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Anti-plane Circular Nano-inclusion Problem with Electric Field Gradient and Strain Gradient Effects

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Abstract: As well known, gradient theories can describe size effects that are important in nano-scale problems. In this paper, we analyze the Eshelby-type anti-plane inclusion problem embedded in infinite dielectric body by considering both strain gradient and electric field gradient effects to account for the size effect and high-order electromechanical coupling effect. The size-dependent Eshelby and Eshelby-like tensor, strain, stress, electric field and electric displacement components are derived explicitly by means of Green's function method. Theoretical results indicate that strain and electric field are decoupled for anti-plane inclusion problem while stress field and electric displacement are coupled through strain gradient and electric field gradient. Based on the general form, the expressions for a special case of circular inclusion are obtained analytically. Numerical results reveal that when the inclusion radius becomes small, the gradient effects are significantly important and should not be ignored. The values approach asymptotically to classical solutions as increase of inclusion size. And the high-order electromechanical coupling effect in non-piezoelectric material (centrosymmetric dielectrics) can be equivalent to piezoelectricity of conditional piezoelectric materials when the inclusion size is small.

Keywords: Strain gradient effect, Electric field gradient effect, Nano-inclusion, Electromechanical coupling.

1 Introduction

Developments of nanotechnology in recent decades motivate progress in research of nanocomposites. Nano-scale inclusion, as an essential part of nanocomposites, has been investigated through various theories which are proposed to account for size effect [Wang, Pan and Feng (2008)]. To name a few, couple stress elasticity [Wang, Pan and Feng (2008); Lubarda (2003)], micropolar theory [Cheng and

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He (1995)], strain gradient elasticity [Zhang and Sharma (2005)], electric gradient theory [Yang , Zhou and Li (2006); Mindlin (1968)], surface theory [Sharma and Ganti (2004); Yang, Hu and Shen (2012)] and so on. Herein special attention is focused on gradient theories.

Recently, Gao and Park (2007) provided explicit variational formulation for a simplified strain gradient elasticity theory (SSGET) which contains only one material length scale parameter. Then size-dependent Eshelby tensors of spherical, plane strain, cylindrical, ellipsoidal and anti-plane inclusion based on SSGET were obtained in a series of articles [Gao and Ma (2009); Ma and Gao (2010); Gao and Ma (2010, 2012)]. Yang , Zhou and Li (2006) analyzed the effect of electric field gradient in anti-plane problem of a circular inclusion in polarized ceramics. These individual gradient theories promote researchers to combine strain gradient and electric field gradient theory into a unified framework. Hu and Shen (2009) proposed a variational principle considering both the electric field gradient and strain gradient, as well as surface effect and electrostatic force for nano-dielectrics. The theoretical framework can describe size effect, electromechanical coupling and surface effect simultaneously. Basing on polarization gradient and strain gradient, Shen and Hu (2010) developed a framework which took flexoelectric effect, surface effect and electrostatic force into account. Flexoelectric effect [Cross (2006); Zubko, Catalan and Tagantsev (2013)], defined as coupling of strain gradient and polarization (direct effect) and coupling of polarization gradient and strain (inverse effect), have been investigated extensively on inclusion [Maranganti, Sharma and Sharma (2006); Sharma, Maranganti and Sharma (2007)], beam [Yan and Jiang (2013); Liang and Shen (2013)], thin-film [Sharma, Landis and Sharma (2010)] and other topics [Liu, Hu and Shen (2012), (2014); Xu, Hu and Shen (2013)]. It is emphasized that polarization gradient and electric field gradient can alternatively be chosen to represent flexoelectric effect [Ma (2008)]. Moreover, electric field gradient shows lower symmetry, and this character makes process and results of solving problems simpler and more explicit than polarization gradient consideration. Until now, Eshelby-type nano-inclusion problem including both strain gradient and electric field gradient effect has not been studied yet. In this study, we analyze anti-plane nano-scale inclusion problem with electric field gradient and strain gradient effects.

The outline of this paper is as follows. In Sec.2 the general formulation is given for isotropic centrosymmetric dielectrics. Then anti-plane nano-inclusion problem is analyzed in Eshelby's frame by means of Green's function method in Sec.3. Sec.4 gives the analytical expression for anti-plane circular inclusion with corresponding numerical results in Sec.5. Finally, a few concluding remarks are provided in Sec.6.

2 General formulas

The governing equations and boundary conditions including strain/electric field gradient effect and surface effect [Hu and Shen (2009)] are derived through the variational principle. Refer to [Hu and Shen (2009)], the general expression for electric Gibbs free energy density function involving the gradients of strain and electric field can be written as:

$$
\begin{aligned}
U_b = & -\frac{1}{2}a_{kl}E_kE_l - \frac{1}{2}b_{ijkl}E_{i,j}E_{k,l} + \frac{1}{2}c_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - e_{ijkl}\varepsilon_{ij}E_{k,l} - d_{ijk}\varepsilon_{ij}E_k \\
& - h_{ijk}E_iE_{j,k} - f_{ijkl}E_iu_{j,kl} + r_{ijklm}\varepsilon_{ij}u_{k,lm} - \eta_{ijkmn}E_{i,j}u_{k,mn} + \frac{1}{2}g_{ijklmn}u_{i,jk}u_{l,mn}
\end{aligned} \tag{1}
$$

Within the assumption of centrosymmetric dielectrics, the odd tensors will vanish according to Neumann principle. The expression of U_b can be simplified as:

$$
\begin{aligned}
U_b = & -\frac{1}{2}a_{kl}E_kE_l - \frac{1}{2}b_{ijkl}E_{i,j}E_{k,l} + \frac{1}{2}c_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - e_{ijkl}\varepsilon_{ij}E_{k,l} \\
& - f_{ijkl}E_iu_{j,kl} + \frac{1}{2}g_{ijklmn}u_{i,jk}u_{l,mn}
\end{aligned} \tag{2}
$$

where **a** and **c** are the classical second-order dielectric permittivity tensor and four-order elasticity constant tensors respectively. **f** is the flexoelectric tensor while **e** is the converse flexoelectric tensor according to definitions in Shu et al. (2011). **b** and **g** represent purely electric field and elastic nonlocal effect respectively. **u** and **E** are the displacement and the electric field tensor, while the subscript comma indicates differentiation with respect to the spatial variables. Additionally $\boldsymbol{\varepsilon}$ is the strain tensor defined as usual.

According to Hu and Shen (2009), the constraints relation are defined as:

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{3}$$

$$E_i = -\phi_{,i} \tag{4}$$

$$w_{ijm} = \varepsilon_{ij,m} = \frac{1}{2}(u_{i,jm} + u_{j,im}) \tag{5}$$

$$V_{ij} = E_{i,j} = -\phi_{,ij} \tag{6}$$

where ϕ is the electrostatic potential. **w** and **V** are the strain gradient tensor and the electric field gradient tensor respectively. The symmetry of the tensors are showed as: $\varepsilon_{ij} = \varepsilon_{ji}$, $w_{ijm} = w_{jim}$, $V_{ij} = V_{ji}$.

Under the infinitesimal deformation assumption, the constitutive equations can be expressed as:

$$\sigma_{ij} = \frac{\partial U_b}{\partial \varepsilon_{ij}} = c_{ijkl}\varepsilon_{kl} - e_{ijkl}v_{kl} \tag{7}$$

$$\tau_{ijm} = \frac{\partial U_b}{\partial w_{ijm}} = -f_{kijm}E_k + g_{ijmknl}w_{knl} \tag{8}$$

$$D_i = -\frac{\partial U_b}{\partial E_i} = a_{ij}E_j + f_{ijkl}w_{jkl} \tag{9}$$

$$Q_{ij} = -\frac{\partial U_b}{\partial V_{ij}} = b_{ijkl}v_{kl} + e_{klij}\varepsilon_{kl} \tag{10}$$

The governing equations are written as:

$$\left(\sigma_{ij} - \tau_{ijm,m}\right)_{,j} + F_i = 0 \tag{11}$$

$$(D_i - Q_{ij,j})_{,i} - \rho = 0 \tag{12}$$

F_i and ρ are external body force and body electric charge, respectively. By substituting Eqs. (7)-(10) into Eq. (11) and Eq. (12), the general governing equations based on **u** and ϕ can be derived as:

$$c_{ijkl}u_{k,lj} + e_{ijkl}\phi_{,klj} - f_{kijm}\phi_{,kmj} - g_{ijmknl}u_{k,nlmj} + f_i = 0 \tag{13}$$

$$-a_{ij}\phi_{,ji} + f_{ijkl}u_{j,kli} + b_{ijkl}\phi_{,klji} - e_{klij}u_{k,lji} - \rho = 0 \tag{14}$$

For isotropic dielectric material, the coefficients **a**, **b**, **c** and **g** are listed as follows:

$$\left.\begin{array}{l} a_{ij} = a\delta_{ij} \\ c_{ijkl} = c_{12}\delta_{ij}\delta_{kl} + c_{44}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \\ b_{ijkl} = b_{12}\delta_{ij}\delta_{kl} + b_{44}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \\ g_{ijmknl} = c_{12}l^2\delta_{ij}\delta_{kn}\delta_{lm} + c_{44}l^2(\delta_{ik}\delta_{jn}\delta_{lm} + \delta_{in}\delta_{jk}\delta_{lm}) \end{array}\right\} \tag{15}$$

The symmetries of **c** and **g** are same with coefficients in Ref. [Gao and Ma (2009)], where c_{12}, c_{44} are the lame constants and l^2 having the dimension of length squared is the strain gradient coefficient introduced for isotropic elasticity in Ref. [Gao and Park (2007)]. Namely, l is characteristic length of materials which may be related to atomic distance, lattice size, microstructure or others. Refer to [Yang, Zhou and Li (2006)], **b** has the same symmetry as **c** as required by symmetry of materials. The non-zero components of **a** are reduced to a_{11}, a_{22} and a_{33}, which also have the relation $a_{11} = a_{22} = a_{33} = a$ due to isotropy. For flexoelectric tensor **f** and converse flexoelectric tensor **e**, Shu et al. (2011) and Quang and He (2011) detailedly investigated the symmetry of the flexoelectric tensors (direct and converse) and the number of independent material parameters for a given symmetry class in different ways. The all non-zero components of **e** and **f** for cubic crystals can be listed as below [Shu et al. (2011)]:

$$\begin{aligned}
&e_{1111} = e_{2222} = e_{3333} \equiv e_{11} \\
&e_{1122} = e_{2211} = e_{1133} = e_{3311} = e_{2233} = e_{3322} \equiv e_{12} \\
&e_{1221} = e_{1212} = e_{1331} = e_{1313} = e_{2332} = e_{2323} \\
&= e_{2121} = e_{2112} = e_{3131} = e_{3113} = e_{3232} = e_{3223} \equiv e_{44}
\end{aligned} \tag{16}$$

$$\begin{aligned}
&f_{1111} = f_{2222} = f_{3333} \equiv f_{11} \\
&f_{1122} = f_{2121} = f_{1133} = f_{3131} = f_{2233} = f_{3232} \\
&= f_{1212} = f_{2211} = f_{1313} = f_{3311} = f_{2323} = f_{3322} \equiv f_{12} \\
&f_{1221} = f_{2112} = f_{1331} = f_{3113} = f_{2332} = f_{3223} \equiv f_{44}
\end{aligned} \tag{17}$$

Herein, e_{11}, e_{12}, e_{44} and f_{11}, f_{12}, f_{44} are the independent converse flexoelectric and flexoelectric constants respectively. For isotropic materials, the number of independent parameters will reduce from three to two. In other words, there exists a relationship between these non-zero independent components [Shu et al. (2011)] as: $f_{44}+2f_{12}=f_{11}$, $e_{12}+2e_{44}=e_{11}$. This will be used in the following derivations.

3 Anti-plane problem

3.1 Green's function solution

For an anti-plane problem, the displacement and electric potential components are represented by:

$$\left.\begin{aligned}
&u_1 = u_2 = 0 \quad u_3 = u_3(x_1,x_2) \\
&\phi = \phi(x_1,x_2)
\end{aligned}\right\} \tag{18}$$

By substituting Eqs. (15)-(18) into Eq. (13) and Eq. (14), after some simplification and calculation, the final governing equations for anti-plane problems can be expressed as:

$$\nabla^2 u_3 - l^2 \nabla^2 \nabla^2 u_3 = -\frac{F_3}{c_{44}} \tag{19}$$

$$\nabla^2 \phi - \frac{b_{11}}{a} \nabla^2 \nabla^2 \phi = -\frac{\rho}{a}. \tag{20}$$

By using two-dimensional (2-D) Fourier transforms [Gao and Ma (2012)], the Green's function for the above two equations can be easily obtained as, respectively:

$$G^1(\mathbf{x}) = -\frac{1}{2\pi c_{44}}[In|\mathbf{x}| + K_0(\frac{|\mathbf{x}|}{l})] \tag{21}$$

$$G^2(\mathbf{x}) = -\frac{1}{2\pi a}[In|\mathbf{x}| + K_0(\frac{|\mathbf{x}|}{m})] \tag{22}$$

where $|\mathbf{x}| = \sqrt{x_1^2 + x_2^2}$. We define $m = \sqrt{\frac{b_{11}}{a}}$, which has the unit of length. Physically similar with l, m represents the characteristic length depending on what structural effects of materials we focus on. $K_0(\cdot)$ is the modified Bessel function of the second kind of the zeroth order.

3.2 Solutions of inclusion problem

Consider an infinite homogeneous isotropic dielectric body containing an arbitrary shape Eshelby-type inclusion [Eshelby (1957); Mura (1987)]. The body is absence of both body force and body charge but with eigenstrain $\boldsymbol{\varepsilon}^*(\mathbf{x})$, eigenstrain gradient $\mathbf{w}^*(\mathbf{x})$, eigenelectric field $\mathbf{E}^*(\mathbf{x})$, and eigenelectric field gradient $\mathbf{V}^*(\mathbf{x})$ prescribed only inside the inclusion. On account of anti-plane problem, the non-vanishing components of eigenfields are as follows:

$$\left.\begin{array}{l} \varepsilon^*_{\alpha\beta} = \varepsilon^*_{\beta\alpha} = 0,\ \varepsilon^*_{33} = 0,\ \varepsilon^*_{3\alpha} = \varepsilon^*_{\alpha 3} \neq 0, \\ w^*_{\alpha\beta i} = w^*_{\beta\alpha i} = 0,\ w^*_{33i} = 0,\ w^*_{3\alpha\beta} = w^*_{\alpha 3\beta} \neq 0, \\ E^*_\alpha \neq 0,\ E^*_3 = 0, \\ V^*_{\alpha\beta} \neq 0,\ V^*_{\alpha 3} = 0,\ V^*_{3i} = 0. \end{array}\right\} \tag{23}$$

Following Eshelby's formalism for inclusion problem, the constitutive equations can be revised to:

$$\begin{array}{l} \sigma_{ij} = c_{ijkl}(\varepsilon_{kl} - \varepsilon^*_{kl}) - e_{ijkl}(V_{kl} - V^*_{kl}) \\ \tau_{ijm} = -f_{kijm}(E_k - E^*_k) + g_{ijmknl}(w_{knl} - w^*_{knl}) \\ D_i = a_{ij}(E_j - E^*_j) + f_{ijkl}(w_{jkl} - w^*_{jkl}) \\ Q_{ij} = b_{ijkl}(V_{kl} - V^*_{kl}) + e_{klij}(\varepsilon_{kl} - \varepsilon^*_{kl}) \end{array} \tag{24}$$

Substituting Eq. (24) into Eqs. (11)-(12) then leads to:

$$\begin{array}{l} c_{ijkl}u_{k,lj} + e_{ijkl}\phi_{,klj} - f_{kijm}\phi_{,kmj} - g_{ijmknl}u_{k,nlmj} \\ = c_{ijkl}\varepsilon^*_{kl,j} - e_{ijkl}V^*_{kl,j} + f_{kijm}E^*_{k,mj} - g_{ijmknl}w^*_{knl,mj} \end{array} \tag{25}$$

$$\begin{array}{l} -a_{ij}\phi_{,ji} + f_{ijkl}u_{j,kli} + b_{ijkl}\phi_{,klji} - e_{klij}u_{k,lji} \\ = a_{ij}E^*_{j,i} + f_{ijkl}w^*_{jkl,i} - b_{ijkl}V^*_{kl,ji} - e_{klij}\varepsilon^*_{kl,ji} \end{array} \tag{26}$$

Similarly, by using Eq. (15)-(17) and Eq. (23), Eq. (25) and (26) can be simplified as:

$$\nabla^2 u_3 - l^2\nabla^2\nabla^2 u_3 = 2(\varepsilon^*_{31,1} + \varepsilon^*_{32,2}) - 2l^2(w^*_{311,11} + w^*_{312,12} + w^*_{321,21} + w^*_{322,22}) \tag{27}$$

$$\nabla^2\phi - \frac{b_{11}}{a}\nabla^2\nabla^2\phi = -(E^*_{1,1}+E^*_{2,2}) + \frac{b_{12}}{a}(V^*_{11,11}+V^*_{22,11}+V^*_{11,22}+V^*_{22,22}) + \frac{2b_{44}}{a}(V^*_{11,11}+V^*_{12,21}+V^*_{21,12}+V^*_{22,22}) \tag{28}$$

By comparing Eq.(27) and (28) with Eq.(19) and (20), it can be said that, body force F_3 is replaced by:

$$F_3 = -2c_{44}(\varepsilon^*_{31,1}+\varepsilon^*_{32,2}) + 2c_{44}l^2(w^*_{311,11}+w^*_{312,12}+w^*_{321,21}+w^*_{322,22})] \tag{29}$$

and body electric charge is equivalent to:

$$\rho = a\left(E^*_{1,1}+E^*_{2,2}\right) - b_{12}\left(V^*_{11,11}+V^*_{22,11}+V^*_{11,22}+V^*_{22,22}\right) - 2b_{44}\left(V^*_{11,11}+V^*_{12,21}+V^*_{21,12}+V^*_{22,22}\right) \tag{30}$$

According to the definition of Green's function, the solution of Eq. (27) can be expressed in terms of $G^1(\mathbf{x}-\mathbf{y})$ and $F_3(\mathbf{y})$ as:

$$\begin{aligned} u_3(\mathbf{x}) &= \int_{-\infty}^{+\infty} G^1(\mathbf{x}-\mathbf{y})F_3 d\mathbf{y} \\ &= -2c_{44}\int_{-\infty}^{+\infty} G^1(\mathbf{x}-\mathbf{y})(\varepsilon^*_{31,1}+\varepsilon^*_{32,2})d\mathbf{y} \\ &+ 2c_{44}l^2\int_{-\infty}^{+\infty} G^1(\mathbf{x}-\mathbf{y})(w^*_{311,11}+w^*_{312,12}+w^*_{321,21}+w^*_{322,22})d\mathbf{y} \end{aligned} \tag{31}$$

By integrating (31) by parts, and using the divergence theorem and the relation: $\frac{\partial G^i(\mathbf{y}-\mathbf{x})}{\partial \mathbf{y}} = -\frac{\partial G^i(\mathbf{y}-\mathbf{x})}{\partial \mathbf{x}}$, we have:

$$\begin{aligned} u_3(\mathbf{x}) = &-2c_{44}\int_{-\infty}^{+\infty}\left[\frac{\partial G^1(\mathbf{x}-\mathbf{y})}{\partial x_1}\varepsilon^*_{31} + \frac{\partial G^1(\mathbf{x}-\mathbf{y})}{\partial x_2}\varepsilon^*_{32}\right]d\mathbf{y} \\ &+ 2c_{44}l^2\left[\int_{-\infty}^{+\infty} w^*_{311}\frac{\partial^2 G^1(\mathbf{x}-\mathbf{y})}{\partial^2 x_1}d\mathbf{y} + \int_{-\infty}^{+\infty}(w^*_{312}+w^*_{321})\frac{\partial^2 G^1(\mathbf{x}-\mathbf{y})}{\partial x_1\partial x_2}d\mathbf{y} \right. \\ &\left. + \int_{-\infty}^{+\infty} w^*_{322}\frac{\partial^2 G^1(\mathbf{x}-\mathbf{y})}{\partial^2 x_2}d\mathbf{y}\right] \end{aligned} \tag{32}$$

Now the specific case of uniform eigenstrain $\varepsilon^*(\mathbf{x})$, uniform eigenstrain gradient $\mathbf{w}^*(\mathbf{x})$, uniform eigenelectric field $\mathbf{E}^*(\mathbf{x})$, and uniform eigenelectric field gradient

$\mathbf{V}^*(\mathbf{x})$ inside the inclusion occupied by the domain Ω_I are considered. Taking into account this fact, Eq. (32) can be further derived to:

$$\begin{aligned} u_3(\mathbf{x}) = & -2c_{44}[\varepsilon^*_{31}\frac{\partial}{\partial x_1}\int\limits_{\Omega_I} G^1(\mathbf{x}-\mathbf{y})\mathrm{d}\mathbf{y} + \varepsilon^*_{32}\frac{\partial}{\partial x_2}\int\limits_{\Omega_I} G^1(\mathbf{x}-\mathbf{y})\mathrm{d}\mathbf{y}] \\ & + 2c_{44}l^2[w^*_{311}\frac{\partial^2}{\partial^2 x_1}\int\limits_{\Omega_I} G^1(\mathbf{x}-\mathbf{y})\mathrm{d}\mathbf{y} + (w^*_{312}+w^*_{321})\frac{\partial^2}{\partial x_1 \partial x_2}\int\limits_{\Omega_I} G^1(\mathbf{x}-\mathbf{y})\mathrm{d}\mathbf{y} \\ & + w^*_{322}\frac{\partial^2}{\partial^2 x_2}\int\limits_{\Omega_I} G^1(\mathbf{x}-\mathbf{y})\mathrm{d}\mathbf{y}] \end{aligned} \tag{33}$$

Denote $\int\limits_{\Omega_I} G^1(\mathbf{x}-\mathbf{y})\mathrm{d}\mathbf{y} = \langle G^1 \rangle$, and the symbol $\langle \bullet \rangle_{,i}$ indicates differentiation with respect to component of $\mathbf{x}$. Thus Eq. (33) is rewritten as:

$$\begin{aligned} u_3(\mathbf{x}) = & -2c_{44}(\varepsilon^*_{31}\langle G^1 \rangle_{,1} + \varepsilon^*_{32}\langle G^1 \rangle_{,2}) \\ & + 2c_{44}l^2(w^*_{311}\langle G^1 \rangle_{,11} + (w^*_{312}+w^*_{321})\langle G^1 \rangle_{,12} + w^*_{322}\langle G^1 \rangle_{,22}) \end{aligned} \tag{34}$$

Basing on the constraints relation, the strain field can be readily obtained as:

$$\begin{aligned} & \varepsilon_{13}(\mathbf{x}) = \varepsilon_{31}(\mathbf{x}) = \frac{1}{2}u_{3,1}(\mathbf{x}) \\ & = S_{1331}\varepsilon^*_{31} + S_{1332}\varepsilon^*_{32} + T_{13311}w^*_{311} + T_{13312}w^*_{312} + T_{13321}w^*_{321} + T_{13322}w^*_{322} \end{aligned} \tag{35}$$

$$\begin{aligned} & \varepsilon_{23}(\mathbf{x}) = \varepsilon_{32}(\mathbf{x}) = \frac{1}{2}u_{3,2}(\mathbf{x}) \\ & = S_{2331}\varepsilon^*_{31} + S_{2332}\varepsilon^*_{32} + T_{23311}w^*_{311} + T_{23312}w^*_{312} + T_{23321}w^*_{321} + T_{23322}w^*_{322} \end{aligned}$$

where:

$$\begin{aligned} & S_{1331} = -c_{44}\langle G^1 \rangle_{,11} \quad S_{1332} = -c_{44}\langle G^1 \rangle_{,21} = S_{2331} \quad S_{2332} = -c_{44}\langle G^1 \rangle_{,22} \\ & T_{13311} = c_{44}l^2\langle G^1 \rangle_{,111} \quad T_{13312} = T_{13321} = c_{44}l^2\langle G^1 \rangle_{,121} = T_{23311} \\ & T_{23322} = c_{44}l^2\langle G^1 \rangle_{,222} \quad T_{23312} = T_{23321} = c_{44}l^2\langle G^1 \rangle_{,221} = T_{13322} \end{aligned} \tag{36}$$

Similarly, the solution of Eq. (28) is given as follows:

$$
\begin{aligned}
\phi(\mathbf{x}) &= \int G^2(\mathbf{x}-\mathbf{y})\rho \mathrm{d}\mathbf{y} \\
&= a\int G^2(\mathbf{x}-\mathbf{y})(E^*_{1,1}+E^*_{2,2})\mathrm{d}\mathbf{y} - b_{12}\int G^2(\mathbf{x}-\mathbf{y})(V^*_{11,11}+V^*_{22,11}+V^*_{11,22}+V^*_{22,22})\mathrm{d}\mathbf{y} \\
&\quad - 2b_{44}\int G^2(\mathbf{x}-\mathbf{y})(V^*_{11,11}+V^*_{12,21}+V^*_{21,12}+V^*_{22,22})\mathrm{d}\mathbf{y} \\
&= aE^*_1\langle G^2\rangle_{,1} + aE^*_2\langle G^2\rangle_{,2} - b_{12}(V^*_{11}+V^*_{22})(\langle G^2\rangle_{,11}+\langle G^2\rangle_{,22}) \\
&\quad - 2b_{44}[V^*_{11}\langle G^2\rangle_{,11} + (V^*_{12}+V^*_{21})\langle G^2\rangle_{,12} + V^*_{22}\langle G^2\rangle_{,22}]
\end{aligned}
\tag{37}
$$

In which, $\int_{\Omega_I} G^2(\mathbf{x}-\mathbf{y})\mathrm{d}\mathbf{y} = \langle G^2\rangle$. Using the constraints relation yields:

$$
\begin{aligned}
E_1 = -\phi_{,1}(\mathbf{x}) = M_{11}E^*_1 + M_{12}E^*_2 + N_{111}V^*_{11} + N_{112}V^*_{12} + N_{121}V^*_{21} + N_{122}V^*_{22} \\
E_2 = -\phi_{,2}(\mathbf{x}) = M_{21}E^*_1 + M_{22}E^*_2 + N_{211}V^*_{11} + N_{212}V^*_{12} + N_{221}V^*_{21} + N_{222}V^*_{22}
\end{aligned}
\tag{38}
$$

where:

$$
\begin{aligned}
&M_{11} = -a\langle G^2\rangle_{,11} \quad M_{12} = -a\langle G^2\rangle_{,12} = M_{21} \quad M_{22} = -a\langle G^2\rangle_{,22} \\
&N_{111} = (b_{12}+2b_{44})\langle G^2\rangle_{,111} + b_{12}\langle G^2\rangle_{,221} \\
&N_{122} = (b_{12}+2b_{44})\langle G^2\rangle_{,221} + b_{12}\langle G^2\rangle_{,111} \\
&N_{112} = N_{121} = 2b_{44}\langle G^2\rangle_{,121} \quad N_{211} = (b_{12}+2b_{44})\langle G^2\rangle_{,112} + b_{12}\langle G^2\rangle_{,222} \\
&N_{222} = (b_{12}+2b_{44})\langle G^2\rangle_{,222} + b_{12}\langle G^2\rangle_{,112} \quad N_{212} = N_{221} = 2b_{44}\langle G^2\rangle_{,122}
\end{aligned}
\tag{39}
$$

From Eq. (35) and Eq. (38), we can explicitly see that strain field and electric filed are uncoupled for this specific anti-plane problem. In other words, strain field is only induced by the prescribed eigenstrain and eigenstrain gradient, while electric filed is just linked to eigenelectric field and eigenelectric field gradient. Based on constitutive laws Eq. (24), the expressions of stress field $\sigma(\mathbf{x})$ and electric displacement $\mathbf{D}(\mathbf{x})$ can be obtained simply. Unlike the classical results of anti-plane problem where only shear stress and in-plane electric displacement exist, all the components of stress and electric displacement show up in this case. They are listed below as:

$$
\left.
\begin{aligned}
&\sigma_{11}(\mathbf{x}) = -e_{11}(V_{11}(\mathbf{x}) - V^*_{11}) - e_{12}(V_{22}(\mathbf{x}) - V^*_{22}) \\
&\sigma_{12}(\mathbf{x}) = -2e_{44}(V_{12}(\mathbf{x}) - V^*_{12}) \\
&\sigma_{13}(\mathbf{x}) = 2c_{44}(\varepsilon_{13}(\mathbf{x}) - \varepsilon^*_{13}) \\
&\sigma_{22}(\mathbf{x}) = -e_{12}(V_{11}(\mathbf{x}) - V^*_{11}) - e_{11}(V_{22}(\mathbf{x}) - V^*_{22}) \\
&\sigma_{23}(\mathbf{x}) = 2c_{44}(\varepsilon_{23}(\mathbf{x}) - \varepsilon^*_{23}) \\
&\sigma_{33}(\mathbf{x}) = -e_{12}(V_{11}(\mathbf{x}) + V_{22}(\mathbf{x}) - V^*_{11} - V^*_{22})
\end{aligned}
\right\}
\tag{40}
$$

$$\left.\begin{array}{l} D_1 = a(E_1(\mathbf{x}) - E_1^*) \\ D_2 = a(E_2(\mathbf{x}) - E_2^*) \\ D_3 = 2f_{12}(w_{311}(\mathbf{x}) + w_{322}(\mathbf{x}) - w_{311}^* - w_{322}^*) \end{array}\right\} \tag{41}$$

Furthermore we notice that the "new" stress components are induced merely by the corresponding electric field gradient components, and the "new" electric displacement components are only caused by the strain gradient.

4 An anti-plane circular inclusion

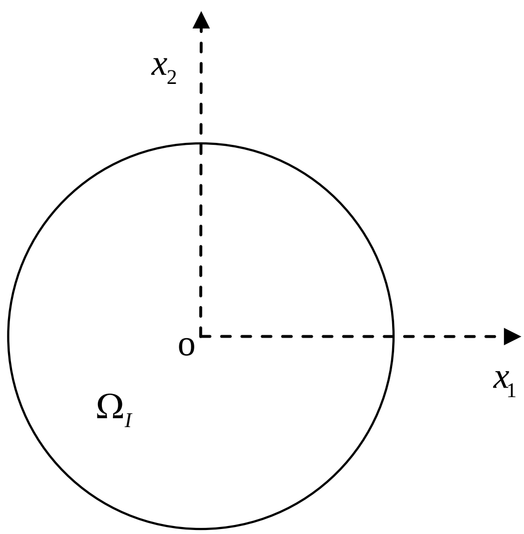

Figure 1: Schematic of a circular inclusion.

Consider a circular inclusion of radius R with a domain Ω_I as show in Fig.1. The analytical expressions of integral of Green's function are already given in Gao and Ma (2012) as:

$$\left\langle G^1 \right\rangle = -\frac{1}{c_{44}}\left\{(\frac{1}{2}R^2 InR - \frac{R^2}{2} + \frac{x^2}{2}) + lxK_0(\frac{x}{l})I_1(\frac{x}{l}) + lI_0(\frac{x}{l})[-RK_1(\frac{R}{l}) + xK_1(\frac{x}{l})]\right\} \tag{42}$$

For $\mathbf{x}$ inside the inclusion, and

$$\left\langle G^1 \right\rangle = -\frac{R}{c_{44}}[\frac{1}{2}RInx + lK_0(\frac{x}{l})I_1(\frac{R}{l})] \tag{43}$$

For **x** outside the inclusion, we can obtain analogously

$$\langle G^2\rangle = -\frac{1}{a}\left\{(\frac{1}{2}R^2InR - \frac{R^2}{2} + \frac{x^2}{2}) + lxK_0(\frac{x}{m})I_1(\frac{x}{m}) + mI_0(\frac{x}{m})[-RK_1(\frac{R}{m}) + xK_1(\frac{x}{m})]\right\} \tag{44}$$

for **x** inside the inclusion, and

$$\langle G^2\rangle = -\frac{R}{a}[\frac{1}{2}RInx + mK_0(\frac{x}{m})I_1(\frac{R}{m})] \tag{45}$$

for **x** outside the inclusion. Here $x = |\mathbf{x}| = \sqrt{x_1^2 + x_2^2}$, I_0, I_1, K_0, K_1 are the modified Bessel function of the first kind of the zeroth order, the first kind of the first order, the second kind of the zeroth order, and the second kind of first order respectively. When $\langle G\rangle$ is only the function of the distance x, differential relations of each order using chain rule are shown in Appendix A. Then fourth-order Eshelby tensor and the fifth-order Eshelby-like tensor for elastic field are written as:

$$\begin{aligned} S_{1331} &= -\frac{c_{44}}{x}[\frac{x_1^2}{x}\langle G^1\rangle'' + (1 - \frac{x_1^2}{x^2})\langle G^1\rangle'] \\ S_{1332} &= -\frac{c_{44}x_1x_2}{x^2}(\langle G^1\rangle'' - \frac{1}{x}\langle G^1\rangle') = S_{2331} \\ S_{2332} &= -\frac{c_{44}}{x}[\frac{x_2^2}{x}\langle G^1\rangle'' + (1 - \frac{x_2^2}{x^2})\langle G^1\rangle'] \end{aligned} \tag{46}$$

and

$$\begin{aligned} T_{13311} &= \frac{c_{44}l^2}{x^3}(x_1^3\langle G^1\rangle^{(3)} + 3x_1x(1 - \frac{x_1^2}{x^2})\langle G^1\rangle'' + 3x_1(\frac{x_1^2}{x^2} - 1)\langle G^1\rangle') \\ T_{13312} = T_{13321} = T_{23311} &= \frac{c_{44}l^2}{x^3}(x_1^2x_2\langle G^1\rangle^{(3)} + xx_2(1 - \frac{3x_1^2}{x^2})\langle G^1\rangle'' + x_2(\frac{3x_1^2}{x^2} - 1)\langle G^1\rangle') \\ T_{23312} = T_{23321} = T_{13322} &= \frac{c_{44}l^2}{x^3}(x_2^2x_1\langle G^1\rangle^{(3)} + xx_1(1 - \frac{3x_2^2}{x^2})\langle G^1\rangle'' + x_1(\frac{3x_2^2}{x^2} - 1)\langle G^1\rangle') \\ T_{23322} &= \frac{c_{44}l^2}{x^3}(x_2^3\langle G^1\rangle^{(3)} + 3x_2x(1 - \frac{x_2^2}{x^2})\langle G^1\rangle'' + 3x_2(\frac{x_2^2}{x^2} - 1)\langle G^1\rangle') \end{aligned} \tag{47}$$

The second order Eshelby tensor and the third order Eshelby-like tensor are shown as:

$$\begin{aligned} M_{11} &= -\frac{a}{x}[\frac{x_1^2}{x}\langle G^2\rangle'' + (1 - \frac{x_1^2}{x^2})\langle G^2\rangle'] \\ M_{12} &= -\frac{ax_1x_2}{x^2}(\langle G^2\rangle'' - \frac{1}{x}\langle G^2\rangle') = M_{21} \\ M_{22} &= -\frac{a}{x}[\frac{x_2^2}{x}\langle G^2\rangle'' + (1 - \frac{x_2^2}{x^2})\langle G^2\rangle'] \end{aligned} \tag{48}$$

and

$$\begin{aligned}
&N_{111} = \tfrac{1}{x^3}[(b_{11}x_1^3 + b_{12}x_1x_2^2)\left\langle G^2\right\rangle^{(3)} + xx_1(3b_{11} + b_{12} - 3\tfrac{b_{11}x_1^2+b_{12}x_2^2}{x^2})\left\langle G^2\right\rangle^{''} \\
&+x_1(3\tfrac{b_{11}x_1^2+b_{12}x_2^2}{x^2} - 3b_{11} - b_{12})\left\langle G^2\right\rangle^{'}] \\
&N_{122} = \tfrac{1}{x^3}[(b_{12}x_1^3 + b_{11}x_1x_2^2)\left\langle G^2\right\rangle^{(3)} + xx_1(3b_{12} + b_{11} - 3\tfrac{b_{12}x_1^2+b_{11}x_2^2}{x^2})\left\langle G^2\right\rangle^{''} \\
&+x_1(3\tfrac{b_{12}x_1^2+b_{11}x_2^2}{x^2} - 3b_{12} - b_{11})\left\langle G^2\right\rangle^{'}] \\
&N_{112} = \tfrac{2b_{44}}{x^3}[x_1^2x_2\left\langle G^2\right\rangle^{(3)} + xx_2(1 - \tfrac{3x_1^2}{x^2})\left\langle G^2\right\rangle^{''} + x_2(\tfrac{3x_1^2}{x^2} - 1)\left\langle G^2\right\rangle^{'}] = N_{121} \\
&N_{211} = \tfrac{1}{x^3}[(b_{11}x_2^3 + b_{12}x_2x_1^2)\left\langle G^2\right\rangle^{(3)} + xx_2(3b_{11} + b_{12} - 3\tfrac{b_{11}x_2^2+b_{12}x_1^2}{x^2})\left\langle G^2\right\rangle^{''} \\
&+x_2(3\tfrac{b_{11}x_2^2+b_{12}x_1^2}{x^2} - 3b_{11} - b_{12})\left\langle G^2\right\rangle^{'}] \\
&N_{222} = \tfrac{1}{x^3}[(b_{12}x_2^3 + b_{11}x_2x_1^2)\left\langle G^2\right\rangle^{(3)} + xx_2(3b_{12} + b_{11} - 3\tfrac{b_{12}x_2^2+b_{11}x_1^2}{x^2})\left\langle G^2\right\rangle^{''} \\
&+x_2(3\tfrac{b_{12}x_2^2+b_{11}x_1^2}{x^2} - 3b_{12} - b_{11})\left\langle G^2\right\rangle^{'}] \\
&N_{212} = \tfrac{2b_{44}}{x^3}[x_2^2x_1\left\langle G^2\right\rangle^{(3)} + xx_1(1 - \tfrac{3x_2^2}{x^2})\left\langle G^2\right\rangle^{''} + x_1(\tfrac{3x_2^2}{x^2} - 1)\left\langle G^2\right\rangle^{'}] = N_{221}
\end{aligned} \tag{49}$$

where $b_{11} = b_{12} + 2b_{44}$. Using recursion formulas of modified Bessel function (seen in Appendix B), we can readily get:

$$\begin{aligned}
&\left\langle G^1\right\rangle^{'} = -\frac{1}{c_{44}}[\frac{1}{2}x - RK_1(\frac{R}{l})I_1(\frac{x}{l})] \\
&\left\langle G^1\right\rangle^{''} = -\frac{1}{c_{44}}\left\{\frac{1}{2} - RK_1(\frac{R}{l})[\frac{1}{l}I_0(\frac{x}{l}) - \frac{1}{x}I_1(\frac{x}{l})]\right\} \\
&\left\langle G^1\right\rangle^{(3)} = -\frac{1}{c_{44}}RK_1(\frac{R}{l})[\frac{1}{lx}I_0(\frac{x}{l}) - (\frac{1}{l^2} + \frac{2}{x^2})I_1(\frac{x}{l})]
\end{aligned} \tag{50}$$

$$\begin{aligned}
&\left\langle G^2\right\rangle^{'} = -\frac{1}{a}[\frac{1}{2}x - RK_1(\frac{R}{m})I_1(\frac{x}{m})] \\
&\left\langle G^2\right\rangle^{''} = -\frac{1}{a}\left\{\frac{1}{2} - RK_1(\frac{R}{m})[\frac{1}{m}I_0(\frac{x}{m}) - \frac{1}{x}I_1(\frac{x}{m})]\right\} \\
&\left\langle G^2\right\rangle^{(3)} = -\frac{1}{a}RK_1(\frac{R}{m})[\frac{1}{mx}I_0(\frac{x}{m}) - (\frac{1}{m^2} + \frac{2}{x^2})I_1(\frac{x}{m})]
\end{aligned} \tag{51}$$

for **x** inside the inclusion;

$$\begin{aligned}
&\left\langle G^1\right\rangle^{'} = -\frac{1}{c_{44}}[\frac{1}{2x}R^2 - RI_1(\frac{R}{l})K_1(\frac{x}{l})] \\
&\left\langle G^1\right\rangle^{''} = -\frac{1}{c_{44}}\left\{-\frac{1}{2x^2}R^2 + RI_1(\frac{R}{l})[\frac{1}{l}K_0(\frac{x}{l}) + \frac{1}{x}K_1(\frac{x}{l})]\right\} \\
&\left\langle G^1\right\rangle^{(3)} = -\frac{1}{c_{44}}\left\{\frac{1}{x^3}R^2 - \frac{R}{x^2l^2}I_1(\frac{R}{l})[xlK_0(\frac{x}{l}) + (x^2 + 2l^2)K_1(\frac{x}{l})]\right\}
\end{aligned} \tag{52}$$

$$\left\langle G^2 \right\rangle' = -\frac{1}{a}[\frac{1}{2x}R^2 - RI_1(\frac{R}{m})K_1(\frac{x}{m})]$$
$$\left\langle G^2 \right\rangle'' = -\frac{1}{a}\left\{-\frac{1}{2x^2}R^2 + RI_1(\frac{R}{m})[\frac{1}{m}K_0(\frac{x}{m}) + \frac{1}{x}K_1(\frac{x}{m})]\right\} \tag{53}$$
$$\left\langle G^2 \right\rangle^{(3)} = -\frac{1}{a}\left\{\frac{1}{x^3}R^2 - \frac{R}{x^2m^2}I_1(\frac{R}{m})[xmK_0(\frac{x}{m}) + (x^2+2m^2)K_1(\frac{x}{m})]\right\}$$

for **x** outside the inclusion. By substituting Eq. (50)-(53) into Eq. (46)-(49), the analytical expression of Eshelby and Eshelby-like tensors for elastic and electric field can be easily obtained. When the eigenstrain, eigenstrain gradient, eigenelectric field and eigenelectric field gradient are given, the corresponding strain and electric field can be derived through Eq. (35) and Eq. (38). Furthermore, using Eq. (40) and Eq. (41) leads to the stress and electric displacement quantities which represent high-order electromechanical coupling to some extent.

5 Numerical results

To numerically illustrate the results obtained above, we now choose NaCl material, which is a centrosymmetric dielectrics, to give a series of numerical results. The flexoelectric constants of NaCl are estimated by means of empirical lattice dynamics [Maranganti and Sharma (2009)]. In nano-inclusion problems, as a typical microstructure, the influences of inclusion size are what we most interested in. Therefore, we choose the characteristic length as $l = m = 2$nm by reference to inclusion size. Fig.2 shows Eshelby tensor incorporating gradient effects for elastic field.

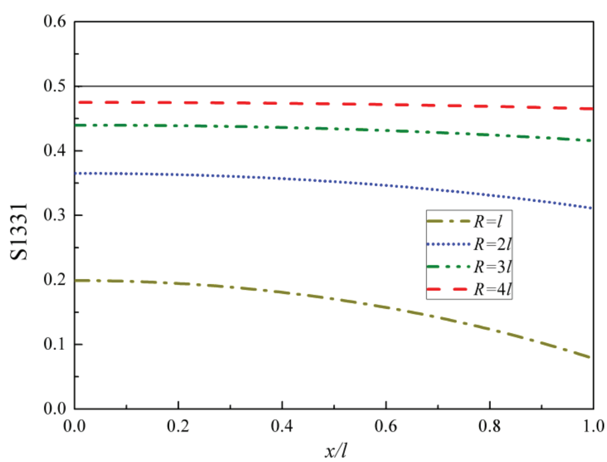

Figure 2: Eshelby tensor of strain along a radial direction of the inclusion.

From Figure 2, it is seen that S_{1331} varies with the position **x** and the inclusion size R, unlike the classical results which is a constant inside the inclusion (the straight line). The differences between the lines to the classical one are due to the strain gradient effects. While R is small, the strain gradient effects are very large and should not be ignored.

However, the values approach gradually to the result of classical elasticity with increase of inclusion radius. It is noted that on account of the decoupling results of anti-plane inclusion problem of dielectrics, our results of elastic Eshelby tensor are in accordance with solutions derived by Gao and Ma (2012) who used the simplified strain gradient elasticity theory (SSGET) sloving Eshelby-type anti-plane strain inclusion problem. Likewise, Fig. 3 gives Eshelby tensor with electric field gradient effects for electric field. We can clearly figure out that M_{11} are no longer uniform, and have similar tendency with S_{1331} for different position and inclusion size.

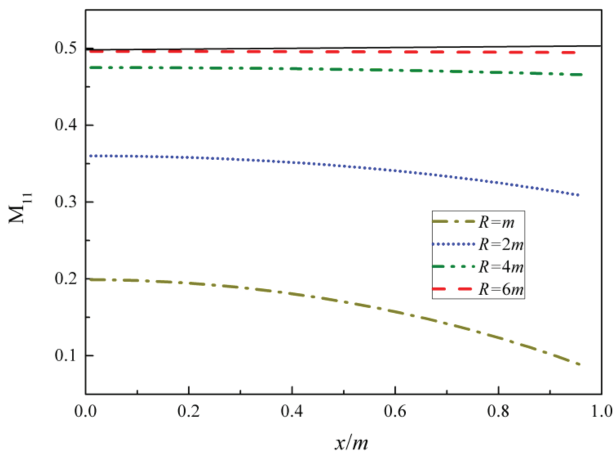

Figure 3: Eshelby tensor of electric field along a radial direction of the inclusion.

To explicitly display our solution, the strain component ε_{13} inside and outside inclusion for various inclusion sizes when prescribed eigenfield containing only ε^*_{13}=0.05 are plotted in Fig. 4. In horizontal axis, $x/R < 1$ and $x/R > 1$ reprensent interior and exterior of inclusion respectively. It is easily observed that the strain is inhomogeneous inside the inclusion and the results for whole inclusion-matrix system converge asymptotically to the classical solution as increase of inclusion size. We also plot the electric component E_1 depending on inclusion size for two fixed point in Fig. 5, where E_1 is normalized by E_1^c. E_1^c is the classical value when electric field gradient effect is absent (i.e., $m = 0$), which is independent of positions

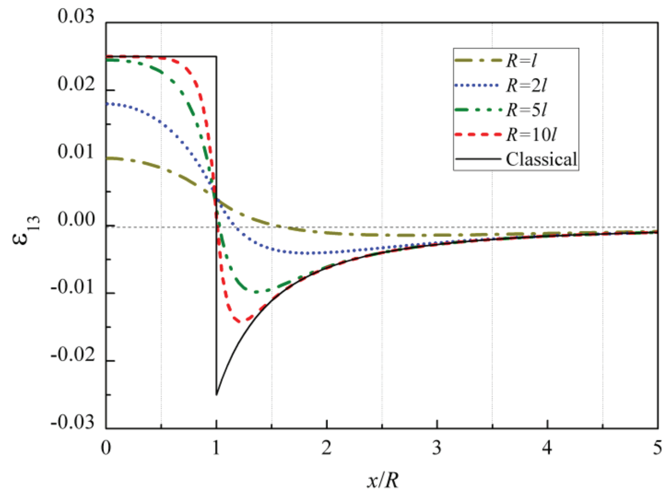

Figure 4: The strain field as a function of position and inclusion size.

inside the inclusion. r represents distance form the center of circular inclusion. It is presented that size effects are intensely strong when inclusion radius R drop to a few nanometers. Besides, electric field gradient effect play a more important role to positon close to interface than near the center of inclusion, which is obviously observed from the curves.

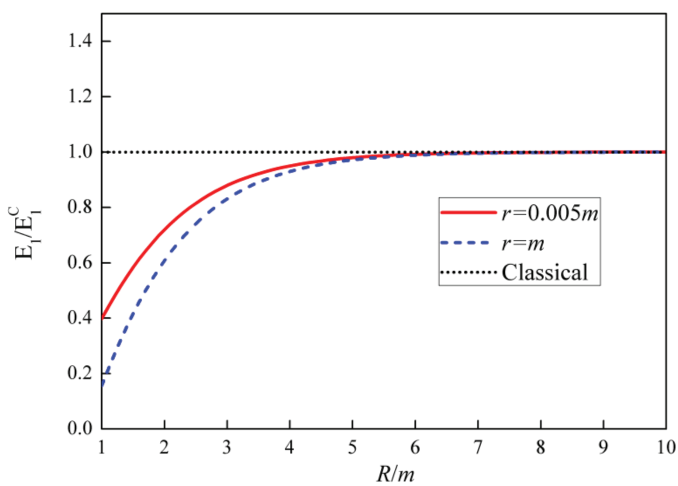

Figure 5: The electric field as a function of inclusion size for fixed positions.

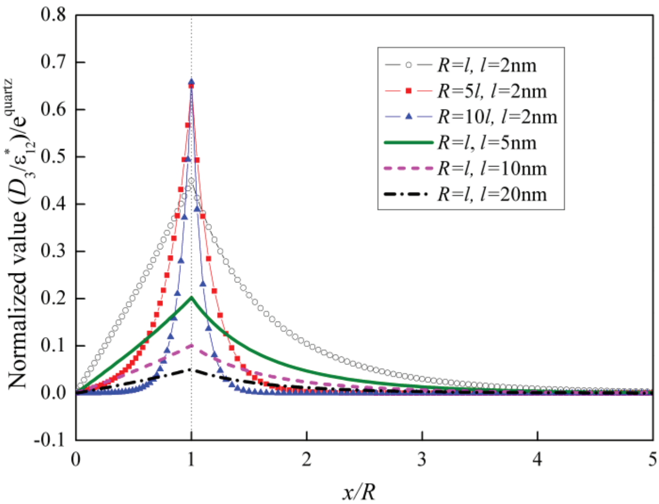

Figure 6: High-order electromechanical coupling coefficient versus piezoelectric coefficient.

Fig. 6 shows high-order electromechanical coupling ability for different inclusion size and characteristic length scales l. "Normalized value" for longitudinal axis is defined as $\left(D_3/\varepsilon_{13}^*\right)\Big/e^{\text{quartz}}$, which represents the ratio of high-order electromechanical coupling coefficient and piezoelectric coefficient of quartz. In order to simplify the calculation, the case of only the uniform eigenstrain ε_{13}^* prescribed within the inclusion is considered. From the figure we can see that D_3 arises in and around inclusion due to the existence of the cooresponding strain gradient, which is demonstrated by the third formula of Eq. (41). The values in and around inclusion gradually apporoach to zero with increase of inclusion size as they should be. In addition, it is noted that D_3 changes a lot for different characteristic length scale parameter of elastic field l representing different material. The strain gradient effects decrease as l increases, and this tendency could be explained in terms of energy. As we know from expression of purely elastic nonlocal effect term, increasing of l brings incresase of elastic energy. On account of total energy conservation, the electric energy will decrease which validly induce the reduction of electric displacement. In the numerical results, the flexoelectric and converse flexoelectric coefficients employed take the order of magnatitude of 10^{-10}, with the units of c/m.

In the present paper we intensively focus on the analytical results. Pure analytical derivations may work for infinite domain inclusion problem while real microstructures are more complexed. Combining of analytical and numerical methods [Dong and Atluri (2012a); Dong and Atluri (2012b); Bishay, Dong and Atluri (2014)] is a powerful tool for direct numerical simulations of realistic problems of real material structures, which provide valuable insights for our future work.

6 Conclusions

In summary, we analyze the elastic and electric field of anti-plane inclusion problem by incorporating both strain and electric field gradient effect for centrosymmetric dielectrics. The Eshelby tensor, Eshelby-like tensor, strain field, electric field, stress and electric displacement are analytically derived by means of Green's function method. Unlike the classical results, all the physical quantities taken in account to gradient effects show strongly size-dependent features. Our results indicate that strain and electric field are decoupled which is reasonable. And more importantly, the high order electroelastic coupling appears between stress and electric field. The components of stress which do not exist in classical case can be induced by the electric gradient, and the out-plane electric displacement component can be caused by the strain gradient, that is an good interpretation of flexoelectric coupling effects. The numerical results reveal that, firstly, the strain and electric field gradient effects are significantly strong and should not be ignored when the inclusion size is small. Secondly, the results gradually approach to the classical solution as increase of the inclusion radius. Thirdly, the high-order electromechanical coupling effect in centrosymmetric dielectrics (non-piezoelectric materials) can be equivalent to piezoelectric effect in traditional piezoelectric materials when the inclusion size is small.

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Appendix A

For a sufficiently smooth function $F(x)$, the following differential relations hold:

$$
\begin{aligned}
&F_{,\alpha} = x_\alpha D_1 F\\
&F_{,\alpha\beta} = x_\alpha x_\beta D_2 F + \delta_{\alpha\beta} D_1 F\\
&F_{,\alpha\beta\gamma} = x_\alpha x_\beta x_r D_3 F + \left\langle \delta_{\alpha\beta} x_r \right\rangle_3 D_2 F\\
&F_{,\alpha\beta\gamma\gamma} = x^2 x_\alpha x_\beta D_4 F + (x^2 \delta_{\alpha\beta} + 6 x_\alpha x_\beta) D_3 F + 4\delta_{\alpha\beta} D_2 F\\
&F_{,\alpha\beta\gamma\chi} = x_\alpha x_\beta x_r x_\chi D_4 F + \left\langle \delta_{\alpha\beta} x_r x_\chi \right\rangle_6 D_3 F + \left\langle \delta_{\alpha\beta}\delta_{\gamma\chi} \right\rangle_3 D_2 F\\
&F_{,\alpha\beta\gamma\theta\chi} = x_\alpha x_\beta x_r x_\chi x_\theta D_5 F + \left\langle \delta_{\alpha\beta} x_r x_\chi x_\theta \right\rangle_{10} D_4 F + \left\langle \delta_{\alpha\beta}\delta_{\gamma\theta} x_\chi \right\rangle_{15} D_3 F\\
&F_{,\alpha\beta\gamma\theta\chi} = x_\alpha x_\beta x_r x^2 D_5 F + (x^2 \left\langle \delta_{\alpha\beta} x_r \right\rangle_3 + 8 x_\alpha x_\beta x_r) D_4 F + \left\langle \delta_{\alpha\beta} x_\gamma \right\rangle_3 D_3 F
\end{aligned}
$$

where,

$$
\begin{aligned}
&D_1 F = \tfrac{F'}{x}\\
&D_2 F = \tfrac{1}{x^2}(F'' - \tfrac{F'}{x})\\
&D_3 F = \tfrac{1}{x^3}(F''' - \tfrac{3F''}{x} + \tfrac{3F'}{x^2})\\
&D_4 F = \tfrac{1}{x^4}(F^{(4)} - \tfrac{6F'''}{x} + \tfrac{15F''}{x^2} - \tfrac{15F'}{x^3})\\
&D_5 F = \tfrac{1}{x^5}(F^{(5)} - \tfrac{10F^{(4)}}{x} + \tfrac{45F'''}{x^2} - \tfrac{105F''}{x^3} + \tfrac{105F'}{x^4})\\
&\left\langle \delta_{\alpha\beta} x_r \right\rangle_3 = \delta_{\alpha\beta} x_r + \delta_{\alpha\gamma} x_\beta + \delta_{\gamma\beta} x_\alpha\\
&\left\langle \delta_{\alpha\beta} x_r x_\chi \right\rangle_6 = \delta_{\alpha\beta} x_r x_\chi + \delta_{\chi\gamma} x_\alpha x_\beta + \delta_{\alpha\chi} x_r x_\beta + \delta_{\alpha\gamma} x_\beta x_\chi + \delta_{\beta\gamma} x_\alpha x_\chi + \delta_{\chi\beta} x_r x_\alpha\\
&\left\langle \delta_{\alpha\beta}\delta_{\gamma\chi} \right\rangle_3 = \delta_{\alpha\beta}\delta_{\gamma\chi} + \delta_{\alpha\gamma}\delta_{\beta\chi} + \delta_{\alpha\chi}\delta_{\beta\gamma}\\
&\left\langle \delta_{\alpha\beta} x_r x_\chi x_\theta \right\rangle_{10} = \delta_{\alpha\beta} x_r x_\chi x_\theta + \delta_{\alpha\theta} x_r x_\chi x_\beta + \delta_{\alpha\gamma} x_\beta x_\chi x_\theta + \delta_{\alpha\chi} x_r x_\beta x_\theta + \delta_{\theta\beta} x_r x_\chi x_\alpha\\
&+\delta_{\gamma\beta} x_\alpha x_\chi x_\theta + \delta_{\chi\beta} x_r x_\alpha x_\theta + \delta_{\theta\gamma} x_\alpha x_\chi x_\beta + \delta_{\theta\chi} x_\alpha x_\beta x_\gamma + \delta_{\gamma\chi} x_\alpha x_\beta x_\theta
\end{aligned}
$$

Appendix B

Recursion formulas of modified Bessel function are expressed as:

For the first kind I, $\begin{cases} I_n^{'} = I_{n-1} - \frac{n}{x} I_n \\ I_n^{'} = I_{n+1} + \frac{n}{x} I_n \end{cases}$

For the second kind K, $\begin{cases} -K_n^{'} = I_{n-1} + \frac{n}{x} K_n \\ -K_n^{'} = K_{n+1} - \frac{n}{x} K_n \end{cases}$, where n is an integral number.