Problems of Micromorphic Elastic Bodies Approached by Lagrange Identity Method

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Abstract: Taking advantage of the flexibility of Lagrange's identity, we prove the uniqueness theorem and some continuous dependence theorems without recourse to any energy conservation law, or to any boundedness assumptions on the constitutive coefficients. Also, we avoid the use of positive definiteness assumptions on the constitutive coefficients, even if these results are related to the difficult mixed problem in elasticity of micromorphic bodies.

Keywords: Lagrange identity; micromorphic; uniqueness; continuous dependence

1   Introduction

The micromorphic continuum theory is used to describe materials which possess a significant microstructure and therefore exhibit scale-dependent behaviour. These microstructures are viewed as so-called microcontinua, which are assumed to be attached to each physical point and may experience both stretch and rotation deformations which are affine throughout the microcontinuum, nevertheless kinematically independent from the deformation on the macroscale.

The micropolar and microstretch continua may be considered as special cases of the micromorphic theory, since here specific constraints apply on the deformation of the microcontinuum. For instance in the case of the micropolar continuum, the microcontinuum may only experience rotation

Not only the different microcontinuum theories are congeneric, additionally, close relations between the latter and other nonlocal theories exist. Particularly the micromorphic and the second-order gradient theory can be transferred into each other by limit considerations.

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Eringen was the one who initiated the theory of micromorphic elastic bodies, as a generalization of the theory of microstretch elastic bodies (see [Eringen (1972, 1990, 1999)]). Micromorphic theory envisions a material body as a continuous collection of deformable particle, each possesses finite size and inner structure, unlike classical continuum mechanics which considers a material body as a continuous collection of material points, each with infinitesimal size and no inner structure.

In micromorphic continua material particles are considered deformable. A material point of this body possess twelve degrees of freedom: three for the motion of particle, three for microrotations and six for the microdeformations of the particle.

This theory aims to eliminate discrepancies between classical elasticity and experiments, since the classical elasticity failed to present acceptable results when the effects of material microstructure were known to contribute significantly to the body's overall deformations, for example, in the case of granular bodies with large molecules (e.g. polymers), graphite or human bones.

Other intended applications of this theory are to composite materials reinforced with chopped fibers and various porous materials.

Recent results of the micromorphic theory has been applied to liquid crystals, theory of turbulence, blood, anisotropic fluids and suspensions (see, for instance, [Eringen (2005, 2009); Wang and Lee (2010)]).

After Eringen has established the theory of microstretch elastic solids, many other papers are concerned with this theory. For instance, Ciarletta in [Ciarletta (1995)] has used the basic results deduced by Eringen in order to investigate the isothermal bending of microstretch elastic plates. Iesan in [Iesan (2002)] prove the existence of a generalized solution in linear dynamic theory of micromorphic thermoelasticity.

The paper [Iesan (2011)] is concerned with the theory of micromorphic thermoelastic solids of degree 1. In the context of this theory, the author sets the equations governing the infinitesimal deformations superposed on large deformations at nonuniform temperature.

Also, the paper [Iesan and Nappa (2005)] of Iesan and Nappa deals with the problem of heat flow in a micromorphic continua. They obtain a new theory of heat for materials with inner structure that permits propagation of heat as thermal waves at finite speed. In our studies [Marin (2010)] and [Marin (2012)] we tackle some questions with regards to these materials.

Previous papers on uniqueness and continuous dependence in elasticity or thermoelasticity had been based almost exclusively on the assumptions that the elasticity tensor or thermoelastic coefficients are positive definite (see, for instance the paper [Wilkes (1980)]).

In other papers, the authors recourse to an energy conservation law, in order to

derive the uniqueness or continuous dependence of solutions. For instance, an uniqueness result was indicated in paper [Green and Laws (1972)] of Green and Laws by supplementing the restrictions arising from thermodynamics with certain definiteness assumptions.

The objective of our study is to examine by a new approach the mixed initial-boundary value problem in the context of elasticity of micromorphic solids. The approach is developed on the basis of Lagrange identity and its consequences. Therefore, we establish the uniqueness and continuous dependence of solutions with respect to body forces and body moments. We also deduce the continuous dependence of solutions of our mixed problem with respect to initial data and, at last, with respect to constitutive coefficients. The results are obtained for bounded regions of the Euclidian three dimensional space.

2   Basic equations

We assume that a bounded region B of three-dimensional Euclidian space R^3 is occupied by a micromorphic elastic body, referred to the reference configuration and a fixed system of rectangular Cartesian axes. Let $\bar{B}$ denote the closure of B and call ∂B the boundary of the domain B. We consider ∂B be a piecewise smooth surface and designate by n_i the components of the outward unit normal to the surface ∂B. Letters in boldface stand for vector fields. We use the notation v_i to designate the components of the vector **v** in the underlying rectangular Cartesian coordinates frame. Superposed dots stand for the material time derivative. We shall employ the usual summation and differentiation conventions: the subscripts are understood to range over integer $(1,2,3)$. Summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate.

The spatial argument and the time argument of a function will be omitted when there is no likelihood of confusion. We refer the motion of the body to a fixed system of rectangular Cartesian axes Ox_i, $i = 1,2,3$. Let us denote by u_i the components of the displacement vector and by φ_{ij} the components of the microdeformation tensor.

As usual, we denote by t_{ij} the components of the stress tensor, σ_{ij} the components of the microstress tensor and m_{kij} the components of the stress moment tensor over B.

The equations of motion in elasticity of micromorphic bodies are (see, for instance, [Iesan and Nappa (2005)]):

$$t_{ji,j} + \rho f_i = \rho \ddot{u}_i,$$
$$m_{kij,k} + t_{ji} - \sigma_{ji} + \rho l_{ij} = I_{jk} \ddot{\varphi}_{ik}, \tag{1}$$

For an anisotropic and homogeneous micromorphic elastic body, the constitutive equations have the form:

$$t_{ij} = A_{ijrs}\,\varepsilon_{rs} + E_{ijrs}\mu_{rs} + F_{ijrsp}\gamma_{rsp},$$
$$\sigma_{ij} = E_{rsij}\,\varepsilon_{rs} + B_{ijrs}\mu_{rs} + G_{ijrsp}\gamma_{rsp}, \tag{2}$$
$$m_{kij} = F_{rsijk}\,\varepsilon_{rs} + G_{rsijk}\mu_{rs} + C_{ijkrsp}\gamma_{rsp},$$

The components of the strain tensors ε_{ij}, μ_{ij} and γ_{ijk} are defined by means of the geometric equations:

$$\varepsilon_{ij} = u_{j,\,i} - \varphi_{ji}, \; \gamma_{ijk} = \varphi_{ij,\,k}$$
$$\mu_{ij} = \frac{1}{2}\left(\varphi_{ij} + \varphi_{ji}\right). \tag{3}$$

The above equations (1)-(3) are satisfied at all points $(x,t) \in B \times (0,\infty)$ and in these equations we have used the following notations: ρ the reference mass density; $I_{ij} = I_{ji}$ the components of microinertia tensor; f_i the components of body force vector; l_{ij} the components of body moment tensor; A_{ijrs}, B_{ijrs}, C_{ijkrsp}, E_{ijrs}, F_{ijrsp} and G_{ijrsp} are the characteristic constitutive coefficients and we assume that these coefficients satisfy the following symmetry relations

$$A_{ijrs} = A_{rsij}, \; B_{ijrs} = B_{rsij} = B_{jirs}, \; C_{ijkrsp} = C_{rspijk},$$
$$E_{rsij} = E_{rsji}, \; G_{ijrsp} = G_{jirsp}. \tag{4}$$

One can assume that a positive constant λ_0 exists such that

$$I_{ij}\xi_i\xi_j \geq \lambda_0\xi_i\xi_i, \; \forall \xi_i. \tag{5}$$

We denote by t_i the components of surface traction and by m_{ij} the components of surface moments, which at regular points of the surface ∂B are defined by $t_i = t_{ji}n_j$, $m_{ij} = m_{kij}n_k$, n_i are the components of the outward unit normal of the surface ∂B. Along with the system of equations (1)-(3) we consider the following initial conditions:

$$u_i(x,0) = a_i(x), \; \dot{u}_i(x,0) = b_i(x),$$
$$\varphi_{ij}(x,0) = \beta_{ij}(x), \; \dot{\varphi}_{ij}(x,0) = \phi_{ij}(x), \tag{6}$$

for $\in \bar{B}$, and the following boundary conditions (t_0 is some instant that may be infinite)

$$u_i = \bar{u}_i, \; \text{on} \; \partial B_1 \times [0,t_0), \; t_i = \bar{t}_i, \; \text{on} \; \partial B_1^c \times [0,t_0)$$
$$\varphi_{ij} = \bar{\varphi}_{ij}, \; \text{on} \; \partial B_2 \times [0,t_0), \; m_{ij} = \bar{m}_{ij}, \; \text{on} \; \partial B_2^c \times [0,t_0), \tag{7}$$

where $\bar{u}_i$, $\bar{t}_i$, $\bar{\varphi}_{ij}$ and $\bar{m}_{ij}$ are prescribed continuous functions on their domain of definition. Also, ∂B_1 and ∂B_2 with respective complements ∂B_1^c and ∂B_2^c are subsets of the surface ∂B such that

$$\partial B_1 \cap \partial B_1^c = \partial B_2 \cap \partial B_2^c = \emptyset$$

$$\partial B_1 \cup \partial B_1^c = \partial B_2 \cup \partial B_2^c = \partial B$$

We assume that a_i, b_i, β_{ij}, ϕ_{ij}, $\bar{u}_i$, $\bar{t}_i$ $\bar{\varphi}_{ij}$ and $\bar{m}_{ij}$ prescribed functions in their domains. Taking into account the constitutive equations (3), from (1) and (2) we obtain the following system of equations

$$
\begin{aligned}
\rho \ddot{u}_i &= \left(A_{ijrs}\varepsilon_{rs}\right)_{,j} + \left(E_{ijrs}\mu_{rs}\right)_{,j} + \left(F_{ijrsp}\gamma_{rsp}\right)_{,j} + \rho f_i, \\
I_{jk}\ddot{\varphi}_{ik} &= \left(F_{rsijk}\varepsilon_{rs}\right)_{,k} + \left(G_{rsijk}\mu_{rs}\right)_{,k} + \left(C_{ijkrsp}\gamma_{rsp}\right)_{,k} + A_{ijrs}\varepsilon_{rs} \\
&\quad + E_{ijrs}\left(\varepsilon_{rs} - \mu_{rs}\right) + \left(F_{ijrsp} - G_{ijrsp}\right)\gamma_{rsp} - B_{ijrs}\mu_{rs} + \rho l_{ij}
\end{aligned}
\tag{8}
$$

By a solution of the mixed initial boundary value problem of the theory of elasticity of micromorphic bodies in the cylinder $\Omega_0 = B \times [0, t_0)$ we mean an ordered array (u_i, φ_{ij}) which satisfies the system of equations (8) for all $(x, t) \in \Omega_0$, the boundary conditions (7) and the initial conditions (6).

3 Main result

Let us consider $f(t,x)$ and $g(t,x)$ two functions assumed to be twice continuously differentiable with respect to the time variable t. By direct calculations, it is easy to deduce that

$$\frac{d}{dt}\left(f\dot{g} - \dot{f}g\right) = \dot{f}\dot{g} + f\ddot{g} - \ddot{f}g - \dot{f}\dot{g} = f\ddot{g} - \ddot{f}g.$$

For the sake of simplicity, the spatial argument and the time argument of the functions $f(t,x)$ and $g(t,x)$ are omitted because there is no likelihood of confusion.

In the above equality, we substitute the functions $f(t,x)$ and $g(t,x)$ by the functions $U_i(x,t)$ and $V_i(x,t)$, which are, also, assumed to be twice continuously differentiable with respect to the time variable and then we obtain the following well known Lagrange's identity:

$$
\begin{aligned}
&\int_B \rho(x)\left[U_i(x,t)\dot{V}_i(x,t) - \dot{U}_i(x,t)V_i(x,t)\right]dV \\
&= \int_0^t \int_B \rho(x)\left[U_i(x,s)\ddot{V}_i(x,s) - \ddot{U}_i(x,s)V_i(x,s)\right]dV\,ds \\
&\quad + \int_B \rho(x)\left[U_i(x,0)\dot{V}_i(x,0) - \dot{U}_i(x,0)V_i(x,0)\right]dV
\end{aligned}
\tag{9}
$$

Let us denote by $(u_i^{(\alpha)}, \varphi_{ij}^{(\alpha)})$, $(\alpha = 1, 2)$ two solutions of the mixed initial boundary value problem defined by (8) (6) and (7) which correspond to the same boundary and initial data, but to different body forces and body moments, $(f_i^{(\alpha)}, l_{ij}^{(\alpha)})$, $(\alpha = 1, 2)$. We introduce the following notations:

$$v_i = u_i^{(2)} - u_i^{(1)}, \ \psi_{ij} = \varphi_{ij}^{(2)} - \varphi_{ij}^{(1)} \tag{10}$$

We are now in position to prove first basic result.

Theorem 1. *For the differences* (v_i, ψ_{ij}) *of two solutions of the mixed initial boundary value problem* (8), (6) *and* (7), *the Lagrange identity becomes*:

$$2 \int_B \left[\rho v_i(t) \dot{v}_i(t) + I_{jk} \psi_{ij}(t) \dot{\psi}_{ik}(t) \right] dV$$

$$= \int_0^t ds \int_B \rho \left[v_i(2t-s)\mathscr{F}_i(s) - v_i(s)\mathscr{F}_i(2t-s) + \right. \tag{11}$$

$$\left. + \psi_{ij}(2t-s)\mathscr{L}_{ij}(s) - \psi_{ij}(s)\mathscr{L}_{ij}(2t-s) \right] dV$$

where we have used the notations

$$\mathscr{F}_i = f_i^{(2)} - f_i^{(1)}, \ \mathscr{L}_{ij} = l_{ij}^{(2)} - l_{ij}^{(1)}. \tag{12}$$

Proof. Clearly, since of the linearity of the problem defined by (8), (6) and (7), we deduce that the differences (v_i, ψ_{ij}) from (10) represent the solution of a mixed initial boundary value problem anallogous to (8), (6) and (7) in which the loads are $\mathscr{F}_i$ and $\mathscr{L}_{ij}$ from (12), but the corresponding initial conditions and the corresponding boundary conditions become homogeneous. By setting

$$U_i(x,s) = v_i(x,s), \ V_i(x,s) = v_i(x, 2t-s), \ s \in [0, 2t], \ t \in [0, t_0/2),$$

then the identity (9), after some straightforward calculus, becomes

$$2 \int_B \rho v_i(t) \dot{v}_i(t) dV = \int_0^t ds \int_B \rho [v_i(2t-s)\ddot{v}_i(s) - \ddot{v}_i(2t-s)v_i(s)] dV \tag{13}$$

where we have used the fact that the initial and boundary data are null. If we employ the symmetry conditions of the coefficients I_{ij} such that we can write

$$I_{jk} \frac{d}{dt} \left[W_{ik}(t) \dot{\psi}_{ij}(t) - \dot{W}_{ik}(t) \psi_{ij}(t) \right] = I_{jk} \left[W_{ik}(t) \ddot{\psi}_{ij}(t) - \ddot{W}_{ik}(t) \psi_{ij}(t) \right].$$

Because the differences satisfy null initial data, the following identity is obtained

$$\int_B I_{jk} \frac{d}{dt} \left[W_{ik}(t) \dot{\psi}_{ij}(t) - \dot{W}_{ik}(t) \psi_{ij}(t) \right] dV = \int_0^t \int_B I_{jk} \left[W_{ik}(t) \ddot{\psi}_{ij}(t) - \ddot{W}_{ik}(t) \psi_{ij}(t) \right] dV ds.$$

In this equality we use the substitution

$$W_i(s) \rightarrow \psi_{ij}(2t - s), \ s \in [0, 2t], \ t \in [0, t_0/2)$$

which leads us to the equality

$$2 \int_B I_{jk} \psi_{ik}(t) \psi_{ij}(t) dV = \int_0^t \int_B I_{jk} [\psi_{ik}(2t-s) \ddot{\psi}_{ij}(s) - \ddot{\psi}_{ik}(2t-s) \psi_{ij}(s)] dV ds. \tag{14}$$

We will add member by member relations (13) and (14) thus obtained

$$2 \int_B \left[\rho v_i(t) \dot{v}_i(t) + I_{jk} \psi_{ik}(t) \psi_{ij}(t) \right] dV$$
$$= \int_0^t \int_B \{ \rho [v_i(2t-s) \ddot{v}_i(s) - \ddot{v}_i(2t-s) v_i(s)] + \tag{15}$$
$$+ I_{jk} [\psi_{ik}(2t-s) \ddot{\psi}_{ij}(s) - \ddot{\psi}_{ik}(2t-s) \psi_{ij}(s)] \} dV ds.$$

We shall eliminate the inertial terms on the right-hand side of the relation (15) by means of the equations of motion for the differences (v_i, ψ_{ij}). So, in view of the equation $(1)_1$, we have

$$\rho [v_i(2t - s) \ddot{v}_i(s) - \ddot{v}_i(2t - s) v_i(s)]$$
$$= [v_i(2t-s)t_{ji}(s) - v_i(s)t_{ji}(2t-s)]_{,j}$$
$$+ [t_{ji}(2t-s)v_{j,i}(s) - t_{ji}(s)v_{j,i}(2t-s)] \tag{16}$$
$$+ \rho [\mathscr{F}_i(s)v_i(2t - s) - \mathscr{F}_i(2t - s)v_i(s)]$$

Taking into account the equation (1), we have

$$I_{jk}[\psi_{ik}(2t-s) \ddot{\psi}_{ij}(s) - \ddot{\psi}_{ik}(2t-s) \psi_{ij}(s)]$$
$$= \psi_{ik}(2t-s) [m_{jik,j}(s) + t_{ki}(s) - \sigma_{ki}(s) + \rho l_{ki}(s)]$$
$$- \psi_{ij}(s) [m_{kij,k}(2t-s) + t_{ij}(2t-s) - \sigma_{ij}(2t-s) + \rho l_{ij}(2t-s)]$$
$$= [\psi_{ik}(2t-s)m_{jik}(s) - \psi_{ik}(s)m_{jik}(2t-s)]_{,j} \tag{17}$$
$$- m_{jik}(s)\gamma_{jik}(2t+s) + m_{jik}(2t-s)\gamma_{jik}(s)$$
$$+ t_{ki}(s)\psi_{ik}(2t-s) - t_{ij}(2t-s)\psi_{ij}(s) + \sigma_{ij}(2t-s)\psi_{ij}(s)$$
$$- \sigma_{ki}(s)\psi_{ik}(2t-s) + \rho [\mathscr{L}_{ij}(s)\psi_{ij}(2t-s) - \mathscr{L}_{ij}(2t-s)\psi_{ij}(s)]$$

If we add the relations (16) and (17) we are led to the equality

$$\rho \left[v_i(2t-s)\ddot{v}_i(s) - \ddot{v}_i(2t-s)v_i(s) \right]$$
$$+ I_{jk} \left[\psi_{ik}(2t-s)\ddot{\psi}_{ij}(s) - \ddot{\psi}_{ik}(2t-s)\psi_{ij}(s) \right]$$
$$= \left[v_i(2t-s)t_{ji}(s) - v_i(s)t_{ji}(2t-s) \right]_{,j}$$
$$+ \left[\psi_{ik}(2t-s)m_{jik}(s) - \psi_{ik}(s)m_{jik}(2t-s) \right]_{,j}$$
$$- m_{jik}(s)\gamma_{jik}(2t-s) + m_{jik}(2t-s)\gamma_{jik}(s) \tag{18}$$
$$+ t_{ij}(2t-s)\varepsilon_{ij}(s) - t_{ij}(s)\varepsilon_{ij}(2t-s)$$
$$+ \sigma_{ij}(2t-s)\mu_{ij}(s) - \sigma_{ij}(s)\mu_{ij}(2t-s)$$
$$+ \rho \left[\mathscr{F}_i(s)v_i(2t-s) - \mathscr{F}_i(2t-s)v_i(s) \right]$$
$$+ \rho \left[\mathscr{L}_{ij}(s)\psi_{ij}(2t-s) - \mathscr{L}_{ij}(2t-s)\psi_{ij}(s) \right]$$

In the right side of the identity (18) we use geometric equations (2) and then obtain

$$\rho \left[v_i(2t-s)\ddot{v}_i(s) - \ddot{v}_i(2t-s)v_i(s) \right]$$
$$+ I_{jk} \left[\psi_{ik}(2t-s)\ddot{\psi}_{ij}(s) - \ddot{\psi}_{ik}(2t-s)\psi_{ij}(s) \right]$$
$$= \left[v_i(2t-s)t_{ji}(s) - v_i(s)t_{ji}(2t-s) \right]_{,j}$$
$$+ \left[\psi_{ik}(2t-s)m_{jik}(s) - \psi_{ik}(s)m_{jik}(2t-s) \right]_{,j}$$
$$+ \rho \left[\mathscr{F}_i(s)v_i(2t-s) - \mathscr{F}_i(2t-s)v_i(s) \right]$$
$$+ \rho \left[\mathscr{L}_{ij}(s)\psi_{ij}(2t-s) - \mathscr{L}_{ij}(2t-s)\psi_{ij}(s) \right]$$
$$- F_{rsjik}\varepsilon_{rs}(s)\gamma_{jik}(2t-s) - G_{rsjik}\mu_{rs}(s)\gamma_{jik}(2t-s)$$
$$- C_{jikrsp}\gamma_{rsp}(s)\gamma_{jik}(2t-s) + F_{rsjik}\varepsilon_{rs}(2t-s)\gamma_{jik}(s) \tag{19}$$
$$+ G_{rsjik}\mu_{rs}(2t-s)\gamma_{jik}(s) + C_{jikrsp}\gamma_{rsp}(2t-s)\gamma_{jik}(s)$$
$$+ A_{ijrs}\varepsilon_{rs}(2t-s)\varepsilon_{ij}(s) + E_{ijrs}\mu_{rs}(2t-s)\varepsilon_{ij}(s)$$
$$+ F_{ijrsp}\gamma_{rs}(2t-s)\varepsilon_{ij}(s) - A_{ijrs}\varepsilon_{rs}(s)\varepsilon_{ij}(2t-s)$$
$$- E_{ijrs}\mu_{rs}(s)\varepsilon_{ij}(2t-s) - F_{ijrsp}\gamma_{rsp}(s)\varepsilon_{ij}(2t-s)$$
$$+ E_{rsij}\varepsilon_{rs}(2t-s)\mu_{ij}(s) + B_{ijrs}\mu_{rs}(2t-s)\mu_{ij}(s)$$
$$+ G_{ijrsp}\gamma_{rsp}(2t-s)\mu_{ij}(s) - E_{rsij}\varepsilon_{rs}(s)\mu_{ij}(2t-s)$$
$$- B_{ijrs}\mu_{rs}(s)\mu_{ij}(2t-s) - G_{ijrsp}\gamma_{rsp}(s)\mu_{ij}(2t-s)$$

In identity (19) we use the symmetry relations (4) so that it becomes

$$
\begin{aligned}
&\rho\left[v_i(2t-s)\ddot{v}_i(s) - \ddot{v}_i(2t-s)v_i(s)\right] \\
&+ I_{jk}\left[\psi_{ik}(2t-s)\ddot{\psi}_{ij}(s) - \ddot{\psi}_{ik}(2t-s)\psi_{ij}(s)\right] \\
&= \left[v_i(2t-s)t_{ji}(s) - v_i(s)t_{ji}(2t-s)\right]_{,j} \\
&+ \left[\psi_{ik}(2t-s)m_{jik}(s) - \psi_{ik}(s)m_{jik}(2t-s)\right]_{,j} \\
&+ \rho\left[\mathscr{F}_i(s)v_i(2t-s) - \mathscr{F}_i(2t-s)v_i(s)\right] \\
&+ \rho\left[\mathscr{L}_{ij}(s)\psi_{ij}(2t-s) - \mathscr{L}_{ij}(2t-s)\psi_{ij}(s)\right]
\end{aligned}
\tag{20}
$$

Substituting the result of (20) in equality (15) we use the divergence theorem then take into account the fact that we have null initial data and null boundary data. Thus we obtain equality (11) and the proof of Theorem 1 is complete. With the help of identity (11) we will prove the uniqueness of solution of the mixed initial-boundary value problem which consists of equations (8), boundary data (7) and initial data (6).

Thus, in the following Theorem 2 we first prove the uniqueness of solution of the mentioned problem.

Theorem 2. *Assume that the symmetry relations* (4) *are satisfied. Then the mixed initial-boundary value problem for micromorphic elastic body has at most one solution.*

Proof. Suppose, by contrary, that our problem has two solution $(u_i^{(\alpha)},\ \varphi_{ij}^{(\alpha)})$, $\alpha = 1, 2$. If we denote by

$$
v_i = u_i^{(2)} - u_i^{(1)}, \ \psi_{ij} = \varphi_{ij}^{(2)} - \varphi_{ij}^{(1)}
\tag{21}
$$

then we should prove that

$$
v_i(x,t) = \psi_{ij}(x,t) = 0, \ \forall\ (x,t) \in B \times [0,t_0).
\tag{22}
$$

Because of linearity, the differences defined in (21) also represent a solution to our problem, but corresponding to null body force and null body moment. In this particular case, the identity (11) received the form

$$
\int_B \left[\rho v_i(t)\dot{v}_i(t) + I_{jk}\psi_{ij}(t)\dot{\psi}_{ik}(t)\right] dV = 0.
$$

If we integrate this equality on the interval $[0,\tau]$, $\tau \in [0,\ t_0/2)$ we are led to

$$
\int_B \left[\rho v_i(\tau)v_i(\tau) + I_{jk}\psi_{ij}(\tau)\psi_{ik}(\tau)\right] dV = 0
\tag{23}
$$

Because of the conditions imposed to the density ρ and microinertia I_{ij}, identity (23) ensures that

$$v_i(x,t) = \psi_{ij}(x,t) = 0, \ \forall \ (x,t) \in B \times [0,t_0/2]. \tag{24}$$

If t_0 is infinite, then the proof of Theorem 2 is completed. Where t_0 is finite, proof of Theorem 2 continues as follows. We set

$$v_i(x,t_0/2) = \dot{v}_i(x,t_0/2) = 0, \ \psi_{ij}(x,t_0/2) = \dot{\psi}_{ij}(x,t_0/2) = 0$$

and repeat the above procedure on the interval $[t_0/2, \ t_0/2 + t_0/4]$ so we can extend the conclusion (24) on $B \times [0, 3t_0/4)$ and so on. As a result the differences $(v_i(x,t), \ \psi_{ij}(x,t))$ are null on $B \times [0,t_0)$ and this concludes the proof of Theorem 2. In the next step we can formulate and prove a result on continuous dependence with regard to body force and body moment. This feature refers to the solution of the mixed initial boundary value problem consisting in system of differential equations (8), initial conditions (6) and the boundary conditions (7).

Theorem 3. *Let* $(u_i^{(\alpha)}, \ \varphi_{ij}^{(\alpha)})$ *($\alpha = 1, 2$) be two solutions of the above mixed problem which correspond to the same initial and boundary data but to different body force and body moment* $(f_i^{(\alpha)}, \ l_{ij}^{(\alpha)})$ *($\alpha = 1, 2$). We will use the notation*

$$\mathscr{F}_i = f_i^{(2)} - f_i^{(1)}, \ \mathscr{L}_{ij} = l_{ij}^{(2)} - l_{ij}^{(1)}.$$

If there is $t_* \in (0,t_0)$ *such that,*

$$\int_0^{t_*} \int_B \rho \mathscr{F}_i(t)\mathscr{F}_i(t)dVdt \le M_1^2, \ \int_0^{t_*}\int_B \rho \mathscr{L}_{ij}(t)\mathscr{L}_{ij}(t)dVdt \le M_1^2$$
$$\int_0^{t_*}\int_B \rho u_i(t)u_i(t)dVdt \le K^2, \ \int_0^{t_*}\int_B I_{jk}\varphi_{ik}(t)\varphi_{ij}(t)dVdt \le M^2 \tag{25}$$

then we have the following estimate of solutions (for $\tau \in [0,t_0/2)$*)*

$$\int_B \left[\rho v_i(\tau)v_i(\tau) + I_{jk}\psi_{ij}(\tau)\psi_{ik}(\tau) \right] dV$$
$$\le t_* K \left[\int_0^{t_*} \int_B \rho \mathscr{F}_i(t)\mathscr{F}_i(t)dVdt \right]^{1/2} + t_* M \left[\int_0^{t_*} \int_B \rho \mathscr{L}_{ij}(t)\mathscr{L}_{ij}(t)dVdt \right]^{1/2} \tag{26}$$

Proof. The proof will be based on the identity (11). For each integral in the right-side of this identity, we apply the Schwarz's inequality.

For instance

$$
\int_0^t ds \int_B \rho v_i(2t-s)\mathscr{F}_i(s)dVds
$$

$$
\leq \left[\int_0^t \int_B \rho \mathscr{F}_i(\tau)\mathscr{F}_i(\tau)dVd\tau\right]^{1/2} \left[\int_0^t \int_B \rho u_i(2t-\tau)u_i(2t-\tau)dVd\tau\right]^{1/2}
$$

$$
= \left[\int_0^t \int_B \rho \mathscr{F}_i(\tau)\mathscr{F}_i(\tau)dVd\tau\right]^{1/2} \left[\int_t^{2t} \int_B \rho u_i(\tau)u_i(\tau)dVd\tau\right]^{1/2} \tag{27}
$$

$$
\leq K \left[\int_0^{t_*} \int_B \rho \mathscr{F}_i(\tau)\mathscr{F}_i(\tau)dVd\tau\right]^{1/2}
$$

where, at last, we have used the substitution $2t - \tau \to \tau$. Proceeding in a similar manner to second integral in the right-side of identity (11), we will obtain an estimate similar to (27). Finally, we integrate the resulting inequality over $[0, \tau]$, $\tau \in [0, t_*/2]$ and we are led to the estimate (26) which conclude Theorem 3. Now, we will state and prove a continuous dependence result with regard to initial data.

Theorem 4. *Suppose that the symmetry relations* (4) *are satisfied and consider*

$$
\left(u_i^{(1)},\ \varphi_{ij}^{(1)}\right),\ \left(u_i^{(1)} + v_i,\ \varphi_{ij}^{(1)} + \psi_{ij}\right)
$$

two solutions of our mixed problem which correspond to the same body force and body moment, to the same boundary data, but to different initial data

$$
\left(a_i^1,\ b_i^1,\ \beta_{ij}^1,\ \phi_{ij}^1\right),\ \left(a_i^2,\ b_i^2,\ \beta_{ij}^2,\ \phi_{ij}^2\right)
$$

where

$$
a_i^2 = a_i^1 + \mathscr{A}_i,\ b_i^2 = b_i^1 + \mathscr{B}_i,\ \beta_{ij}^2 = \beta_{ij}^1 + \mathscr{C}_{ij},\ \phi_{ij}^2 = \phi_{ij}^1 + \mathscr{D}_{ij}
$$

Perturbations $(\mathscr{A}_i,\ \mathscr{B}_i,\ \mathscr{C}_{ij},\ \mathscr{D}_{ij})$ *are subject to the following restrictions*

$$
\int_B \rho\left(\mathscr{A}_i\mathscr{A}_i + \mathscr{B}_i\mathscr{B}_i\right)dV \leq M_3^2,\ \int_B \rho\left(\mathscr{C}_{ij}\mathscr{C}_{ij} + \mathscr{D}_{ij}\mathscr{D}_{ij}\right)dV \leq M_4^2.
$$

We define the functions $U_i(x,t),\ \Phi_{ij}(x,t)$ *by*

$$
U_i(x,t) = \int_0^t\int_0^s v_i(x,\tau)d\tau ds,\ \Phi_{ij}(x,t) = \int_0^t\int_0^s \psi_{ij}(x,\tau)d\tau ds \tag{28}
$$

If the functions $U_i(x,t)$, $\Phi_{ij}(x,t)$ satisfy the restrictions (25)$_2$, then is valid the estimate

$$\int_B \left[\rho U_i(t) U_i(t) + I_{jk} \Phi_{ij}(t) \Phi_{ik}(t) \right] dV$$

$$\leq t_* K \left[\left(t_* + \frac{t_*}{2} \right) \int_B \rho \mathscr{A}_i \mathscr{A}_i dV + \left(\frac{t_*^2}{2} + \frac{t_*^3}{3} \right) \int_B \rho \mathscr{B}_i \mathscr{B}_i dV \right]^{1/2} \tag{29}$$

$$+ t_* M \left[\left(t_* + \frac{t_*}{2} \right) \int_B I_{jk} \mathscr{C}_{ik} \mathscr{C}_{ij} dV + \left(\frac{t_*^2}{2} + \frac{t_*^3}{3} \right) \int_B dV + I_{jk} \mathscr{D}_{ik} \mathscr{D}_{ij} dV \right]^{1/2}$$

Proof. If we integrate by parts the integrals in (28) we obtain

$$U_i(x,t) = \int_0^t (t-s) v_i(x,s) ds, \; \Phi_{ij}(x,t) = \int_0^t (t-s) \psi_{ij}(x,s) ds \tag{30}$$

Clarly, the difference functions $(v_i(x,t), \psi_{ij}(x,t))$ satisfy the equations of motion as in (8), but with null body force and null body moment $f_i = l_{ij} = 0$. Also, it is easy to prove that the difference functions satisfy the following initial conditions

$$v_i(x,0) = \mathscr{A}_i(x), \; \dot{v}_i(x,0) = \mathscr{B}_i(x),$$
$$\psi_{ij}(x,0) = \mathscr{C}_{ij}(x), \; \dot{\psi}_{ij}(x,0) = \mathscr{D}_{ij}(x).$$

By direct calculations, we deduce that the functions $(v_i(x,t), \psi_{ij}(x,t))$ defined in (28) satisfy the equations of motion as in (8), but with the following body force and body moment

$$f_i(x,t) = \mathscr{A}_i(x) + t \mathscr{B}_i(x),$$
$$l_{ij}(x,t) = \mathscr{C}_{ij}(x) + t \mathscr{D}_{ij}(x)$$

These latest specification enables us to conclude that the estimate (29) is obtained from estimate (26) of Theorem 3 such that Theorem 4 is concluded.

A further consequence of Theorem 3 is the following continuous dependence result of the solution to the problem (8), (7) and (6) upon constitutive coefficients.

Theorem 5. *Suppose that the symmetry relations* (4) *are satisfied and consider*

$$\left(u_i^{(1)}, \varphi_{ij}^{(1)} \right), \; \left(u_i^{(1)} + v_i, \varphi_{ij}^{(1)} + \psi_{ij} \right)$$

two solutions of our mixed problem which correspond to the same body force and body moment, to the same boundary data, to the same initial data, but to different

constitutive coefficients, respectively

$$\left(A_{ijrs}^{(1)},\ E_{ijrs}^{(1)},\ F_{ijrsp}^{(1)},\ B_{ijrs}^{(1)},\ G_{ijrsp}^{(1)},\ C_{ijkrsp}^{(1)} \right)$$

$$\left(A_{ijrs}^{(1)} + \mathscr{A}_{ijrs}, E_{ijrs}^{(1)} + \mathscr{E}_{ijrs}, F_{ijrsp}^{(1)} + \mathscr{F}_{ijrsp}, \right.$$

$$\left. B_{ijrs}^{(1)} + \mathscr{B}_{ijrs}, G_{ijrsp}^{(1)} + \mathscr{G}_{ijrsp}, C_{ijkrsp}^{(1)} + \mathscr{C}_{ijkrsp} \right)$$

If the difference functions $(v_i(x,t),\ \psi_{ij}(x,t))$ satisfy the restrictions (25), then any solution $(u_i(x,t),\ \varphi_{ij}(x,t))$ of the initial boundary value problem for elastic micromorphic bodies that satisfies the condition

$$\int_0^{t_*} \int_B \Big(u_{i,j} u_{i,j} + u_{i,jk} u_{i,jk} + \dot{u}_{i,j} \dot{u}_{i,j} +$$

$$+ \varphi_{ij,k} \varphi_{ij,k} + \varphi_{ij,km} \varphi_{ij,km} + \dot{\varphi}_{ij,k} \dot{\varphi}_{ij,k} \Big)\, dV\, dt \le M_5^2$$

depends continuously on th constitutive coefficients on interval $[0,\ t_*/2]$ with regard to the measure

$$\int_B \Big[\rho v_i(t) v_i(t) + I_{jk} \psi_{ik}(t) \psi_{ij}(t) \Big]\, dV$$

Proof. By direct calculations we are led to the conclusion that the differences of two solution of our mixed problem, that is $(v_i(x,t),\ \psi_{ij}(x,t))$, satisfy null initial conditions, null boundary conditions and the equations of motion (8) in which the constitutive coefficients are actually

$$\left(A_{ijrs}^{(1)},\ E_{ijrs}^{(1)},\ F_{ijrsp}^{(1)},\ B_{ijrs}^{(1)},\ G_{ijrsp}^{(1)},\ C_{ijkrsp}^{(1)} \right)$$

Also, the body force ρf_i and body moment ρl_{ij}, in the equations of motion, have the following expressions

$$\rho f_i = \left(\mathscr{A}_{ijrs} \varepsilon_{rs}^{(2)} + \mathscr{E}_{ijrs} \mu_{rs}^{(2)} + \mathscr{F}_{ijrsp} \gamma_{rsp}^{(2)} \right)_{,j}$$

$$\rho l_{ij} = \left(\mathscr{F}_{rsijk} \varepsilon_{rs}^{(2)} + \mathscr{G}_{rsijk} \mu_{rs}^{(2)} + \mathscr{C}_{ijkrsp} \gamma_{rsp}^{(2)} \right)_{,j}$$

$$+ \mathscr{A}_{ijrs} \varepsilon_{rs}^{(2)} + \mathscr{E}_{ijrs} \mu_{rs}^{(2)} + \mathscr{F}_{ijrsp} \gamma_{rsp}^{(2)}$$

$$- \mathscr{E}_{rsij} \varepsilon_{rs}^{(2)} - \mathscr{B}_{ijrs} \mu_{rs}^{(2)} - \mathscr{G}_{ijrsp} \gamma_{rsp}^{(2)}$$

$\varepsilon_{rs}^{(2)}$, $\mu_{rs}^{(2)}$ and $\gamma_{rsp}^{(2)}$ are the strain tensors which correspond to the second considered solution, that is,

$$\left(u_i^{(1)} + v_i,\ \varphi_{ij}^{(1)} + \psi_{ij} \right)$$

Considering these specifications, we deduce that the problem has become analogous to the problem of Theorem 4. As a consequence, taking into account the estimates (29) and (26) we are led to the desired result and the proof of Theorem 5 is completed.

4   Conclusion

Usualy, the uniqueness and continuous dependence results, with regard to solutions of mixed initial boundary value problem, are obtained imposing strong restrictions. In our study these results are obtained without recourse to any conservation law or to boundedness assumptions on the constitutive coefficients. Also, we avoid to use the hypothesis that constitutive coefficients are positive definite tensors.

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