 CMC, vol.42, no.3, pp.227-244, 2014

An Integrated Fracture Mechanics Based Approach for Non-Linear Analysis of Lightly Reinforced Concrete Beams

Ananthalakshmi K. Iyer[1], **A. Rama Chandra Murthy**[2], **Smitha Gopinath**[2] **and Nagesh R. Iyer**[3]

Abstract: A non-linear fracture mechanics based approach is proposed to depict a typical fracture mechanism from initiation to growth, eventually leading to failure. This concept is developed for a lightly reinforced beam in flexure. The proposed model integrates the existing methodology of a Stress Intensity Factor equilibrium equation with the bridging forces developed in concrete cover and rebar. The model and solution algorithm outlined presents an elaborate understanding of the mechanism involved and is significant in predicting the behaviour of flexural members. The analysis is performed using MATLAB programming. The proposed approach ensures a maximum tolerable crack length and crack width for flexural members to prevent a catastrophic failure. Such an approach has the potential to serve as an analysis and design tool for reinforced concrete components subjected to normal conditions and towards deciding rehabilitation and strengthening measures.

Keywords: Fracture model, RCC Beam, Stress intensity factor, Crack propagation.

1 Introduction

Concrete has been one of the most commonly used building materials all over the world. However, concrete is a quasi-brittle material with high compressive strength, low tensile strength and toughness, owing to which, cracks of different size and shape occur during service and construction processes. This reduces the durability of concrete structures and lifetime is shortened. Hence it is necessary to study

[1] M.E. (Structural Engineering), Government College of Technology, Coimbatore – 641 013
[2] Scientist, CSIR-Structural Engineering Research Centre, CSIR Campus, Taramani, Chennai - 600 113
[3] Director, CSIR-Structural Engineering Research Centre, CSIR Campus, Taramani, Chennai - 600 113

the crack propagation in concrete structures. Fracture mechanics is a rapidly developing field that has great potential for application to concrete structural design. Non-linear Fracture Mechanics (NLFM) is largely concerned with the inelastic effects that are present in the vicinity of a stressed crack tip.

2 Background

In the present economic climate, design of structural components to meet the more stringent limits of serviceability, ultimate strengths and durability seem to be more appealing with respect to the sustainability scenario. Ashmawi et al. (1992) developed a crack control design model to obtain the minimum area of tension steel for a RC beam in flexure. Jan (1992) related the fracturing phenomena in RC slab to the flexural stiffness to develop an analytical tool for its design and analysis. Shah et al. (1995) developed a fracture mechanics approach to predict cracking of RC members subjected to tension by balancing the rates of change of the strain energy, the debonding energy, and the sliding energy. By combining the concept of fictitious crack model and conventional bending theory, Hillerborg (1988), developed a model by taking into account the effect of compressive strain localization. Bosco and Carpinteri (1992) proposed a fracture mechanics model for bending of RC beams where concrete is assumed to be a linear elastic material and steel is considered to be elastic-plastic. The effect of reinforced steel bars is simulated by a closing force whose magnitude is determined by a compatibility condition. Baluch (1992) modified this model to include the nonlinear behaviour of concrete under compression. Alaee and Karihaloo (2003) developed an analytical model to predict the moment resistance of the retrofitted beams by taking a fracture mechanics approach. Raghu Prasad et al. (2005) developed a one-dimensional analysis and design procedure for reinforced concrete structures, generally based on yield phenomena and the plastic flow of steel in tension and concrete in compression. Wu et al.(2006) developed an analytical model to predict the effective fracture toughness, K_{IC} of concrete based on the fictitious crack model. Taking into account Lagrange Multiplier Method, the maximum load, P_{max} was obtained, as well as the critical effective crack length, a_c. Roeslar et al.(2007) developed a finite element-based cohesive zone model using bilinear softening to predict the monotonic load versus crack mouth opening displacement curve of geometrically similar notched concrete specimens. Wu et al. (2010) proposed analytical solution to predict the load-bearing capacity of CFRP sheet-strengthened cracked concrete beams with reasonable accuracy. Wallin (2013) proposed a novel LEFM based estimate of the effective stress intensity factor and the effective crack growth at maximum load in a fracture mechanics test to obtain a simple power law approximation of the effective K–R curve applicable to the description of not only different size specimens, but also specimens with varying geometry.

3 Review on Existing Fracture Models

The existing approaches to the failure of beams in bending are studied. In general, it is accepted that reinforced concrete beams show distinct fracture characteristics due to their quasi brittle nature and fail by steel yielding. In particular, lightly reinforced beams in three-point-bending fail by a single crack across the central cross section, as opposed to normally reinforced beams in which multiple or distributed cracking occurs prior to collapse. The question of minimum reinforcement points to,

1. Stability of a system of interacting cracks, which ensures that the cracks will remain densely spaced, and
2. Avoidance of snapback in bending at a cross-section with a single crack

The first problem controls the spacing of flexure cracks which are critical to control their width. A certain minimum reinforcement is required to prevent a large crack spacing, causing a large crack opening. This problem is important for serviceability under normal loads and small overloads. The second problem which is important for preventing sudden catastrophic failure without warning, where the resistance to external moment is majorly taken up by concrete is discussed through various models as follows.

All LEFM models are rooted in the model first proposed by Carpinteri. Carpinteri's model follows the basic superposition approach where the reinforced beam with a crack of length $'a'$ subjected to bending is approximated by a beam subjected to the bending moment and to the steel force applied remotely from the crack plane. Next, the steel action is decomposed in a standard way into a bending moment and a centric force. With this decomposition, the stress intensity factor and the additional rotation caused by the crack θ_c is calculated. Carpinteri assumed that the steel behaviour was elastic-perfectly plastic, and that the crack was closed while the steel remained elastic. Therefore the crack growth takes place only when the steel yields and the stress intensity factor reach its critical value simultaneously.

The major limitation of this model is that, due to the simplifications involved in the derivation, the crack cannot grow while the steel remains elastic and does not slip whereas in reality it does slip.

Ashmawi et al. (1992) modified Carpinteri's model to include the nonlinear behaviour of concrete under compression and obtained the minimum area of tension steel for a RC beam in flexure to safely sustain a design moment within the prescribed limit of permissible crack length. This model was based on an iterative procedure which was developed by satisfying simultaneously the fracture criterion of crack growth and the equilibrium condition at incipient fracture. This model followed the linear variation in strain across the depth of beam, with respect the stress in the uncracked ligament of concrete. The compressive stress in concrete was taken in the form of 'Madrid Parabola' which is similar to the one given in IS 456 2000.

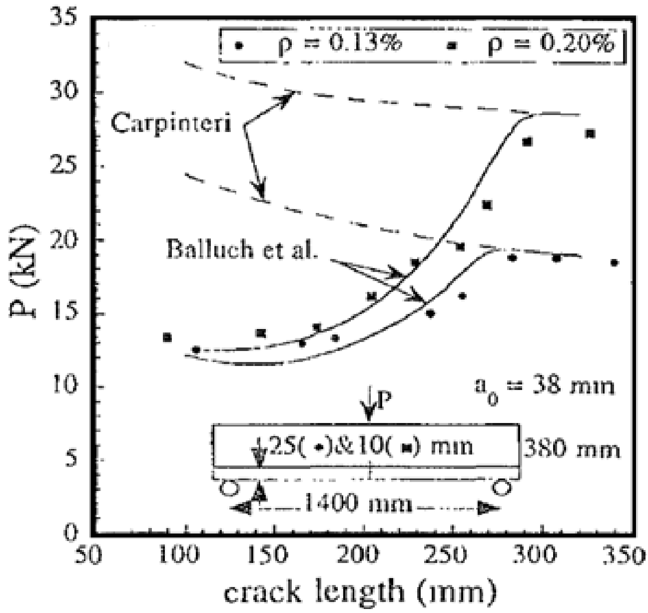

Figure 1: Comparison of Load vs. Crack Length [Ashmawi et al. (1992), Bosco and Carpinteri (1992)].

Baluch model keeps Carpinteri's solution after steel yielding, but relaxes the assumption that the crack remains closed while the steel is elastic. It compares the load-crack length plots for two lightly reinforced notched beams tested in three-point-bending with Carpinteri's model. From Figure. 1, it can be noted that the theoretical predictions by Baluch coincide with that of Carpinteri's model only after the yielding of steel. Also the theory predicts a sharp change of slope upon steel yielding, which happens only in an ideal scenario and not in reality where a rounded transition happens.

The cohesive crack model has been used by several investigators to analyze lightly reinforced beams in three-point bending. All the previous models investigated has established a common point that a single cohesive crack forms at the central section while the concrete in the bulk behaves elastically and the steel is elastic-plastic. Hedadal and Kroon (1991) considered the bond slip and implemented it numerically. They considered the load-displacement plot obtained from pull-out tests to include the effect of reinforcement. It is assumed that the steel displacement u_s is taken as half the crack opening at the level of reinforcement, w_s. This model introduces the action of steel on the concrete as the force F_s concentrated at the surface of the cohesive crack and treats it as a cohesive force with a load-crack opening plot as shown in Figure. 2.

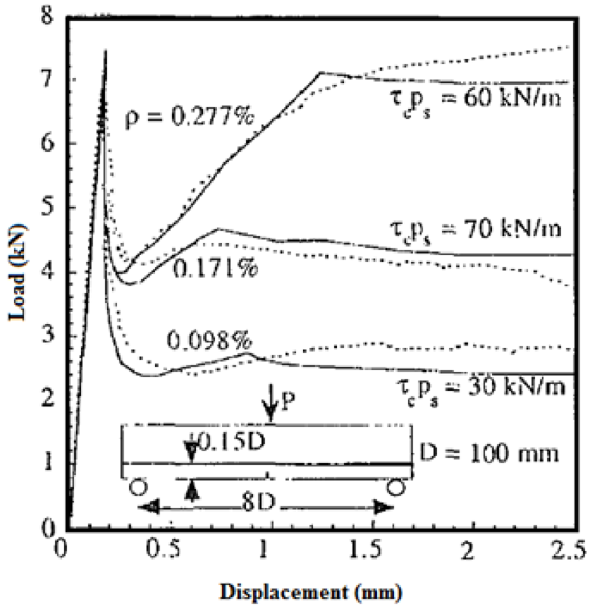

Figure 2: Load-Displacement Plot by Hedadal-Kroon (1991).

Planas et al. (1993) model used a different numerical approach incorporating the effect of the reinforcement by means of internal stresses. This allows considering the steel concrete interaction to be located within the concrete rather than at the surface. In the first approach it is analyzed for the case of perfect bond with the steel concrete interaction represented by two forces acting on the crack faces. The second case depicts the concentrated force of steel at the centre of gravity of the bond stresses at a length of $L_s/2$ and the last case is analyzed by considering the distributed bond stresses at a length L_s. The analysis shown in Figure. 3 confirms that perfect adherence implies a very sharp and high peak. However it also turns out that this peak depends strongly on the width or diameter of the reinforcement. The reason is that in this approach the steel forces are modeled as a nodal force which causes the computational procedures to smear this force roughly over an element width and thus one never deals with the concentrated force but with the distributed force.

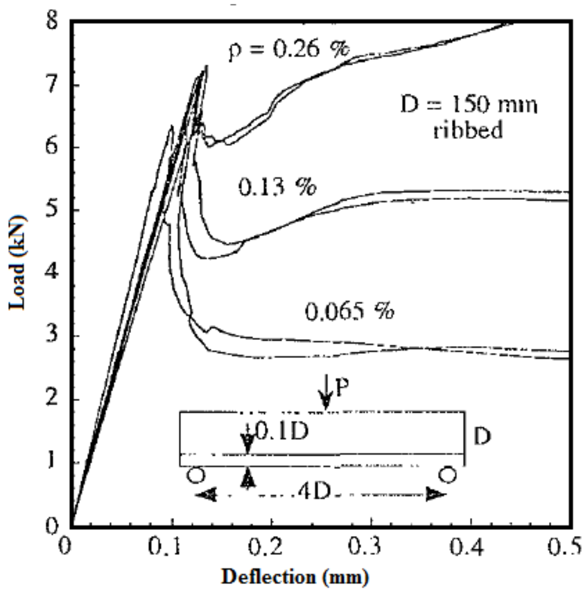

Figure 3: Load-Displacement Plot by Planas et al. (1993).

4 Proposed Analytical Model

All the existing models shown above depict a general pattern with respect to the load vs. deflection plots. A similar pattern is observed using the proposed analytical model. Also this model shows an improved version of crack propagation from initiation to growth comparing with that of the existing Baluch model (1992).The mechanism for the proposed NLFM based model for the flexural failure is explained in Figure. 4, where the free body diagram is shown. It is seen that in addition to the external moment due to external load, there are two distinct forces which resist the fracture in the component.

The resisting forces include the concentrated force from rebar F_s and the tensile strength of the concrete σ_s. The non-linear interaction and behaviour between these two forces are further analysed by considering the following mechanism shown in Figure. 5.

Perfect bond between steel and concrete is assumed in this mechanism. This mech-

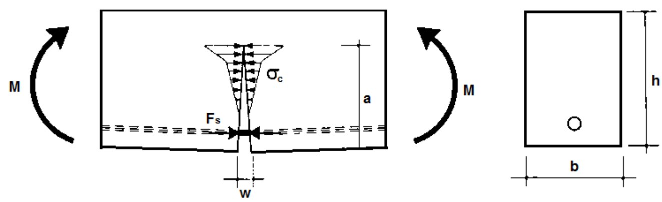

Figure 4: Free Body Diagram for Flexural Crack Propagation.

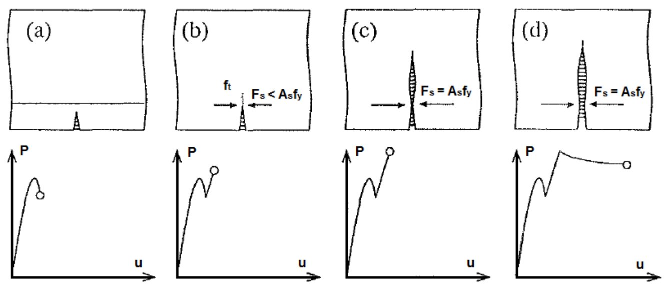

Figure 5: Load vs. Deformation plots corresponding to various stages of crack growth.

anism shows that the cohesive crack growth process in a reinforced concrete beam follows the stages shown in Figure. 5 (a)-(d).

1. The cohesive zone extends to the cover and may go through the first peak if the cover is thick enough.
2. The cohesive crack is pinned by the steel and hardening occurs.
3. The tensile strength is reached at points ahead of reinforcement after which two separate cracks exist at both sides of the reinforcement until the yield strength is reached in the steel.
4. A softening phase begins, with an open crack extending across the reinforcement.

This mechanism is applied to the three-point bend lightly reinforced beam to investigate its non-linear fracture behaviour.

The "cohesive law" given by Dugdale (1960) and Barenblatt (1962), which represents the relation between the crack opening displacement and the traction that resists them is applied. The cohesive law operates up to a critical value of the crack opening displacement where the stress drops to zero. Since the stresses at the strip yield zone are infinite, there cannot be a singularity at the crack tip (the stress intensity factor at the tip of plastic zone must be equal to zero). Thus the plastic zone length r_p is found from the condition that the stress intensity factors from the remote tension and closure stress cancel one another.

$$c = r_p = \frac{\pi K_I^2}{8\sigma_y^2}$$

The relationship between the geometrical properties of a crack and the bridging force exerted by steel crossing the crack is quite complicated to formulate a theoretical approach. The method for calculating the steel bridging force is taken from Kaar and Hognestad (1965) formula.

Kaar and Hognestad (1965) modified the previous empirical relations and established Kaar-Hognestad formulae to predict crack widths on the tension face of flexural members with high strength deformed bars.

$$w_c = 0.115 r_t A_c^{1/4} f_s \times 10^6$$

This empirical relation is used in the proposed analytical model to establish the approximate relation between steel force F_s and the crack opening displacement w_c.

As the stress at the crack tip is finite, the net stress intensity factor at the crack tip must vanish. Thus leading the crack faces to close smoothly near the tip. The plastic correction is obtained from the Dugdale-Barenblatt model as discussed earlier and denoted as zone c near the crack tip which is much smaller than the length of the crack $'a'$.

The net K_I, at the crack tip is obtained by superimposing the stress intensity factors produced at the crack tip by the applied moment (K_{IM}), and the closure forces by steel (K_{IS}) and concrete (K_{IConc}). The condition of finite stress at crack tip K_I is taken as,

$$K_{IM} - K_{IS} - K_{IConc} = 0$$

The negative sign for the last two terms indicates that the stress intensity factor produced by the applied moment is reduced by the various closure forces.

K_{IM} is calculated using the formula by Guinea et al. (1998) and the rest are calculated using directly or with integration with the help of relation proposed by Tada et al. (1985). The stress intensity factor produced at the crack tip by the moment

Mgiven by Guinea et al. (1998).

$$K_{IM} = \frac{M}{h^{3/2}b} Y_M(\xi)$$

where, ξ is *a*/*h* and the geometric factor $Y_M(\xi)$ for a span to depth ratio of 8 is taken as,

$$Y_M(\xi) = \frac{6\xi^{\frac{1}{2}}\left(1.99 + 0.83\xi - 0.31\xi^2 + 0.14\xi^3\right)}{(1-\xi)^{\frac{3}{2}}(1+3\xi)}$$

To compute K_{IConc}, the residual stress in concrete is discretized into infinitely many concentrated forces as shown in Figure.6. Each force is applied over small crack segments of length dy at a distance $y(= y* .h)$ from the bottom of the crack.

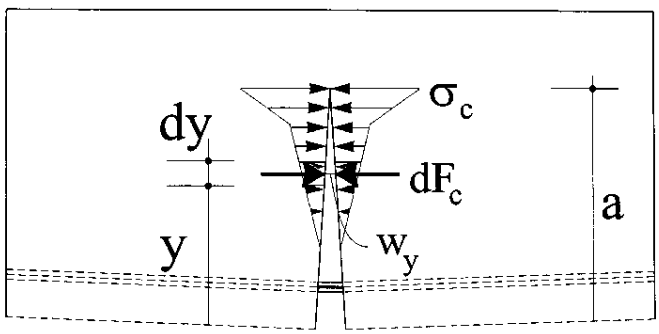

Figure 6: Discretization of Cohesive Stresses in Concrete.

The magnitude of each force is formulated as

$$dF_c = \sigma_c w(1 - y/a) b dy$$

The stress intensity factor produced by this force can be calculated by the relation suggested by Tada et al. (1985).

$$dK_{IConc} = \frac{dFc}{h^{1/2}b} Y_S(\xi, y*)\, dy*$$

Finally this relation is integrated over the length of the crack in concrete.

The stress intensity factor produced by the concentrated steel force, F_S acting at β=*c*/*h* is given by Tada et al. (1985).

$$K_{IS} = \frac{F_S}{h^{1/2}b} Y_S(\xi, \beta)$$

where the geometric factor $Y_S(\xi,\beta)$ for a span to depth ratio of 8 is taken as,

$$Y_S(\xi,\beta)=\left\{\frac{3.52\left(1-\frac{\beta}{\xi}\right)}{(1-\xi)^{\frac{3}{2}}}-\frac{4.35-\frac{5.28\beta}{\xi}}{(1-\xi)^{\frac{1}{2}}}+\left[\frac{1.30-0.30\left(\frac{\beta}{\xi}\right)^{3/2}}{\left(1-\left(\frac{\beta}{\xi}\right)^2\right)^{1/2}}+0.83-\frac{1.76\beta}{\xi}\right]\right.$$
$$\left.\times\left[1-\left(1-\frac{\beta}{\xi}\right)\xi\right]\right\}\frac{2}{\sqrt{\Pi\xi}}$$

For calculating the geometric factor for stress intensity factor due to cohesive forces in concrete $Y_S(\xi,y*)$, $y*$ is replaced with β in the above expression. The theoretical formulations and analysis is performed using the MATLAB software and the flowchart is given in Figure. 7.

5 Results and Discussion

The proposed model is analysed for reinforced concrete beams for two different cover thicknesses and for various percentages of steel. Figures. 8-10 show the various load-deflection plots for the proposed non-linear fracture mechanics based model for different conditions. Figures. 11-13 show the various load-crack length plots for the proposed non-linear fracture mechanics based model for different conditions. Table 1 tabulates the distinct values of parameters involved in the analysis for various conditions.

The results obtained from the analysis show,

1. Non-linear variation with the help of load-displacement plots.
2. Crack initiation and propagation with the help of load-crack length plots.

There is a common peak observed in all the plotted charts from the proposed analytical model which corresponds to a steep increase in the load at the level of reinforcement. This is due to the theory that when the crack propagates through the cover and reaches the rebar, the flexural load is taken up solely by the steel bridging force. Later at consecutive levels there is a gradual increase in load to attain the peak load level once again. Since such results are not observed in real, this theory has to be further improved to get a smooth distribution of forces at the composite level where steel reinforcements are embedded in concrete

It is clearly noted from the load-deflection plots that,

1. A non-linear variation is obtained through the proposed fracture based approach.

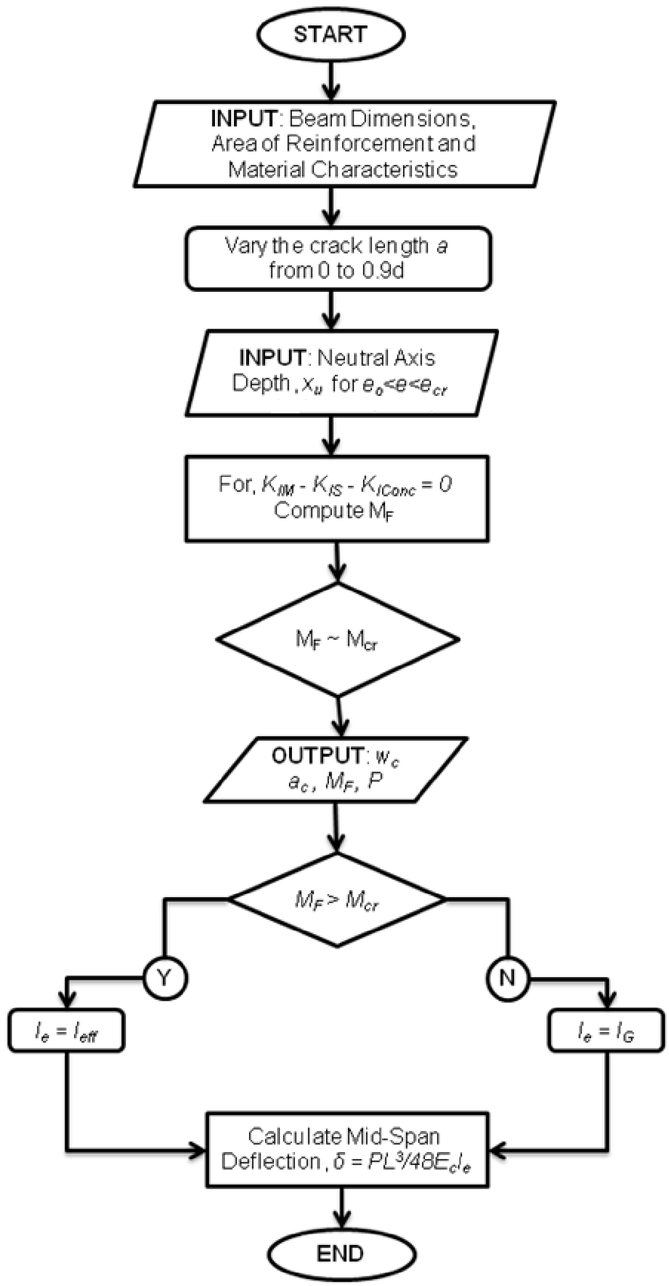

Figure 7: Flowchart for the Proposed NLFM Model.

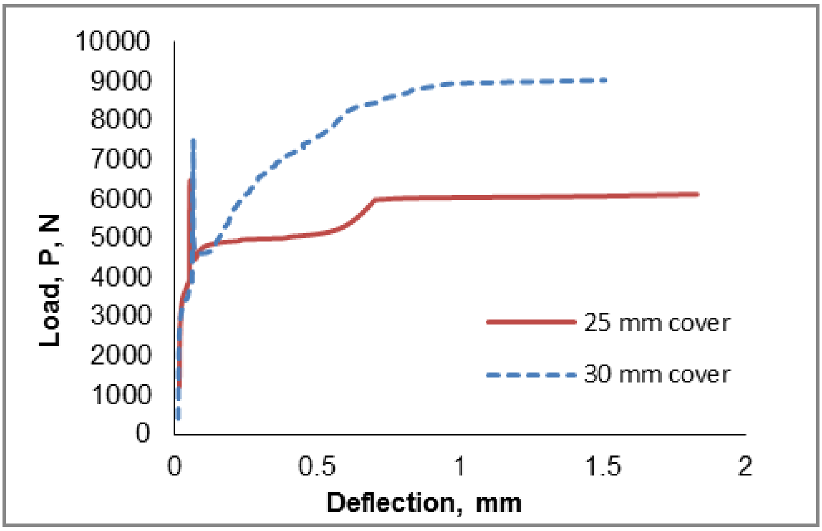

Figure 8: Load-Deflection Plot for 0.785% steel.

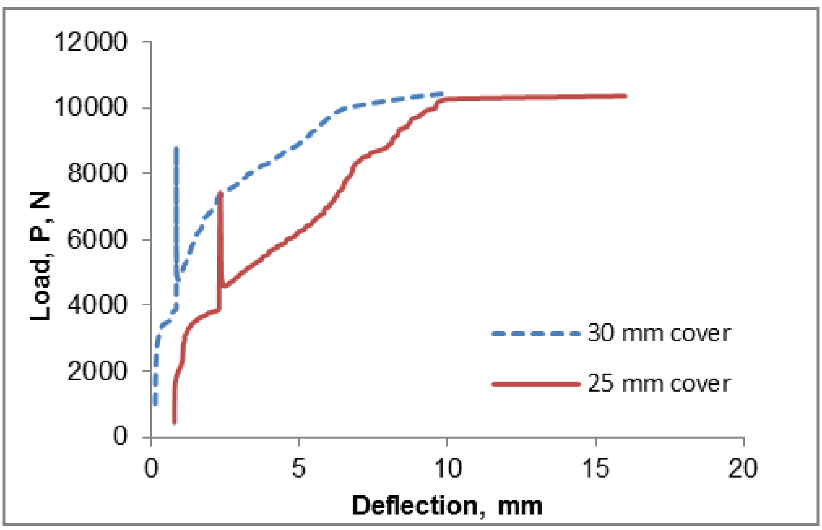

Figure 9: Load-Deflection Plot for 1.131% steel.

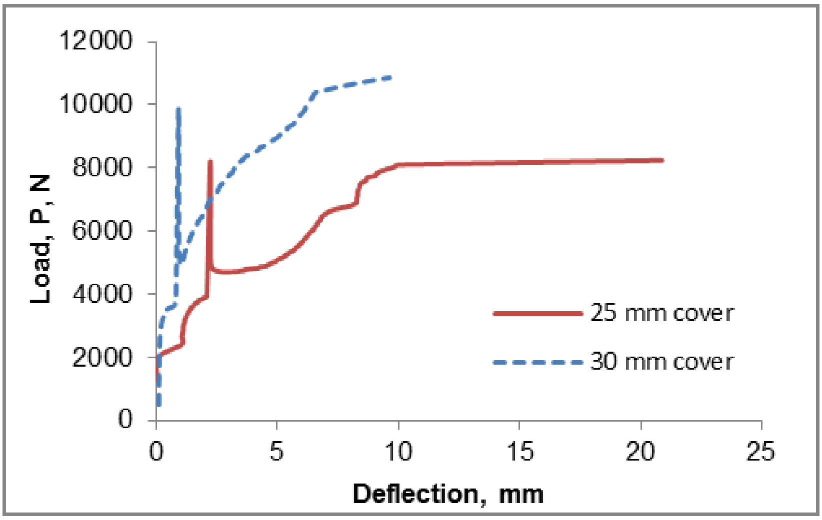

Figure 10: Load-Deflection Plot for 1.571% steel.

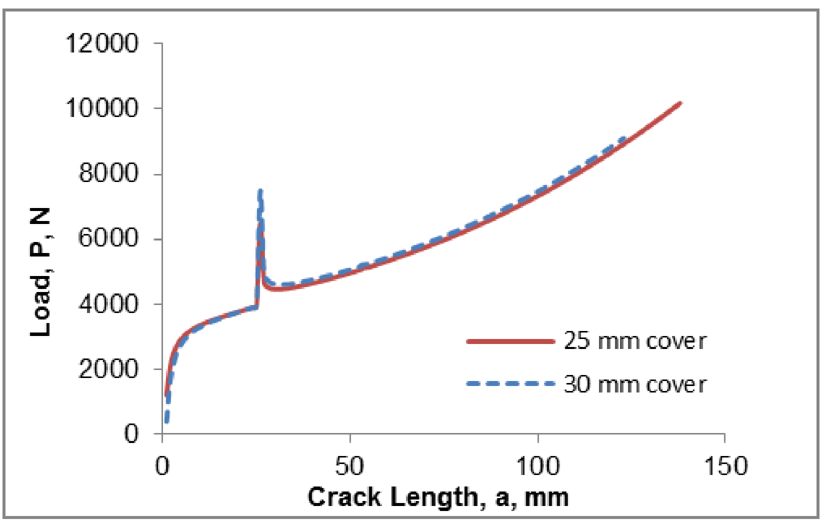

Figure 11: Load-Crack Length Plot for 0.785% steel.

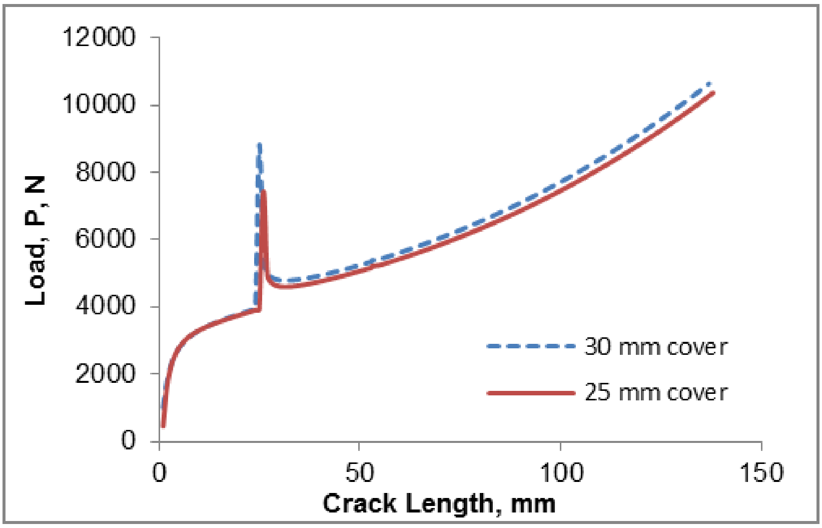

Figure 12: Load-Crack Length Plot for 1.131% steel.

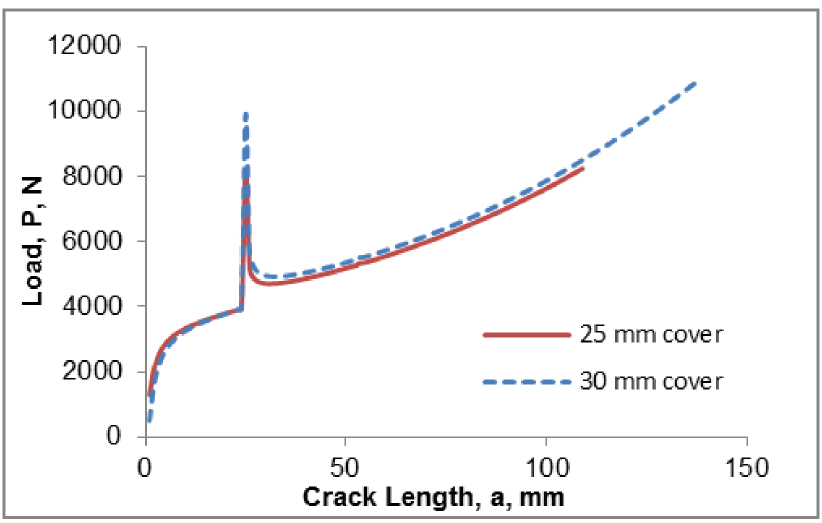

Figure 13: Load-Crack Length Plot for 1.571% steel.

Table 1: Analytical Results for the Proposed Model.

S.No.	Beam Dimension	Cover Thickness	% steel	Fracture Load	Deflection	K_{IS}	K_{Ic}	Crack Width
	mm	mm	%	N	mm	MPa $\sqrt{m}$	MPa $\sqrt{m}$	mm
1	100 x 200 x 1500	25	0.785	6474	0.8	0.587	0.09	0.065
2			1.131	7423	1.3	0.808		0.067
3			1.571	8195	2.0	0.988		0.067
4			2.262	9791	2.2	1.360		0.068
5	100 x 200 x 1500	30	0.785	7484	0.634	0.822	0.09	0.065
6			1.131	8812	0.825	1.132		0.067
7			1.571	9894	0.920	1.384		0.067
8			2.262	12128	1.600	1.904		0.068

2. There is a smaller linear zone which is followed by a non-linear zone due to the cracking in concrete cover.

3. The non-linear zone goes upto the peak at which the crack crosses the reinforcement, after which a U-shaped zone follows. This corresponds to the load taken up by the concrete after the crack crosses the reinforcement.

4. This zone ends at a point at which the reinforcement yields after which it is followed by a long tail. This corresponds to the steel necking and complete slip in the bond.

It is clearly observed from the load-crack length plots that,

1. The initiation of crack is distinctly seen.

2. A sharp peak is followed after the initiation of crack when the crack propagates and crosses the reinforcement.

3. The post-peak behaviour predicted by Baluch's model is similar to the proposed model.

4. The tail end in proposed model corresponds to the yielding of steel.

6 Summary

A non-linear fracture mechanics based approach has been developed with detailed discussions on the mechanisms involved and various theoretical formulations adapted. The analytical model proposed is used for analysing a reinforced concrete beam with various percentages of steel. The cover thickness to reinforcement is varied to evaluate its effect in the contribution to the fracture load.

It is generally seen that as the percentage of steel increases the peak load also increases due to the increase in the stress intensity factor due to steel bridging forces. The variations in the steel bridging forces are the cause for the difference in the peak load for a same specimen. The analysis from the proposed model is plot for the load-deflection vs. load-crack length which show a clear non-linear variation and with the help of which the distinct values of the parameters involved are evaluated. The theoretical values obtained are comparatively lower than the usual field values and thus improvement in load factors has to be further included along with a smooth distribution of load at the level of reinforcement. This study can be further applied to, concrete dams, retaining walls, canals, etc since percentage of steel is usually low and also for rehabilitation and retrofitting measures.

Acknowledgement: The authors thank the staff of the Computational Structural Mechanics Group of CSIR-SERC for their co-operation and suggestions provided during the investigation. This paper is published with the kind permission of the Director, CSIR-Structural Engineering Research Centre, Taramani, Chennai, India.

References

Alaee, F. J.; Karihaloo, B. L. (2003): Fracture model for flexural failure of beams retrofitted with CARDIFRC. *Journal of Engineering Mechanics*, 129,pp. 1028-38.

Ashmawi, W. M.; Baluch, M. H. and Azad, A. K. (1992): Crack control design of reinforced concrete beams in flexure. Concrete Design Based on Fracture Mechanics, Proceedings, ACI Committee 446, pp. 133-46.

Barenblatt, G. I. (1962): The Mathematical Theory of Equilibrium Cracks Formed In Brittle Fracture. *Zhurnal Prikladnoy Mekhaniki i Tekhnicheskoy Fiziki*, 4,pp. 3-56.

Bosco, C.; Carpinteri, A. (1992): *Fracture mechanics evaluation of minimum reinforcement in concrete structures.* Application of fracture mechanics to reinforced concrete, Proc. Int. Workshop, Turin, Italy, 1990, Ed. Elsevier, London, pp.347–377.

Dugdale, D. S. (1960): Yielding of steel sheets containing slits. *J. Mechanics Physics Solids,* vol. 8, pp.100-104.

Guinea, G. V.; Pastor, J. Y.; Planas, J.; Elices, M. (1998): Stress intensity factor, compliance and CMOD for a general three-point-bend beam. *International Journal of Fracture Mechanics*, vol. 89, pp.103–116.

Hedadal, O.; Kroon, I. B. (1991): Lightly Reinforced High Strength Concrete. Master Thesis, University of Aalborg, Denmark.

Hillerborg, A. (1988): Applications to Fracture Mechanics to Concrete. Report TYBM-3030, Lund, Sweden, pp. 1-28.

Kaar, P. H.; Hognestad, E. (1965): High strength bars as concrete reinforcement-Part 7: Control of cracking in T-beam flanges. *Journal of PCA Research and Development Laboratories*, vol. 7, no. 1, pp. 42-53.

Planas, J.; Ruiz, G.; Elices, M. (1993): Fracture Criteria for Concrete: Mathematical Approximations and Experimental Validation. *Engg. Fracture Mechanics*, vol. 35, no. 1/2/3, pp. 87-94.

Raghu Prasad, B. K.; Bharatkumar, B. H.; Murthy, D. S. R.; Narayanan, R.; Gopalakrishnan, S. (2005): Fracture Mechanics Model for Analysis of Plain and Reinforced High-Performance Concrete Beams. *Journal of Engineering Mechanics*, vol. 131, pp. 831-838.

RILEM TC 162-TDF Test and Design Methods for Steel Fibre Reinforced Concrete, 1985.

Roesler, J.; Paulino, G. H.; Park, K.; Gaedicke, C. (2007): Concrete fracture prediction using bilinear softening. *Journal of Cement and Concrete Composites*, vol. 29, pp. 300-12.

Shah, S. P.; Swartz, S. E.; Ouyang, C. (1995): Fracture mechanics of concrete. John Wiley & Sons, New York.

Tada, H.; Paris, P.; Irwin, G. (1985): The stress analysis of cracks handbook. Del Research Corporation, St. Louis, Part 2, pp. 16–17.

Wallin, K. (2013): A New LEFM Based Description of Concrete Fracture and Size Effects. *International Conference on fracture*, Beijing, China.

Wu, Z.; Yang, S.; Hu, X.; Zheng, J. (2006): An analytical model to predict the effective fracture toughness of concrete for three-point bending notched beams. *Journal of Engineering Fracture Mechanics*, vol. 73, pp. 2166–91.

Wu, Z.; Yang, S.; Hu, X.; Zheng, J.; Fan, X.; Shan, J. (2010): Analytical Solution for Fracture Analysis of CFRP Sheet–Strengthened Cracked Concrete Beams. *Journal of Engineering Mechanics*, vol. 136, pp.1202-19.