

Finite Deflection of Slender Cantilever with Predefined Load Application Locus using an Incremental Formulation

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Abstract: In this paper, a class of problems involving space constrained loading on thin beams with large deflections is considered. The loading is such that, the locus of the force application point moves along an arbitrarily predefined path, fixed in space. Both linear elastic as well as elastic-perfectly plastic materials are considered. A simplification is realized using the moment-curvature relationship directly. The governing equation obtained is highly non-linear owing to inclusion of both material and geometric non-linearity. A general algorithm is described to solve the governing equation using an incremental formulation coupled with Runge Kutta 4th order initial value explicit solver. Additionally, the presented method is capable of handling unloading and reverse loading conditions. An example problem where the load application point locus is an inclined straight line is solved to demonstrate the performance of the method. It is found that, the force response due to the inclined locus is stiffer than the vertical locus. This response is akin to dry friction condition on a vertical locus case.

Keywords: Large deflections, Beam bending, Material non-linearity, Geometric non-linearity, plasticity, incremental formulation.

Nomenclature

FAP	force application point: applied force acts at this intermediate point on the deformed beam (Fig. 1)
TPBVP	two point boundary value problem
IVP	initial value problem
x	the horizontal coordinate
w	vertical displacement of beam at any x
w', w''	first and second partial derivatives of w with respect to x respectively
E	Young's modulus

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I	area moment of inertia of beam c/s about neutral axis
l	horizontal distance between fixed support and FAP of cantilever
l_0	horizontal distance between fixed support and the intersection of FAP locus with x axis
δ	vertical displacement at FAP
ψ	angle the tangent to the deformed beam makes with horizontal at FAP
κ	curvature of deformed beam
κ_y	maximum elastic curvature of deformed beam
M	bending moment
M_y	maximum elastic resistive bending moment of beam section
D	tangent modulus of moment-curvature relationship
L	horizontal span in three point bend case of non zero bending moment
R	central vertical force on simply supported three point bending beam
F	resultant force on cantilever at FAP
RK4	Runge Kutta 4 th order method for solving initial value problems

1 Introduction

The subject of the large deflection of linear elastic thin beams has always been of great historical interest for many researchers in the field of applied mathematics and mechanics. This problem is popularly known as the "elastica". The elastica and its variations attracted great minds like Galileo, James Bernoulli, Euler just to name a few, as documented by Levien (2008). In the past few decades the requirement of minimum weight criteria in aerospace industry has led to a renewed interest in the study of flexible structures as noted by Fertis (2006). The simplest flexible structure popular among researchers is that of an elastic cantilever beam undergoing large deflection. Large deflection of a flexible cantilever beam is essentially a two point boundary value problem (TPBVP) whose closed form analytical solutions are limited to evaluation of elliptic integrals. From mathematical point of view the problem solving approach has changed from analytical to semi-analytical and more recently to efficient numerical techniques. To incorporate various engineering constraints like plasticity or frictional contact etc, numerical methods are more easily implementable and hence we see an upsurge in this field of research. However for validation of non-exact procedures, the elliptic integral solutions, wherever possible, remain crucial.

Working to obtain an analytical solution to the elastica, Bisshopp and Drucker (1945) obtained the solution for the large deflection of linear elastic horizontal cantilever beam under a concentrated vertical load at the free end, in terms of elliptic integrals. A comprehensive account of elliptic integral techniques pertaining to

elastica may be found in the work of Frisch-Fay (1962). Theocaris, Paipetis and Paolinelis (1977) investigated large deflection of three point bending considering axial stress and expressed the solutions in the form of elliptic integrals. More recently Zakharov and Okhotkin (2002) solved the elastica for various end conditions using elliptic integrals.

Elliptic integral based analytic solutions are explicit in the end slope of the elastica curve and implicit for load or displacements. This causes iterations to be involved when load or displacement solutions are sought. In order to primarily overcome the inconvenience of implementation caused due to this implicitness of the elliptic integral solutions, various semi analytical techniques are devised. Wang, Chen and Liao (2008) employed a semi-analytical technique called the homotopy analysis method to solve the elastica. Elgohary, Dong, Junkins and Atluri (2014) recently presented scalar homotopy method and applied it to solve the William's toggle which involves geometric non-linearity. Similar to homotopy perturbation method there exists another one called the Adomian decomposition technique and this was used by Banerjee, Bhattacharya and Mallik (2008) to solve the elastica. Ghosh and Roy (2007) employed the method to solve large deflection elastic and monotonic plastic deformation cases. The semi-analytical techniques primarily solves elastic problems. The methods need extensive use of symbolic operation softwares. Hence, though the methods does provide explicit solutions they suffer from the short comings of providing solutions in the form of long expressions with a scope only for relatively simpler problems.

In order to overcome the shortcomings of the analytical techniques, numerical approaches are adopted. Among various numerical methods, though FEM is a very versatile tool in solving structural problems, sometimes other non-FEM based computational techniques do appear to be economical and easier. This is quite apparent while dealing with beam problems involving both geometric and material non-linearity. Some of the non-FEM based numerical research work are indicated here . For a comprehensive review and performance of various pertinent numerical methods, the seminal work of Dong, Alotaibi, Mohiuddine and Atluri (2014) may be consulted. Wang (1969) solved the flexible beam problem under uniformly distributed load by employing the Newton Raphson root finding scheme and numerical integration. Holden (1972) used RK4 method coupled with iterative numerical integration scheme of Simpson to solve for finite deflection profile and critical buckling load of flexible beam-columns, under uniformly distributed lateral and axial loads. Lewis and Monasa (1981) considered Ludwick type material property for a cantilever undergoing large deflection when acted upon by a vertical concentrated load at its free end. They solved the problem by using RK4 method and iterative numerical integration. Lee, Wilson and Oh (1993) investigated the elastica problem

by considering the cantilever to be non-prismatic under combined loading. They solved the problem by coupling Runge-Kutta and Regula-Falsi methods. Wang (1987) solved the flexible inclined cantilever under end load by utilizing perturbation and numerical integration scheme. Nallathambi, Rao and Srinivasan (2010) solved the large deflection of elastic curved beam under tip concentrated follower load problem by iterative shooting method.

Since plastic deformation is common in various engineering applications when a beam bends by a finite amount, a study towards its understanding and implementation is important. Yu and Johnson (1982) coined the terminology 'plastica' indicating an extension of the closed form elastica theory to incorporate plasticity. He solved the plastica: a cantilever under conservative compressive force using the perturbation technique and numerical integration. Subsequent to this, Xiao-qiang and Tongxi (1986) analyzed the entire process of plastic deformation using the plastica theory. For a horizontal beam under vertical tip force they presented numerical solution to unloading process in plastically deformed region.

Using the plastica theory Feng and Tong-xi (1991) studied the influence of end angle of tip load under the assumption of monotonic plastic loading conditions. Huang, Yu, Lu and Lippmann (2003) approached the air bending problem of sheet metal forming by using a completely numerical technique coined as mass spring finite difference model. It is an efficient incremental method based on iterative finite difference scheme. They incorporated the unloading process also in the plastically deformed domain in the formulation. The authors prescribed the end movement and end rotation of the elasto-plastic beam. However, the problem of predefined load application path at an intermediate point of a beam instead of at its ends, may also be of interest. For example, in metal forming process a punch descends down centrally on a symmetrically placed metal sheet over a die. The overhang region of the sheet is drawn into the die region as the deformation process progresses. The cantilever idealization of this problem leads to vertical predefined locus of force application point. Depending on this predefined path, numerous potentially important results may be obtained for complete elasto plastic deformation process. Large deflection, elasto-plastic deformation and contact; all bring in non-linearity to a beam deflection problem. In contact problems involving a beam and a rigid surface, the point of application of contact force on the beam keeps changing in an implicit manner. In order to simplify the analysis, a guess curve in space instead of the locus of actual point of contact, may serve the purpose of explicit approximate solution. A guess curve rendered solution may also be used as the trial solution for a fully implicit algorithm which may reduce the computational cost in terms of number of iterations. In structural softening non-linear problems, implicit formulation becomes complicated owing to requirement of sophisticated strategy of tracing

the equilibrium path. Results from explicit methodology such as the one described here, may serve as fair approximate solution in those cases.

The objective of the current work is to present a general approach to solve explicitly problems in which force is applied along an arbitrarily defined path. For example in three point bending case when support radius is of considerable dimension, the contact point moves towards the loading agency along the roller surface in an implicit manner. This has a stiffening effect in the force v.s. central deflection response. A simple guess path simulating this effect is clearly of interest from simplicity in formulation point of view.

The problem involves both material and geometric non-linearity. Such a problem is referred to, in this paper, as space constrained force problem (SCFP), see Fig. 1. The method formulated is such that one should be able to use it to handle non-monotonic loading directly.

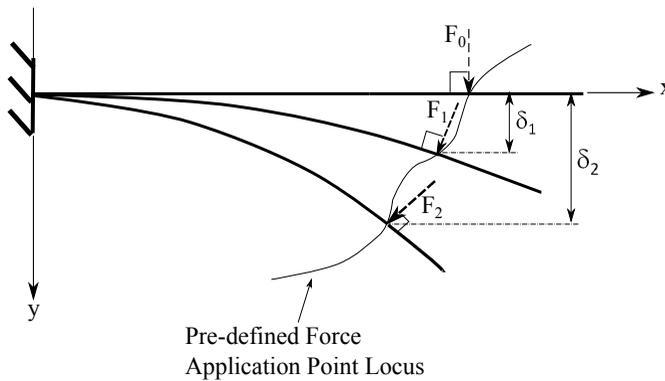


Figure 1: A typical space constrained force problem depiction

The approach is based on an incremental formulation coupled with RK4 solver. The discussed formulation is a direct and easily implementable method of solving non-linear structural problems which renders explicit solution, Bathe and Bolourchi (1979). In SCFP we obtain non-linear differential equation governing the response of the structure. This equation is solved for the incremental kinetic and kinematic quantities at each such time steps for a given increment of load or displacement. In this incremental formulation, the reference configuration for a given step is taken to be that of the equilibrated configuration of the previous step. Based on this approach an example problem is explicitly solved, see Fig. 2. Here, we considered a horizontal cantilever with load acting on it at an intermediate point in such a way that the locus of point of application of the force remained at an inclination of θ with the vertical while the force direction was kept normal to the deformed

axis of the beam. One of the practical applications of SCFP when $\theta = 0$ can be found in sheet metal forming processes as can be seen in Kalpakjian, Schmid and Kok (2008) where a vertically descending punch is positioned symmetrically with respect to a die on which the sheet is placed horizontally. The method is simple to formulate, easy to implement in computationally less intensive facility. This is what is desirable in a press brake control system where faster results are sought with minimal iterations .

The general formulation of SCFP is explained in section 2. In section 3, the numerical technique for the governing differential equation with appropriate boundary conditions for the example SCFP is described. The validation of results with published literature and three point bend experiment is presented in section 4.1. The effect of inclination (clockwise and anti-clockwise) of FAP loci in force response is discussed in section 4.2. In section 4.3, the origin and significance of a non-dimensional parameter is described. The non-monotonic loading response for various inclinations of FAP loci is presented in section 4.4.

2 Problem Formulation

A straight prismatic cantilever beam under an arbitrarily defined locus of force application point (FAP) is shown in Fig. 1. In a particular SCFP, see Fig. 2, the locus is an inclined straight line making an angle θ with the vertical. The follower force is kept perpendicular to the deformed axis of the beam. This force in general could be conservative or follower in other ways too.

For an Euler-Bernoulli¹ beam the kinematic condition is given by :

$$\kappa = \kappa(x, t) = \frac{w''}{\{1 + (w')^2\}^{\frac{3}{2}}} \quad (1)$$

$$w = w(x, t)$$

A general rate independent constitutive law, governing the moment and curvature relationship may be expressed as:

$$D = \frac{dM}{d\kappa} \quad (2)$$

In Eq. (2), D is the flexural rigidity of the structure. Clearly in the linear elastic case it is a constant and is given by EI . In a non-linear elastic case it could be a function of curvature (κ) given by the slope of moment curvature relationship. However in the elasto-plastic case, for plastic loading, it is the slope of moment curvature curve

¹ thin beam formulation within small strain framework leading to moderately large curvature is considered

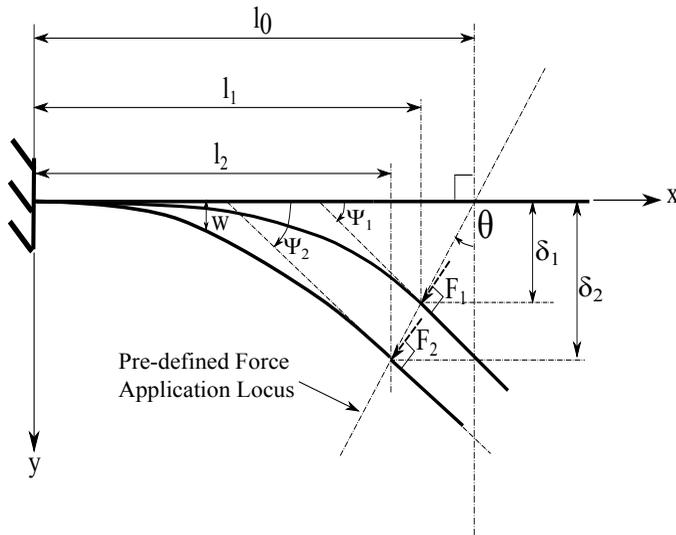


Figure 2: Cantilever for linear inclined FAP locus

and in all other cases of linear elastic loading or unloading, it is a constant and is given by EI . The kinetic relationship in a general case is given by :

$$M = M(x, t) \tag{3}$$

Combining the kinematic condition, constitutive law and the kinetics and then comparing the coefficient of dt , the governing non-linear differential equation for beam deflection is obtained as :

$$\frac{\partial M}{\partial t} = D \frac{\partial \kappa}{\partial t} \tag{4}$$

When the deformation process is quasi static under end displacement controlled loading, the quantities at any time t may be equivalently considered to be functions of $\delta(t)$ instead. Hence the temporal integration of Eq. 4 over a small time step of Δt can be conveniently written as :

$$\Delta M = D \Delta \kappa \tag{5}$$

In which: $\Delta(\cdot)_t = \frac{\partial(\cdot)}{\partial t} \Big|_t \Delta t = \frac{\partial(\cdot)}{\partial \delta} \Big|_{\delta(t)} \Delta \delta$.

Eq. 5 is the linear incremental governing differential equation with increment in displacement w at any x as the primary dependent variable, for the step from time instant t to $t + \Delta t$.

3 Methodology for Numerical Solution

The solution methodology can be efficiently explained with the help of an example problem as shown in Fig. 2. In this particular case the eq. 3 is given by:

$$M = F \cos \psi (l - x) + F \sin \psi (\delta - w) \quad (6)$$

In which $l = l_0 - \delta \tan \theta$

Subsequently, the governing incremental differential equation after non-dimensionalization is given by:

$$\alpha \Delta \bar{w}'' - \beta \Delta \bar{w}' + \gamma \Delta \bar{w} = A \Delta \bar{F} + B \Delta \psi + C \Delta \bar{\delta} \quad (7)$$

Where

$$\Delta((\cdot)''') = (\Delta(\cdot))''', \quad \Delta((\cdot)') = (\Delta(\cdot))', \quad \alpha = \frac{1}{\{1 + (\bar{w}')^2\}^{\frac{3}{2}}}, \quad \beta = \frac{3\bar{w}'' \bar{w}'}{\{1 + (\bar{w}')^2\}^{\frac{5}{2}}}$$

$$\gamma = \frac{\bar{F}}{\bar{D}} \sin \psi, \quad A = \frac{1}{\bar{D}} \{(\bar{x}_{max} - \bar{x}) \cos \psi + (\bar{\delta} - \bar{w}) \sin \psi\}$$

$$B = \frac{\bar{F}}{\bar{D}} \{(\bar{\delta} - \bar{w}) \cos \psi - (\bar{x}_{max} - \bar{x}) \sin \psi\}, \quad C = \frac{\bar{F}}{\bar{D}} (\sin \psi - \cos \psi \tan \theta)$$

$$\bar{x}_{max} = 1 - \bar{\delta} \tan \theta, \quad \bar{w}' = w', \quad \bar{w}'' = l_0 w'', \quad \bar{F} = \frac{F l_0^2}{EI}, \quad \bar{D} = \frac{D}{EI},$$

$$\bar{\delta} = \frac{\delta}{l_0}, \quad \bar{x} = \frac{x}{l_0}, \quad \bar{w} = \frac{w}{l_0}$$

The boundary conditions for the equation is given by:

$$\Delta \bar{w}|_{\bar{x}=0} = 0, \quad \Delta \bar{w}'|_{\bar{x}=0} = 0, \quad \Delta \bar{w}|_{\bar{x}=\bar{x}_{max}} = \Delta \bar{\delta}, \quad \Delta \bar{w}'|_{\bar{x}=\bar{x}_{max}} = \sec^2 \psi \Delta \psi \quad (8)$$

In order to efficiently solve Eq. 7, it is decomposed into three IVPs, defined as:

$$\Delta \bar{w} = v_1 \Delta \bar{F} + v_2 \Delta \psi + v_3 \Delta \bar{\delta} \quad (9)$$

In which

$$\alpha v_1'' - \beta v_1' + \gamma v_1 = A \quad (10)$$

$$\alpha v_2'' - \beta v_2' + \gamma v_2 = B \quad (11)$$

$$\alpha v_3'' - \beta v_3' + \gamma v_3 = C \quad (12)$$

With the initial conditions given by:

$$v_i|_{\bar{x}=0} = 0, \quad v'_i|_{\bar{x}=0} = 0; \quad i = 1, 2, 3 \quad (13)$$

Eq.s 10-12 are solved by using RK4 method in which α , β , γ , A , B and C are evaluated based on previous step. Clearly for the very first step, the displacement field is considered to be trivial and the entire beam is assumed to be in linear elastic state. The conditions specified at $\bar{x} = \bar{x}_{max}$, in the Eq. (8) is invoked to evaluate increment in end angle $\Delta\psi$ and increment in non-dimensional force $\Delta\bar{F}$ from Eqs. (14) and (15) which are obtained from (9) and its space derivative:

$$v_1|_{\bar{x}_{max}}\Delta\bar{F} + v_2|_{\bar{x}_{max}}\Delta\psi = (1 - v_3|_{\bar{x}_{max}})\Delta\bar{\delta} \quad (14)$$

$$v'_1|_{\bar{x}_{max}}\Delta\bar{F} + (v'_2|_{\bar{x}_{max}} - \sec^2\psi)\Delta\psi = -v'_3|_{\bar{x}_{max}}\Delta\bar{\delta} \quad (15)$$

Subsequently, the increment in displacement (Eq. 9) and its derivatives are evaluated. Followed by this, the displacement field is updated and α and β evaluated. Then the increment in curvature (normalized w.r.t l_0) is computed from:

$$\Delta\bar{\kappa} = \alpha\Delta\bar{w}'' - \beta\Delta\bar{w}' \quad (16)$$

With increment in curvature as basic input to a constitutive module along with known state variables pertaining to the previous step, the tangent flexural rigidity \bar{D} is obtained. In the present paper we discuss an elasto-perfectly plastic uniaxial stress strain case with isotropic hardening moment curvature law for a rectangular beam section as shown in Fig.3. The state variables are bending moment, yield moment, curvature and plastic curvature. The yield moment can increase and plastic curvature change possible only when plastic deformation takes place. Employing return mapping algorithm, increment in state variables are computed along with the \bar{D} . Subsequent to invocation of constitutive module, γ , A , B and C are evaluated and the process is continued till the final displacement $\bar{\delta}|_{l_{max}}$ is reached.

The solution of this methodology reduces to the three point bending problem by considering $\theta = 0$ and assuming negligible roller diameter. In the three point bending problem, the normalized central force \bar{R} is given by:

$$\bar{R} = \frac{RL^2}{EI} = 8\bar{F}\cos\psi \quad (17)$$

Where $L = 2l_0$ is the total span of beam under non zero bending moment.

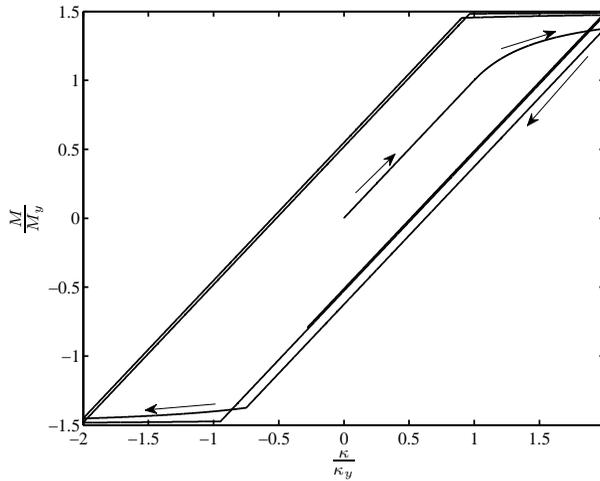


Figure 3: Normalized moment vs. curvature relationship for rectangular cross-section elasto-perfectly plastic beam

4 Results and Discussions

In section 4.1, the validation of the incremental formulation is tested against published literature and experimental outcomes is presented. In section 4.2, the role of inclination angle θ on the reactive force response is discussed. A non-dimensional quantity governing and influencing the responses of SCFP is defined in section 4.3. The unloading response for plastically deformed cases with different inclinations is discussed in the following section.

4.1 Validation

To validate the accuracy of the method in prediction of the deformed profile of a thin beam, the linear elastic cantilever result is compared with published result of Nallathambi, Rao and Srinivasan (2010). In the present way of analysis (incremental) the following inputs are supplied: the end vertical displacement of the cantilever and the horizontal projection of the deformed beam. The outputs are the profile as shown in Fig. 4.1 and the reactive force. The reactive force is found to be 7.6 kN and the profiles are seen to be in excellent agreement.

In quasi static structural problems, instability onset may be defined as the point beyond which the slope of load-displacement curve becomes negative. In this spirit the capability of the method in predicting the onset of instability and beyond is tested by comparing its result with experimental outcome of a three point bend test.

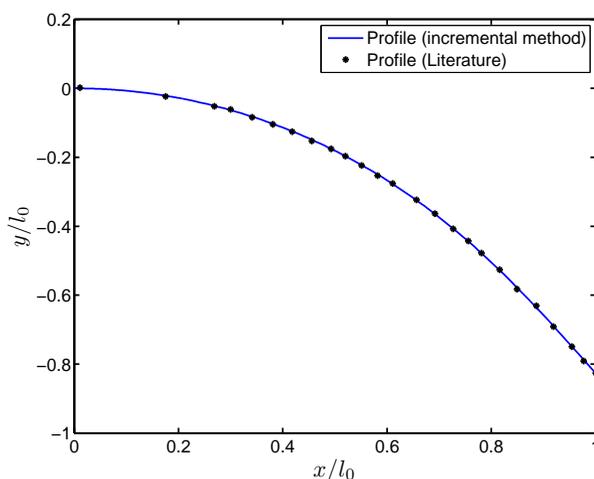


Figure 4: Validation of the incremental method with literature Nallathambi, Rao and Srinivasan (2010)

A three point bend test under displacement control loading (see Fig. 5) is performed on a thin tempered steel sheet on a Servo Hydraulic testing machine from BISS with a maximum capacity of 15 kN and least count of 0.01 N. The support rollers used have diameter of 10 mm. Samples in the form of rectangular strips are used for the bending test. The dimensions of the samples used are 300 mm in length and 20 mm in width with 0.25 mm and 0.3 mm in thickness (for two kinds of samples). Two samples each of 0.25 mm and 0.3 mm thicknesses are tested. The center to center distance between the support rollers is maintained at 98.87 mm. The punch is used to apply load centrally to the samples.

The samples are tested at two cross-head speeds: 1 and 2 mm/min and the responses are found to be independent of loading rates. After complete removal of the load the samples came back to their original shapes and sizes that concluded elastic loading process. The output of the experiment is obtained in the form of load vs. displacement plots. Since the samples slipped between the support rollers as the central deflection increased, the distance between the points of contact of sample to the roller decreased from 98.87 mm to the limiting minimum of 88.87 mm (10 mm is the roller diameter which is deducted from 98.87 mm) when the overhangs could have become almost vertical. These two extreme lengths are considered for normalization of the load-displacement data. The averaged normalized experimental data is presented in Fig. 6 along with the current incremental formulation prediction, for validation. In the equations, friction is not considered and hence it may be

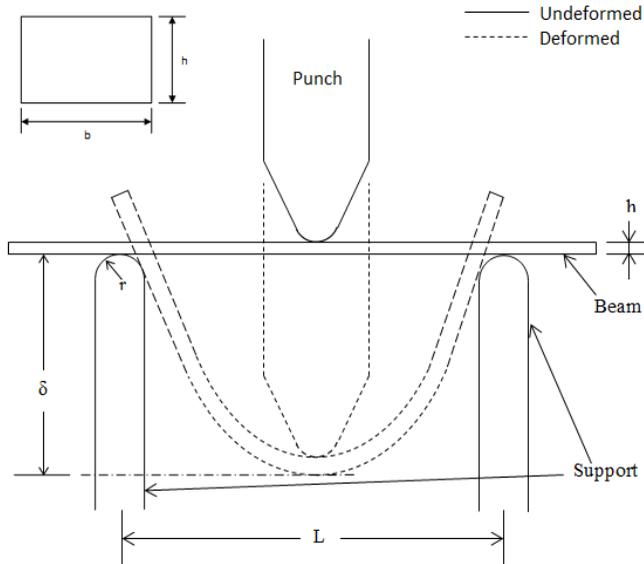


Figure 5: Schematic diagram of the experimental set up

postulated the reason for experimental result to show more stiffness as compared to the incremental results, see Theocaris et al Theocaris, Paipetis and Paolinelis (1977) for details.

4.2 Role of inclination of FAP locus

Depending on the clockwise or anticlockwise sense of θ , the force response varies significantly. If the sense is clockwise then it has a stiffening effect much akin to that of presence of friction see Theocaris, Paipetis and Paolinelis (1977) and is shown in Fig. 7, for elastic deformation. The corresponding bend profiles along with the FAP loci are presented in Fig. 8. Beyond the intersection points of the beams with the corresponding dotted FAP line, the beams may be assumed to be straight with the slope as that at the respective intersections.

4.3 Role of Elastica Parameter: ζ

² The bending moment in the beam may be normalized in two ways, viz. with respect to $\frac{EI}{l_0}$ and M_y .

² ζ is very much the same quantity as first pointed out by Yu et al Yu and Johnson (1982) in describing the elastic parameter

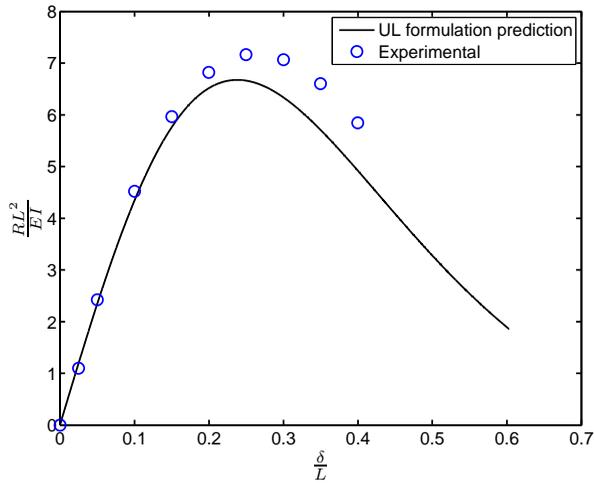


Figure 6: Validity of instability prediction in three point bending of elastic beam

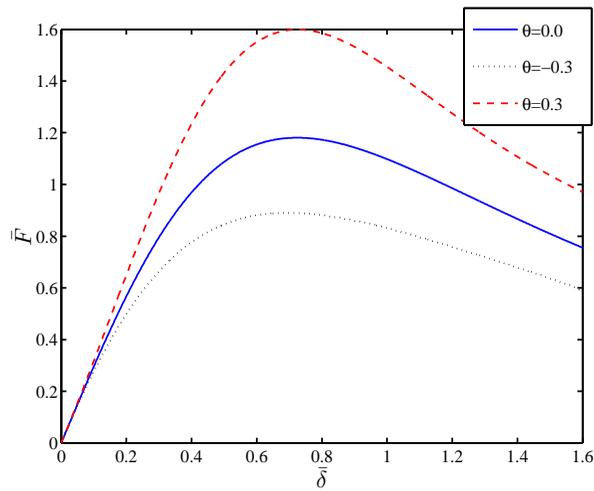


Figure 7: Normalized force vs. free end displacement in elastic case

Let :

$$\bar{M} = \frac{Ml_0}{EI}; \quad M^* = \frac{M}{M_y} \tag{18}$$

The relation between these two normalized bending moments for the case is given

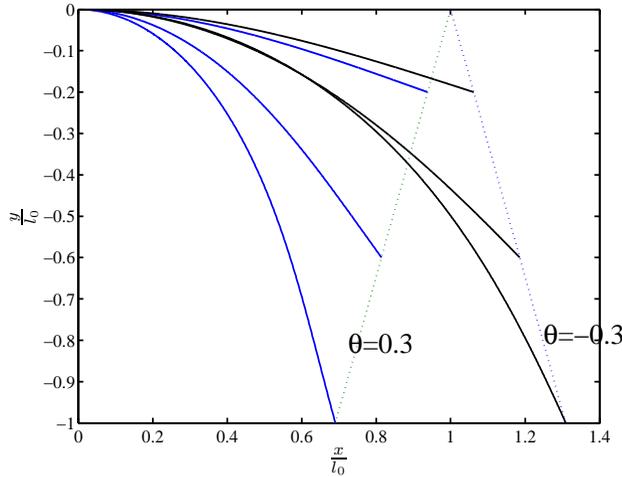


Figure 8: Deflected beam profile for various FAP loci

by:

$$\bar{M} = \left(\frac{M_y l_0}{EI}\right) M^* \quad (19)$$

And hence the elastica parameter is defined as:

$$\zeta = \frac{M_y l_0}{EI} \quad (20)$$

In the Fig. 9, for $\theta = 0$ the bending moment response for various ζ is presented. In a completely elastic bending process the maximum \bar{M} is seen to be 1.43. This is obtained by solving the elastica SCFP with $\theta = 0$. On the other hand, maximum M^* within elastic limit is 1. Hence, the limiting $\zeta_l = 1.43$ when both the maximums are reached simultaneously. When $\zeta > \zeta_l$ the bending will entirely be elastic, and the response will be that of elastica. However when $\zeta < \zeta_l$, the response curve will separate out from the elastica response at $\bar{M} = \zeta$, which follows from Eq. 19. It may be noted here that ζ_l is the maximum bending moment (\bar{M}) of the SCFP elastica solution and hence will be a function of θ or in a more general situation, on the path of FAP locus. This limiting elastica parameter subsequently may be used for design optimization.

4.4 Non-monotonic Loading Response

In Fig. 10 the force displacement response for identical final displacement and ζ is presented for three different inclination of FAP locus. The $\zeta (= 0.67)$ is chosen

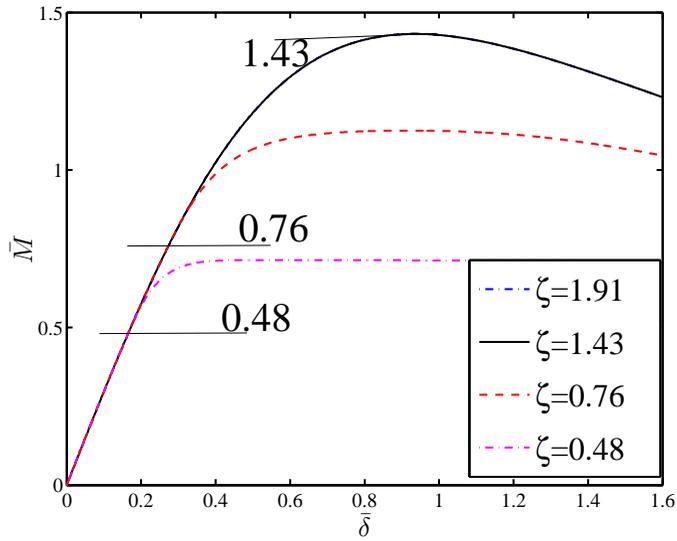


Figure 9: Normalized fixed end moment vs. free end displacement for $\theta = 0$

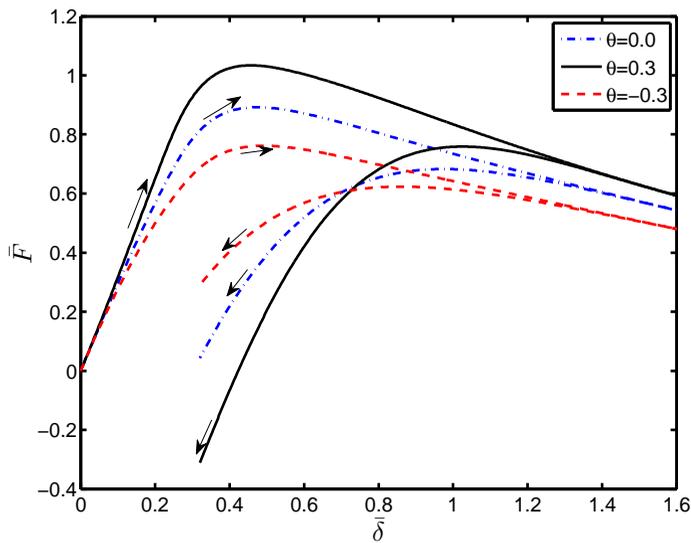


Figure 10: Non-monotonic loading response for various angle of inclinations of FAP locus for same displacement load and $\zeta = 0.67$

arbitrarily which ensured plastic deformation for all the three cases of θ .

In order to understand the influence of path of FAP (here θ) in inducing plastic deformation, a quantity η is defined as :

$$\eta(\theta) = \frac{\zeta_l(\theta) - \zeta}{\zeta_l(\theta)} \quad (21)$$

Where $0 \leq \zeta \leq \zeta_l(\theta)$

As increasing θ has a stiffening effect to \bar{F} as seen from Fig. 7, similar trend can be expected for \bar{M} . And since ζ_l is equal to the maximum of \bar{M} for a given θ , it is intuitive to perceive that $\zeta_l(\theta)$ increases with increase in θ .

From this understanding and considering Eq. 21, it may be postulated that an higher η will lead to larger plastic deformation as can be confirmed from Fig. 10. It can be seen that for $\theta = -0.3$ the plastic deformation is least. And for the most stiff case i.e. when $\theta = 0.3$, tensile or normally outward force is developed at the FAP.

5 Conclusion

A class of problems where the FAP is constrained to move in an arbitrarily pre-defined locus may be solved efficiently using the method proposed in this paper. The condition of FAP locus being a straight line, but inclined with the vertical, is solved here. The complexity of this problem arises from the inclusion of material and geometric non-linearity. Analytical solution to such problems when the material is elasto-plastic is difficult to obtain. In this paper, an incremental method is employed to solve the governing differential equation. Local elastic unloading which may occur in large deflection problems is naturally incorporated in the formulation. A non-dimensional parameter depending on both material and geometry is obtained here via the process of normalization of bending moment by two different ways. This parameter is seen to govern the fixed end moment versus end displacement response in an elasto-plastic case. It precisely defines the point on the elastic response curve where the elasto-plastic curve branches out.

The algorithm presented here, is based on displacement controlled loading. However, owing to the versatile structure of the algorithm, the controlling exciter may easily be interchanged to load or end angle (ψ). It is suitable for problems involving material and/or structural softening. While elliptic integral solutions are always end slope controlled, the method proposed in this paper may be used to explicitly obtain the responses based on other exciter as well. The effect of friction can also be incorporated easily into the governing equation without changing the basic algorithm structure.

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