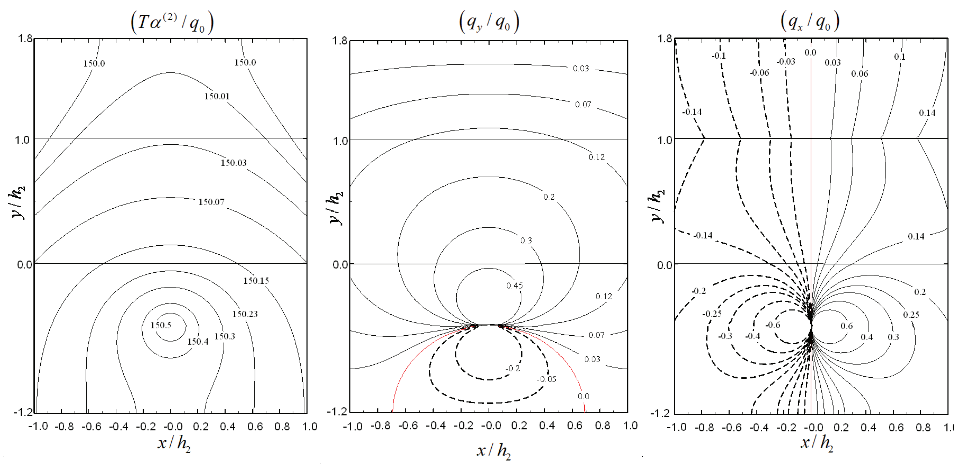

Figure 11: Full-field distribution of temperature, heat flux q_y and heat flux q_x for a three-layer media subjected to a heat source q_0 at $(0, -0.5h_2)$ with $\alpha^{(1)} = 1,\ \alpha^{(2)} = 1,\ \alpha^{(3)} = e,\ \beta^{(1)} = 0,\ \beta^{(2)} = 1,\ \beta^{(3)} = 0$ and $h_1 = 1.2,\ h_2 = 1,\ h_3 = 0.8$.

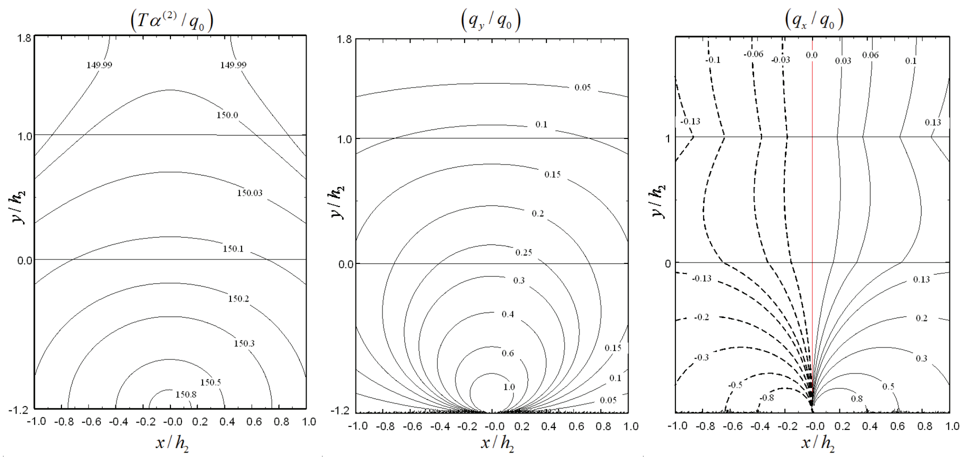

Figure 12: Full-field distribution of temperature, heat flux q_y and heat flux q_x for a three-layer media subjected to a heat source q_0 at $(0,-h_1)$ with $\alpha^{(1)}=1,\ \alpha^{(2)}=1,\ \alpha^{(3)}=e,\ \beta^{(1)}=0,\ \beta^{(2)}=1,\ \beta^{(3)}=0$ and $h_1=1.2,\ h_2=1,\ h_3=0.8$.

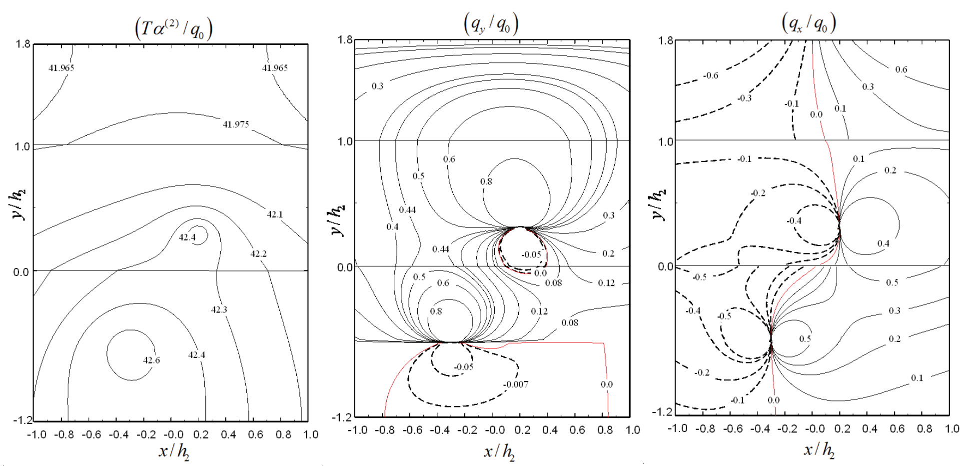

Figure 13: Full-field distribution of temperature, heat flux q_y and heat flux q_x for a three-layer media subjected to a heat source q_0 at $(0.2h_2, 0.3h_2)$and $(-0.3h_2,-0.6h_2)$ with $\alpha^{(1)}=5,\ \alpha^{(2)}=2,\ \alpha^{(3)}=1,\ \beta^{(1)}=2,\ \beta^{(2)}=1,\ \beta^{(3)}=3$ and $h_1=1.2,\ h_2=1,\ h_3=0.8$.

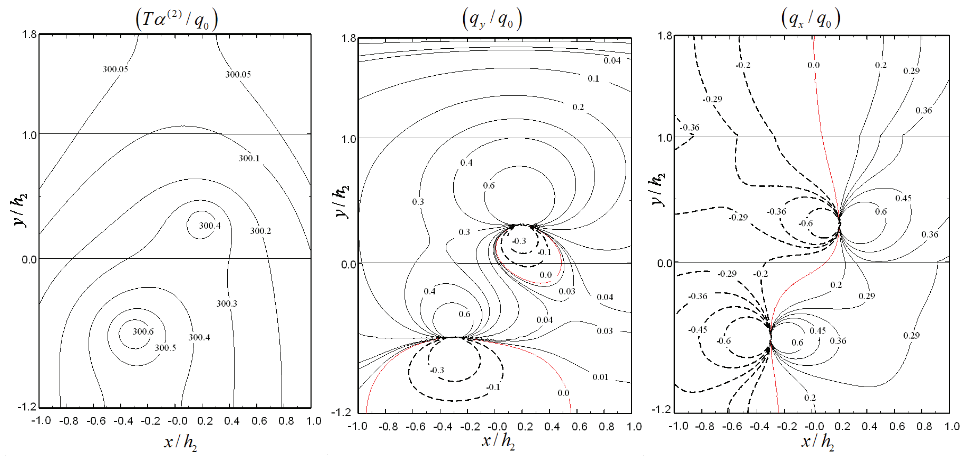

Figure 14: Full-field distribution of temperature, heat flux q_y and heat flux q_x for a three-layer media subjected to a heat source $q_0$at$(0.2h_2, 0.3h_2)$and $(-0.3h_2, -0.6h_2)$with $\alpha^{(1)} = 1$, $\alpha^{(2)} = 1$, $\alpha^{(3)} = e$, $\beta^{(1)} = 0$, $\beta^{(2)} = 1$, $\beta^{(3)} = 0$ and $h_1 = 1.2$, $h_2 = 1$, $h_3 = 0.8$.

7 Conclusions

In this study, a two-dimensional heat conduction problem of nonhomogeneous functionally graded materials with three layers subjected to a heat source is investigated. The conductivities are assumed to be exponential function of coordinates. From the Fourier transform method, the full-field solutions of nonhomogeneous functionally graded materials are presented. The analytical results presented in this study can be easily extended for the problems of multiple heat sources by superposition. If the functionally graded effect is neglected, the results are reduced to solutions of the homogeneous problem. A computational program for numerical calculation of the full field analysis is easily constructed using the analytical solutions. Detailed numerical results of full-field distributions of temperature and heat fluxes with different functionally graded parameters are presented and discussed. One of the objectives for this study is focused on the continuous characteristics of the field quantities at the interface. For the special cases that the conductivities are continuous at the interface, it is shown in this study that all the physical fields (i.e., temperature T, heat fluxes q_x and q_y) are continuous at the interface. Furthermore, the slopes of contour plots of the temperature and heat flux q_y are also continuous.

Acknowledgment

The authors want to express their sincere congratulation to professor Wen-Hwa Chen for receiving the Lifetime Achievement Medal and for his academic contribution on fracture mechanics and computational mechanics. The authors gratefully acknowledge Grant NSC 98-2923-E-002-003-MY3 from the National Science Council, Republic of China, to National Taiwan University.

References

Abu-Alshaikh, I.; Köklüce, B. (2006): One-dimensional transient dynamic response in functionally graded layered media. *J. Eng. Math.*, vol. 54, pp. 17-30.

Chen, W.Q.; Bian, Z.G.; Lv, C.F.; Ding, H.J. (2004): 3D free vibration analysis of a functionally graded piezoelectric hollow cylinder filled with compressible fluid. *Int. J. Solids Struct.*, vol.41, pp. 947-964.

Chiu, T.C.; Erdogan, F. (1999): One-dimensional wave propagation in a functionally graded elastic medium. *J. Sound Vibr.*, vol. 222, pp. 453-487.

Dondero, M.; Cisilino, A.P.; Tomba, J.P. (2011): Numerical design of random micro-heterogeneous materials with functionally-graded effective thermal conductivities using genetic algorithms and the fast boundary element method. *CMES-Comp. Model. Eng.*, vol. 78, pp. 225-245.

Dong, L.; Atluri, S.N. (2012): T-Trefftz Voronoi cell finite elements with elastic/rigid inclusions or voids for micromechanical analysis of composite and porous materials. *CMES-Comp. Model. Eng.*, vol. 83, pp. 183-219.

Durbin, F. (1974): Numerical inversion of Laplace transforms: an efficient improvement to Dubner and Abate's method. *Computer J.*, vol. 17, pp. 371-376.

Erdogan, F.; Wu, B.H. (1995): Crack problems in FGM layers under thermal stresses. *J. Therm. Stresses*, vol.19, pp. 237-265.

Jabbari, M.; Sohrabpour, S.; Eslami, M.R. (2002): Mechanical and thermal stresses in a functionally graded hollow cylinder due to radially symmetric loads. *Int. J. Pres. Ves. Pip.*, vol. 79, pp. 493-497.

Jin, Z.H.; Paulino, G. H. (2001): Transient thermal stress analysis of an edge crack in a functionally graded material. *Int. J. Fracture*, vol. 107, pp. 73-98.

Jin, Z.H.; Batra, R.C. (1996): Stresses intensity relaxation at the tip of an edge crack in a functionally graded material subjected to a thermal shock. *J. Therm. Stresses*, vol.19, pp. 317-339.

Lee, J.M.; Ma, C.C. (2010): Analytical solutions for an antiplane problem of two dissimilar functionally graded magnetoelectroelastic half-planes. *Acta Mechanica*, vol. 212, pp. 21-38.

Ma, C.C.; Lee, J.M. (2009a): Theoretical analysis of in-plane problem in functionally graded nonhomogeneous magnetoelectroelastic bimaterials. *Int. J. Solids Struct.*, vol. 46, pp. 4208-4220.

Ma, C.C.; Lee, J.M. (2009b): Full-field analysis of a functionally graded magnetoelectroelastic nonhomogeneous layered half-plane. *CMES-Comp. Model. Eng.*, vol.54, pp. 87-120.

Ma, C.C.; Lin, Y.H.; Lin, S.H. (2012): Transient wave propagation in a functionally graded slab and multilayered medium subjected to dynamic loadings. *CMC-Computers, Materials, & Continua*, vol.31, pp. 37-64.

Noda, N.; Jin, Z.H. (1994): A crack in functionally gradient materials under thermal shock *Archive of applied mechanics*, vol. 64, pp. 99-110.

Tokovyy, Y.V.; Ma, C.C. (2008): Analysis of 2D non-axisymmetric elasticity and thermoelasticity problems for radially inhomogeneous hollow cylinders. *J. Eng. Math.*, vol. 61, pp. 171–184.

Tokovyy, Y.V.; Ma, C.C. (2009): Analytical solutions to the 2D elasticity and thermoelasticity problems for inhomogeneous planes and half-planes. *Arch. Appl. Mech.*, vol. 79, pp. 441-456.

Wu, C.P.; Chiu, K.H.; Wang, Y.M. (2008): A review on the three-dimensional analytical approaches of multilayered and functionally graded piezoelectric plates and shells. *CMC-Computers, Materials, & Continua* , vol. 8, pp. 93-132.