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Exact Solutions and Mode Transition for Out-of-Plane Vibrations of Non-uniform Beams with Variable Curvature

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Abstract: The two coupled governing differential equations for the out-of-plane vibrations of non-uniform beams with variable curvature are derived via the Hamilton's principle. These equations are expressed in terms of flexural and torsional displacements simultaneously. In this study, the analytical method is proposed. Firstly, two physical parameters are introduced to simplify the analysis. One derives the explicit relations between the flexural and the torsional displacements which can also be used to reduce the difficulty in experimental measurements. Based on the relation, the two governing characteristic differential equations with variable coefficients can be uncoupled into a sixth-order ordinary differential equation in terms of the flexural displacement only. When the material and geometric properties of the beam are in arbitrary polynomial forms, the exact solutions with regard to the out-of-plane vibrations of non-uniform beams with variable curvature can be obtained by the recurrence formula. In addition, the mode transition mechanism is revealed and the influence of several parameters on the vibration of the non-uniform beam with variable curvature is explored.

Keywords: Out-of-plane vibration; Variable curvature; Non-uniform beam; Exact solution; Mode transition.

Nomenclature

A	= cross-sectional area of the beam
b_n	= dimensionless bending rigidity, $E(s)I_n(s)/E(0)I_n(0)$
b_t	= dimensionless shear rigidity, $G(s)J(s)/G(0)J(0)$

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c	= dimensionless curvature, $\kappa(s)/\kappa(0)$
E	= Young's modulus of beam
G	= shear modulus of beam
$I_n(s), I_b(s)$	= area moments of inertia of the beam section about the n and b axes, respectively.
I_t	= dimensionless polar moment of inertia, $\rho(s)J(s)/\rho(0)J(0)$
$J(s)$	= polar moment of inertia of the beam section about the t axis
L	= length of beam
L_n	= slenderness ratio, $L\sqrt{A(0)/I_n(0)}$
m	= dimensionless mass per unit length, $\rho(s)A(s)/\rho(0)A(0)$
R	= arch radius
s	= coordinate along the beam
t	= time variable
T^*	= kinetic energy
v, u, w	= center line displacements of the beam in the tangent, the normal and binormal directions, respectively.
u_t, u_n, u_b	= displacements of the beam in the tangent (t), the normal (n) and the binormal (b) directions, respectively.
y, z	= coordinate from the center line in the normal and binormal direction., respectively.
W^*	= dimensionless displacement, W/L.
V^*	= strain energy
κ	= curvature of the beam, 1/R
θ_0	= arc angle, $\kappa(0)L$
Φ	= torsional angle
Φ^*	= dimensionless torsional angle, Φ
ξ	= dimensionless distance to the root of the beam, s/L
ρ	= mass density per unit volume of beam
Ω	= angular frequency
ω	= dimensionless frequency, $\Omega L^2\sqrt{\rho(0)A(0)/E(0)I_n(0)}$
ς	= slenderness ratio parameter, L_n^2
η	= ratio of shear and bending rigidities, $G(0)J(0)/E(0)I_n(0)$
δ	= ratio of polar and area inertias, $J(0)/I_n(0)$
ε, γ	= normal and shear strains, respectively
σ, τ	= normal and shear stresses, respectively

1 Introduction

Since curved beams are basic structural components, their free vibration behaviors have been studied extensively. Based on the Bernoulli-Euler and Timoshenko beam theories, the studies on the static and dynamic response of straight beam structures are tremendous [Lee and Lin (1996); Lee and Hsu (2007); Lin, Lee, and Lin (2008); Lin (2010)]. For curved beams, an intriguing review can be found in the papers by Childamparam and Leissa (1993); Auciello and De Rosa (1994); Hajianmaleki and Qatu (2013).

When the in-plane vibration of a curved beam is considered, the motion is the coupled flexural-flexural deflections in two directions [Lee and Wu (2009); Ren, Su, and Yan (2010); Lin (2011); Lin, Liauh, Lee, Ho, and Wang (2014); Pandit, Thomas, Patel, and Srinivasan (2015); Lin and Lee (2016)]. A lot of literatures in devotion to this problem can be found. When the out-of-plane vibration of a curved beam is considered, the motion is the coupled flexural-torsional deflections. So far, a few of literature on this problem are presented. The relevant literatures are introduced as follows:

Due to its complexity, several approximated methods are usually applied to solve this system in the literatures. Huang, Tseng, Chang, and Hung (2000) investigated the out-of-plane dynamic analysis of beams with arbitrarily varying curvature and cross-section by dynamic stiffness matrix method. The influences of different curvature and cross-section on the natural frequency are studied. Piovan, Cortidnez, and Rossi (2000) investigated the influence of the shear flexibility, due to bending and warping, over the dynamics of the member by the finite element method. Lee, Oh, Mo, and Lee (2008) investigated the effects of different boundary conditions and different curvature on the natural frequencies and mode shapes by the finite element method. Malekzadeh, Haghighi, and Atashi (2010) studied the effects of temperature rise, boundary conditions, material and geometrical parameters on the natural frequencies by the differential quadrature method. Karami, Yardimoglu, and Inman (2010) studied the out of plane vibrations of spiral beams for micro-scale applications by the Rayleigh's approximate method. Ishaquddin, Raveendranath, and Reddy (2012) investigated the flexure and torsion locking phenomena in out-of-plane deformation of Timoshenko curved beam element. It was found that the definition of low-order polynomial displacement in the strain cause usually the element to lock severely and affect the convergence to the true solution.

In addition, for the curved beam with constant curvature and cross-section, [Tufekci and Dogruer (2006)] presented the exact solution by using the initial value method. It was found that the mode transition phenomenon occurs at certain combinations of curvature and length of the arch. For the curved beam with constant curvature, [Lee and Chao (2000)] presented the exact solution. The influence of taper ratio, center angle and arc length on the first two natural frequencies of the beams is illustrated.

So far, no literature has presented the exact solution for out-of-plane vibration of the curved beam with variable curvature and non-uniform cross-section. Moreover, the mode transition mechanism is not clearly discussed. In this study, the exact solution of this general system will be derived and its mode transition mechanism is also investigated.

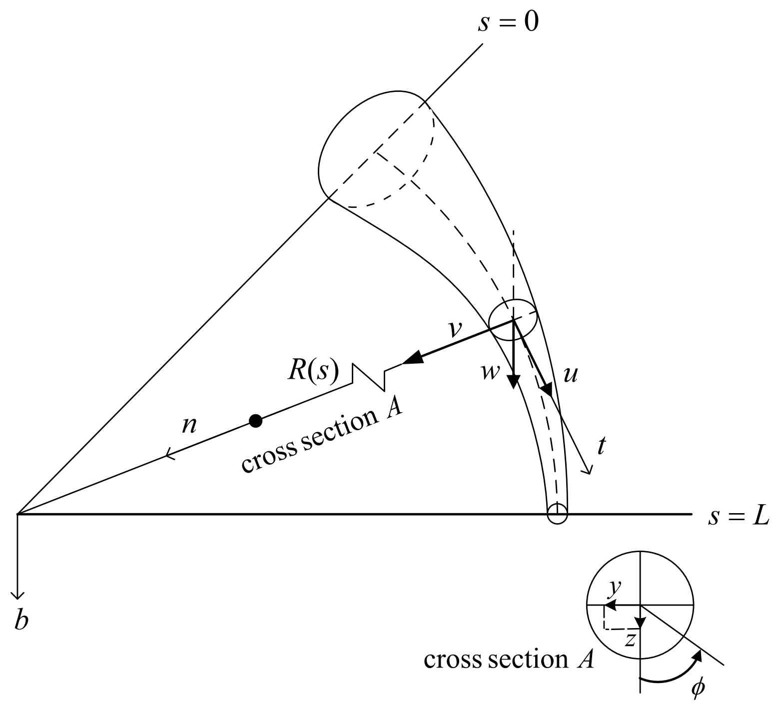

Figure 1: Geometry and coordinate system of a non-uniform beam of variable curvature.

2 Coupled governing equations

Consider the out-of-plane and in-plane motion of a curved non-uniform Bernoulli-Euler beam of variable radius R and circular cross-section, as shown in Figure 1. If the thickness of the beam is small in comparison with the radius of the beam, the displacement fields of the curved beam in curvilinear coordinates are:

$$u_t(s,y,z,t) = u(s,t) - y\left[\kappa(s)u(s,t) + \frac{\partial v(s,t)}{\partial s}\right] - z\frac{\partial w(s,t)}{\partial s}, \tag{1}$$

$$u_n(s,y,z,t) = v(s,t) - z\phi(s,t), \tag{2}$$
$$u_b(s,y,z,t) = w(s,t) + y\phi(s,t), \tag{3}$$

Substituting equations (1), (2) and (3) into the strain-displacement relations in the curvilinear coordinate, the only three non-zero strains, ε_{tt}, γ_{tn} and γ_{tb}, are

$$\begin{aligned}\varepsilon_{tt} &= \frac{1}{1-\kappa y}\left[\frac{\partial u_t}{\partial s} - \kappa u_n\right] \\ &= \frac{1}{1-\kappa y}\left[\left(\frac{\partial u}{\partial s} - \kappa v\right) - y\left(\frac{\partial^2 v}{\partial s^2} + \frac{\partial (ku)}{\partial s}\right) - z\left(\frac{\partial^2 w}{\partial s^2} - \kappa\phi\right)\right],\end{aligned} \tag{4}$$

$$\gamma_{tn} = \frac{\partial u_t}{\partial y} + \frac{1}{1-\kappa y}\left(\frac{\partial u_n}{\partial s} + \kappa u_t\right) = \frac{-z}{1-\kappa y}\left(\frac{\partial \phi}{\partial s} + \kappa\frac{\partial w}{\partial s}\right), \tag{5}$$

$$\gamma_{tb} = \frac{1}{1-\kappa y}\frac{\partial u_b}{\partial s} + \frac{\partial u_t}{\partial z} = \frac{y}{1-\kappa y}\left(\frac{\partial \phi}{\partial s} + \kappa\frac{\partial w}{\partial s}\right), \tag{6}$$

If the thickness of the beam is small in comparison with the radius of the beam, then y is small in comparison with R. The three strains can be reduced to

$$\varepsilon_{tt} = \left(\frac{\partial u}{\partial s} - \kappa v\right) - y\left(\frac{\partial(\kappa u)}{\partial s} + \frac{\partial^2 v}{\partial s^2}\right) - z\left(\frac{\partial^2 w}{\partial s^2} - \kappa\phi\right), \tag{7}$$

$$\gamma_{tn} = -z\left(\frac{\partial \phi}{\partial s} + \kappa\frac{\partial w}{\partial s}\right), \tag{8}$$

$$\gamma_{tb} = y\left(\frac{\partial \phi}{\partial s} + \kappa\frac{\partial w}{\partial s}\right). \tag{9}$$

Employing the three stress and strain relations, the three stresses are

$$\sigma_{tt} = E\left[\left(\frac{\partial u}{\partial s} - \kappa v\right) - y\left(\frac{\partial(\kappa u)}{\partial s} + \frac{\partial^2 v}{\partial s^2}\right) - z\left(\frac{\partial^2 w}{\partial s^2} - \kappa\phi\right)\right], \tag{10}$$

$$\tau_{tn} = -G\left[z\left(\frac{\partial \phi}{\partial s} + \kappa\frac{\partial w}{\partial s}\right)\right], \tag{11}$$

$$\tau_{tb} = G\left[y\left(\frac{\partial \phi}{\partial s} + \kappa\frac{\partial w}{\partial s}\right)\right]. \tag{12}$$

The strain energy of the beam is

$$\begin{aligned}V^* &= \frac{1}{2}\int_0^L\int_A \sigma_{tt}\varepsilon_{tt}\,dA\,ds + \frac{1}{2}\int_0^L\int_A \tau_{tn}\gamma_{tn}\,dA\,ds + \frac{1}{2}\int_0^L\int_A \tau_{tb}\gamma_{tb}\,dA\,ds \\ &= \frac{1}{2}\int_0^L\int_A E\left[\left(\frac{\partial u}{\partial s} - \kappa v\right)^2 + y^2\left(\frac{\partial(\kappa u)}{\partial s} + \frac{\partial^2 v}{\partial s^2}\right)^2 + z^2\left(\frac{\partial^2 w}{\partial s^2} - \kappa\phi\right)^2\right.\end{aligned}$$

$$-2z\left(\frac{\partial u}{\partial s}-\kappa v\right)\left(\frac{\partial(\kappa u)}{\partial s}+\frac{\partial^2 v}{\partial s^2}\right)-2y\left(\frac{\partial u}{\partial s}-\kappa v\right)\left(\frac{\partial^2 w}{\partial s^2}-\kappa\phi\right) \tag{13}$$
$$+2yz\left(\frac{\partial(\kappa u)}{\partial s}+\frac{\partial^2 v}{\partial s^2}\right)\left(\frac{\partial^2 w}{\partial s^2}-\kappa\phi\right)\Bigg]dAds$$
$$+\frac{1}{2}\int_0^L\int_A G\left[z^2\left(\frac{\partial\phi}{\partial s}+\kappa\frac{\partial w}{\partial s}\right)^2+y^2\left(\frac{\partial\phi}{\partial s}+\kappa\frac{\partial w}{\partial s}\right)^2\right]dAds,$$

Since the cross section of the beam considered is doubly symmetric, the first moment of area in each square bracket vanishes.

Therefore, the strain energy is simplified as

$$V^*=\frac{1}{2}\int_0^L\left[EA\left(\frac{\partial u}{\partial s}-\kappa v\right)^2+EI_b\left[\frac{\partial(\kappa u)}{\partial s}+\frac{\partial^2 v}{\partial s^2}\right]^2+EI_n\left(\frac{\partial^2 w}{\partial s^2}-\kappa\phi\right)^2\right.$$
$$\left.+GJ\left(\frac{\partial\phi}{\partial s}+\kappa\frac{\partial w}{\partial s}\right)^2\right]ds. \tag{14}$$

The kinetic energy of the system is

$$T^*=\frac{1}{2}\int_0^L\int_A\rho\left[\frac{\partial u_t}{\partial t}\cdot\frac{\partial u_t}{\partial t}+\frac{\partial u_n}{\partial t}\cdot\frac{\partial u_n}{\partial t}+\frac{\partial u_b}{\partial t}\cdot\frac{\partial u_b}{\partial t}\right]dsdA$$
$$=\frac{1}{2}\int_0^L\int_A\rho\left[\left(\frac{\partial u}{\partial t}\right)^2+\left(\frac{\partial v}{\partial t}\right)^2-2z\left(\frac{\partial v}{\partial t}\right)\left(\frac{\partial\phi}{\partial t}\right)+z^2\left(\frac{\partial\phi}{\partial t}\right)^2\right. \tag{15}$$
$$\left.+\left(\frac{\partial w}{\partial t}\right)^2+2y\left(\frac{\partial w}{\partial t}\right)\left(\frac{\partial\phi}{\partial t}\right)+y^2\left(\frac{\partial\phi}{\partial t}\right)^2\right]dsdA,$$

Similarly, the kinetic energy is simplified as

$$T^*=\frac{1}{2}\int_0^L\left\{\rho A\left[\left(\frac{\partial u}{\partial t}\right)^2+\left(\frac{\partial v}{\partial t}\right)^2+\left(\frac{\partial w}{\partial t}\right)^2\right]+\rho J\left(\frac{\partial\phi}{\partial t}\right)^2\right\}ds. \tag{16}$$

Applying the Hamilton's principle, the governing differential equations and the associated boundary conditions for the system can be derived. It can be shown that if the cross-section of the beam is doubly symmetric about the n and b axes, the in-plane and the out-of-plane vibrations non-uniform beams with variable curvature are independent.

The governing differential equations for the out-of-plane vibrations are

$$\frac{\partial}{\partial s}\left[GJ\kappa\left(\frac{\partial\phi}{\partial s}+\kappa\frac{\partial w}{\partial s}\right)\right]-\frac{\partial^2}{\partial s^2}\left[EI_n\left(\frac{\partial^2 w}{\partial s^2}-\kappa\phi\right)\right]-\rho A\frac{\partial^2 w}{\partial t^2}=0, \tag{17}$$

$$EI_n\kappa\left(\frac{\partial^2 w}{\partial s^2}-\kappa\phi\right)+\frac{\partial}{\partial s}\left[GJ\left(\frac{\partial\phi}{\partial s}+\kappa\frac{\partial w}{\partial s}\right)\right]-\rho J\frac{\partial^2\phi}{\partial t^2}=0, \tag{18}$$

and the associated boundary conditions are at $s=0$ and L:

$$GJ\kappa\left(\frac{\partial\phi}{\partial s}+\kappa\frac{\partial w}{\partial s}\right)-\frac{\partial}{\partial s}\left[EI_n\left(\frac{\partial^2 w}{\partial s^2}-\kappa\phi\right)\right]=0 \text{ or } w=0, \tag{19}$$

$$EI_n\left(\frac{\partial^2 w}{\partial s^2}-\kappa\phi\right)=0 \text{ or } \frac{\partial w}{\partial s}=0, \tag{20}$$

$$GJ\left(\frac{\partial\phi}{\partial s}+\kappa\frac{\partial w}{\partial s}\right)=0 \text{ or } \phi=0. \tag{21}$$

3 Free vibration

For time harmonic vibrations with angular frequency Ω, one assumes

$$w(s,t)=W(s)e^{i\Omega t}, \tag{22}$$

$$\phi(s,t)=\Phi(s)e^{i\Omega t}. \tag{23}$$

The coupled governing characteristic differential equations for the out-of-plane vibrations of non-uniform beams with variable curvature are

$$\frac{d}{ds}\left[GJ\kappa\left(\frac{d\Phi}{ds}+\kappa\frac{dW}{ds}\right)\right]-\frac{d^2}{ds^2}\left[EI_n\left(\frac{d^2W}{ds^2}-\kappa\Phi\right)\right]+\rho A\Omega^2 W=0, \tag{24}$$

$$EI_n\kappa\left(\frac{d^2W}{ds^2}-\kappa\Phi\right)+\frac{d}{ds}\left[GJ\left(\frac{d\Phi}{ds}+\kappa\frac{dW}{ds}\right)\right]+\rho J\Omega^2\Phi=0, \tag{25}$$

where the primes denote differentiation with respect to the s variable.
The associated boundary conditions are at $s=0$ and L:

$$GJ\kappa\left(\frac{d\Phi}{ds}+\kappa\frac{dW}{ds}\right)-\frac{d}{ds}\left[EI_n\left(\frac{d^2W}{ds^2}-\kappa\Phi\right)\right]=0 \text{ or } W=0, \tag{26}$$

$$EI_n\left(\frac{d^2W}{ds^2}-\kappa\Phi\right)=0 \text{ or } \frac{dW}{ds}=0, \tag{27}$$

$$GJ\left(\frac{d\Phi}{ds}+\kappa\frac{dW}{ds}\right)=0 \text{ or } \Phi=0. \tag{28}$$

In terms of the dimensionless quantities in the nomenclature table, the dimensionless governing characteristic differential equations and boundary conditions are shown below, respectively,

$$\eta\theta_0\left[b_t c\left(\Phi^{*\prime}+\theta_0 cW^{*\prime}\right)\right]'-\left[b_n\left(W^{*\prime\prime}-\theta_0 c\Phi^*\right)\right]''+m\omega^2W^*=0, \tag{29}$$

$$\theta_0 b_n c\left(W^{*\prime\prime}-\theta_0 c\Phi^*\right)+\eta\left[b_t\left(\Phi^{*\prime}+\theta_0 cW^{*\prime}\right)\right]'+\frac{\delta}{\varsigma}I_t\omega^2\Phi^*=0. \tag{30}$$

The associated boundary conditions are

$$\eta\theta_0 b_t c\left(\Phi^{*\prime}+\theta_0 cW^{*\prime}\right)-\left[b_n\left(W^{*\prime\prime}-\theta_0 c\Phi^*\right)\right]'=0 \text{ or } W^*=0, \tag{31}$$

$$b_n\left(W^{*\prime\prime}-\theta_0 c\Phi^*\right)=0 \text{ or } W^{*\prime}=0, \tag{32}$$

$$\eta b_t\left(\Phi^{*\prime}+\theta_0 cW^{*\prime}\right)=0 \text{ or } \Phi^*=0. \tag{33}$$

4 Solution method

4.1 Governing Differential Equations in terms of displacement W^* only

To uncouple the governing characteristic differential equations (24) and (25), one defines the following two physical parameters:

$$F_{bW}=\frac{d}{ds}\left(GJ\kappa^2\frac{dW}{ds}\right)-\frac{d^2}{ds^2}\left(EI_n\frac{d^2W}{ds^2}\right)+\rho A\Omega^2 W, \tag{34}$$

$$T_{tW}=EI_n\kappa\left(\frac{d^2W}{ds^2}\right)+\frac{d}{ds}\left(GJ\kappa\frac{dW}{ds}\right), \tag{35}$$

where F_{bw} and T_{tw} are the forces per unit arc length in the b direction and the torque per unit arc length in the t direction, caused by the flexural deflection parameter W, respectively. The corresponding non-dimensional parameters are

$$F^*_{bW}=\eta\theta_0^2\left(b_t c^2 W^{*\prime}\right)'-\left(b_n W^{*\prime\prime}\right)''+m\omega^2 W^*, \tag{36}$$

$$T^*_{tW}=\theta_0 b_n cW^{*\prime\prime}+\eta\theta_0\left(b_t cW^{*\prime}\right)', \tag{37}$$

The two coupled governing characteristic differential equations (29) and (30) can be expressed in terms of F^*_{bW} and T^*_{tW}

$$x\left(\eta\theta_0 b_t c+\theta_0 b_n c\right)\Phi^{*\prime\prime}+[\eta\theta_0\left(b_t c\right)'+2\theta_0\left(b_n c\right)']\Phi^{*\prime}+\theta_0\left(b_n c\right)''\Phi^*=-F^*_{bW}, \tag{38}$$

$$\eta b_t\Phi^{*\prime\prime}+\eta\left(b_t\right)'\Phi^{*\prime}+\left(\frac{\delta}{\varsigma}I_t\omega^2-\theta_0^2 b_n c^2\right)\Phi^*=-T^*_{tW}. \tag{39}$$

Defining $A=b_t c, B=b_n c$, and $\alpha=\eta A+B$, equations (38), (39) can be further rewritten as

$$L(\theta_0 c\Phi^*)=-\frac{c}{\alpha}F^*_{bW}. \tag{40}$$

$$L_1(\theta_0 c\Phi^*)=-\frac{\theta_0 c}{\eta b_t}T^*_{tW}. \tag{41}$$

where

$$L = D^2 + \beta_1 D + \beta_2,\ \ L_1 = D^2 + \beta_3 D + \beta_4, \tag{42}$$

in which

$$\beta_1 = \left(\frac{\alpha}{c^2}\right)' \frac{c^2}{\alpha} + \frac{B'}{\alpha},\ \ \beta_2 = \frac{B''}{\alpha} - \frac{c''}{c} - \left(\frac{\alpha}{c^2}\right)' \frac{\left(c^2\right)'}{2\alpha} - \frac{B'c'}{\alpha c},$$
$$\beta_3 = \left(\frac{b_t}{c}\right)' \frac{c^2}{b_t},\ \ \beta_4 = \frac{\delta I_t \omega^2 - \theta_0^2 \varsigma b_n c^2}{\varsigma} - \left(\frac{b_t}{c}\right)' \frac{\left(c^2\right)'}{2b_t} - \frac{c''}{c}, \tag{43}$$

and D^n denotes the nth order differential operator with respect to ξ. It should be mentioned that the operators at the left hand side of equations (40)–(41) are the second-order and the second-order differential equations with variable coefficients in terms of Φ^*, respectively, the operators at the right hand side of equation (42) are the fourth-order and the second-order differential equations with variable coefficients in terms of W^*, respectively.

First, one subtracts equation (41) from equation (40) and the following first-order differential equation is obtained.

$$L_2(\theta_0 c \Phi^*) = \frac{1}{\beta_1 - \beta_3}\left(\frac{\theta_0 c}{\eta b_t} T_{tW}^* - \frac{c}{\alpha} F_{bW}^*\right), \tag{44}$$

where

$$L_2 = D + \frac{\beta_2 - \beta_4}{\beta_1 - \beta_3}, \tag{45}$$

To further uncouple differential equations (40) and (44), one reduces the order of differential operators on Φ^*. Now one defines a first-order differential operator R_1 first,

$$R_1 = D + \beta_1 - \frac{\beta_2 - \beta_4}{\beta_1 - \beta_3}, \tag{46}$$

After applying operator R_1 on equation (44), then subtracting it from equation (40), one can uncouple the W^* and Φ^* parameters and express Φ^* explicitly in terms of W^* and its first five order derivatives.

$$\theta_0 c \Phi^* = \frac{(R_1 - \beta_1 + \beta_3)}{a_0}\left[\frac{c}{\alpha(\beta_1 - \beta_3)}\right] F_{bW}^* - \frac{R_1}{a_0}\left[\frac{\theta_0 c}{\eta b_t(\beta_1 - \beta_3)}\right] T_{tW}^*, \tag{47}$$

where

$$a_0 = \left[\beta_2 + \left(\frac{\beta_2 - \beta_4}{\beta_1 - \beta_3}\right)^2 - \beta_1\left(\frac{\beta_2 - \beta_4}{\beta_1 - \beta_3}\right) - \left(\frac{\beta_2 - \beta_4}{\beta_1 - \beta_3}\right)'\right]. \tag{48}$$

Substituting equation (47) and its differentiation into equation (44), one obtains the uncoupled sixth-order governing characteristic differential equation in terms of the dimensionless flexural displacement parameter W^*.

$$\begin{aligned}&\left[(D+\frac{\beta_2-\beta_4}{\beta_1-\beta_3})\frac{(R_1-\beta_1+\beta_3)}{a_0}+1\right]\left[\frac{cF^*_{bW}}{\alpha(\beta_1-\beta_3)}\right]\\&\quad-\left[(D+\frac{\beta_2-\beta_4}{\beta_1-\beta_3})\frac{R_1}{a_0}+1\right]\left[\frac{\theta_0 cT^*_{tW}}{\eta b_t(\beta_1-\beta_3)}\right]=0.\end{aligned} \tag{49}$$

4.2 Boundary conditions in terms of displacement W^* only

Equation (47) shows the explicit relation between W^* and Φ^* and its first five order derivatives. The other explicit relations are derived in the following.

When equation (47) is substituted into equation (44), one can obtain an explicit relation between $(\theta_0 c\Phi^*)'$ and W^* and its first five order derivatives

$$\begin{aligned}(\theta_0 c\Phi^*)'=&\left[1-\frac{\beta_2-\beta_4}{\beta_1-\beta_3}\times\frac{R_1}{a_0}\right]\left[\frac{\theta_0 cT^*_{tW}}{\eta b_t(\beta_1-\beta_3)}\right]\\&-\left[1+\frac{\beta_2-\beta_4}{\beta_1-\beta_3}\times\frac{(R_1-\beta_1+\beta_3)}{a_0}\right]\left[\frac{cF^*_{bW}}{\alpha(\beta_1-\beta_3)}\right].\end{aligned} \tag{50}$$

The associated boundary conditions, in terms of W^* and its first five order derivatives, can be obtained by substituting explicit relations equations (47) and (50), into equations (31)–(33). So far, the original coupled system in terms of the displacement W^* and the torsional angle Φ^* has been transformed into one element in terms of the displacement W^* only. Further, the fundamental solutions of the transformed equation (49) can be derived hereafter.

4.3 Exact fundamental solutions

In the previous sections, the uncoupled governing characteristic differential equation (49) in terms of displacement W^* only can be expressed as a sixth-order differential equation with variable coefficients in the form of

$$\sum_{i=0}^{6} e_i(\xi)\, W^{*(6-i)}(\xi)=0,\ \ \xi\in(0,1). \tag{51}$$

In general, the exact fundamental solutions of a sixth-order differential equation with variable coefficients are not available. However, if the coefficients of the differential equation, which involve the material properties and geometric parameters, can be expressed in the polynomial form, a power series representation of the fundamental solutions can be constructed by the method of Frobenius.

Therefore, in the case that all the coefficients are in the polynomial forms, i.e.,

$$e_p(\xi) = \sum_{j=0}^{m_p} a_{p,j} (\xi - \xi_0)^j, \quad p = 0, 1, \ldots, 6, \tag{52}$$

where ξ_0 is a constant and $0 < \xi_0 < 1$ and m_p is the number of the terms in the series, then one can assume the six fundamental solutions of the differential equation in the form:

$$W_i^*(\xi) = \sum_{q=0}^{\infty} A_{q,i} (\xi - \xi_0)^q, \quad i = 0, 1, \ldots, 5, \tag{53}$$

where

$$\begin{aligned}
&A_{0,0} = 1, \; A_{1,0} = A_{2,0} = A_{3,0} = A_{4,0} = A_{5,0} = 0;\\
&A_{1,1} = 1, \; A_{0,1} = A_{2,1} = A_{3,1} = A_{4,1} = A_{5,1} = 0;\\
&A_{2,2} = 1/2, \; A_{0,2} = A_{1,2} = A_{3,2} = A_{4,2} = A_{5,2} = 0;\\
&A_{3,3} = 1/6, \; A_{0,3} = A_{1,3} = A_{2,3} = A_{4,3} = A_{5,3} = 0;\\
&A_{4,4} = 1/24, \; A_{0,4} = A_{1,4} = A_{2,4} = A_{3,4} = A_{5,4} = 0;\\
&A_{5,5} = 1/120, \; A_{0,5} = A_{1,5} = A_{2,5} = A_{3,5} = A_{4,5} = 0.
\end{aligned} \tag{54}$$

Consequently, the six fundamental solutions (53) can be rewritten as

$$W_i^*(\xi) = \frac{1}{i!}(\xi - \xi_0)^i + \sum_{q=6}^{\infty} A_{q,i} (\xi - \xi_0)^q, \quad i = 0, 1, \ldots, 5. \tag{55}$$

Substituting equations (54), (55) into equation (51), collecting all the coefficients of like powers, the following recurrence formula can be obtained:

$$\begin{aligned}
A_{q,i} &= -\frac{1}{n! a_{0,0}} \sum_{l=0}^{n-1} l! a_{n-l,0} A_{l,i}, q = n,\\
A_{q,i} &= -\frac{(q-n)!}{q! a_{0,0}} \left[\sum_{l=0}^{n-1} \frac{(q-n+l)!}{(q-n)!} a_{n-l,0} A_{q-n+l,i} \right.\\
&\quad \left. + \sum_{l=0}^{n} \sum_{j=1}^{q-n} \frac{(q-n+l-j)!}{(q-n-j)!} a_{n-l,j} A_{q-n+l-j,i} \right], \quad q > n.
\end{aligned} \tag{56}$$

With this recurrence formula, one can generate the six exact fundamental solutions of the governing characteristic differential equation. They satisfy the following normalization condition at the origin of the coordinate system.

$$\begin{bmatrix} W_0^* & W_0^{*\prime} & W_0^{*\prime\prime} & W_0^{*\prime\prime\prime} & W_0^{*(4)} & W_0^{*(5)} \\ W_1^* & W_1^{*\prime} & W_1^{*\prime\prime} & W_1^{*\prime\prime\prime} & W_1^{*(4)} & W_1^{*(5)} \\ W_2^* & W_2^{*\prime} & W_2^{*\prime\prime} & W_2^{*\prime\prime\prime} & W_2^{*(4)} & W_2^{*(5)} \\ W_3^* & W_3^{*\prime} & W_3^{*\prime\prime} & W_3^{*\prime\prime\prime} & W_3^{*(4)} & W_3^{*(5)} \\ W_4^* & W_4^{*\prime} & W_4^{*\prime\prime} & W_4^{*\prime\prime\prime} & W_4^{*(4)} & W_4^{*(5)} \\ W_5^* & W_5^{*\prime} & W_5^{*\prime\prime} & W_5^{*\prime\prime\prime} & W_5^{*(4)} & W_5^{*(5)} \end{bmatrix}_{\xi=0} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{57}$$

After substituting the homogenous solution which is a linear combination of the six fundamental solutions into the associated boundary conditions in terms of displacement W^* only, the frequency equation and the natural frequencies of the beam can be obtained.

5 Limiting cases

5.1 Non-uniform beams with constant curvature

For the beam which has constant curvature, the coefficients will become

$$c = 1,\ A = b_t,\ B = b_n,\ \alpha = \eta b_t + b_n,$$
$$\beta_1 = \frac{\eta b_t' + 2b_n'}{\eta b_t + b_n},\ \beta_2 = \frac{b_n'}{\eta b_t + b_n},\ \beta_3 = \frac{b_t'}{b_t},\ \beta_4 = \frac{\delta I_t \omega^2 - \theta_0^2 \varsigma b_n}{\varsigma}. \tag{58}$$

Moreover, equations (36)–(37) are reduced to

$$F_{bW}^* = \eta\theta_0^2\left(b_t W^{*\prime}\right)' - \left(b_n W^{*\prime\prime}\right)'' + m\omega^2 W^*,\ T_{tW}^* = \theta_0 b_n W^{*\prime\prime} + \eta\theta_0\left(b_t W^{*\prime}\right)'. \tag{59}$$

Substituting equations (58)–(59) into equation (49), the corresponding sixth-order governing characteristic differential equation in terms of W^* only is obtained.

5.2 Uniform beams with variable curvature

For the beam which is uniform, the coefficients will become as follow

$$b_n = b_t = m = I_t = 1,\ A = B = c,\ \alpha = c(\eta+1),$$
$$\beta_1 = -\frac{\eta c'}{c(\eta+1)},\ \beta_2 = \frac{b_n'}{\eta b_t + b_n},\ \beta_3 = -c',\ \beta_4 = \frac{\delta\omega^2 - \theta_0^2\varsigma c^2}{\varsigma} + \frac{(c')^2 + c'}{c} \tag{60}$$

Moreover, equations (36)–(37) are reduced to

$$F_{bW}^* = \eta\theta_0^2\left(\mathrm{c}^2 W^{*\prime}\right)' - W^{*\prime\prime\prime\prime} + m\omega^2 W^*,\ T_{tW}^* = (\eta+1)\theta_0 c W^{*\prime\prime} + \eta\theta_0 c' W^{*\prime}. \tag{61}$$

Substituting equations (60)–(61) into equation (50), the corresponding sixth-order governing characteristic differential equation in terms of W^* only is obtained.

5.3 Uniform beam with constant curvature

Further, for an uniform beam with constant curvature, the corresponding governing equation becomes

$$W^{*''''''} + \left(2\theta_0^2 + \frac{\delta\omega^2}{\varsigma\eta}\right) W^{*''''} + \left(\theta_0^4 - \omega^2 - \frac{\delta\theta_0^2\omega^2}{\varsigma}\right) W^{*''} + \left(\frac{\delta\theta_0^2\omega^2 - \delta\omega^4}{\varsigma\eta}\right) W^* = 0. \tag{62}$$

which is exactly the same as given by Lee and Chao (2000).

5.4 Straight bar

If the straight beam is considered, the arc angle $\theta_0 = 0$. Based on equations (36), (47), the governing equation of flexural vibration of a straight beam is obtained

$$\left(b_n W^{*''}\right)'' - m\omega^2 W^* = 0. \tag{63}$$

Moreover, because of $\theta_0 = 0$, equation (37) becomes

$$T_{tW}^* = 0 \tag{64}$$

Substituting equation (64) into equation (39), the governing equation of torsional vibration of a straight bar is obtained

$$\left(\eta b_t \Phi^{*'}\right)' + \frac{\delta}{\varsigma} I_t \omega^2 \Phi^* = 0 \tag{65}$$

6 Numerical results and discussion

In the following, the natural frequencies of linearly curvature and tapered beams of circular cross-section are studied. The diameter d changes linearly with coordinate s, $d(s) = d_0(1 + \varepsilon s/L)$. The corresponding dimensionless mass per unit length is $m(\xi) = (1 - \varepsilon\xi)^2$. The corresponding dimensionless bending, shear rigidities and inertia of mass per unit length are $b_n(\xi) = b_t(\xi) = I_t(\xi) = (1 - \varepsilon\xi)^4$, respectively. The curvature of beam varies linearly, $c(\xi) = 1 + \chi\xi$.

Table 1 demonstrates comparison of the first out-of-plane natural frequencies of clamped-clamped uniform beams with constant curvature by the present method with those in the literatures [Lee and Chao (2000); Volterra and Morell (1961)]. It is found that their results are very consistent.

It is well known that if the cross-section of the beam is doubly symmetric and straight uniform beam, then the torsional and flexural vibrations are uncoupled.

Table 1: The first out-of-plane natural frequencies of clamped-clamped curved beams [$m = b_n = b_t = I_t = c = 1$] *: Lee and Chao (2000); #: Volterra and Morell (1961).

θ_0 (degree)	R	$\eta(0) = 1/0.615$			$\eta(0) = 1.0$		
		#	$*$	Present	#	$*$	Present
0	∞	...	22.373	22.373	...	22.373	22.373
90	50	...	20.840	20.832	...	20.694	20.683
180	50	18.379	18.361	18.358	18.132	18.128	18.124
270	50	17.767	17.765	17.764	16.877	16.875	16.873

The exact natural frequency of torsional vibration of the clamped-clamped straight bar is [Inman (1994)]

$$\omega_m = m\pi \frac{\eta}{\delta} L_n, \;\; m = 1, 2, \ldots \tag{66}$$

which are proportional to the slenderness ratio L_n. In addition, the first three exact natural frequency of flexural vibration of the clamped-clamped straight bar are $\{22.373, 61.673, 120.903\}$ which are independent to the slenderness ratio L_n. These results are shown in Figure 2.

Figure 2 demonstrates also the influence of the slenderness ratio L_n on the first three dimensionless natural frequencies ω of the clamped-clamped curved uniform beam. It is found that the first mode of the curved beam is the flexural one which is independent to the slenderness ratio L_n. Moreover, for small slenderness ratio L_n, the second mode is dominated by the torsional one. When the slenderness ratio L_n is increased over the critical value L_{nc}, the torsional mode will transform into the flexural one. However, for small slenderness ratio L_n the third mode is dominated by the torsional one. When the slenderness ratio L_n is increased, the torsional mode will transform into the flexural one. Finally, the slenderness ratio L_n is further increased over the critical value L_{nc}, the flexural mode will transform back into the torsional one. It is concluded that when the slenderness ratio L_n is increased over the critical value L_{nc}, there exists the transition between the higher torsional and flexural modes. Unfortunately, [Lee, Oh, Mo, and Lee (2008)] made the reverse conclusion that the effect of slenderness ratio L_n on the flexural frequency was significant, but slight on the torsional one.

Figure 3 demonstrates the influence of the slenderness ratio on the first three dimensionless natural frequencies ω of the clamped-clamped tapered beam with different tapered ratio ε. It is found that the $\omega - L_n$ curves with $\varepsilon = 0$ and 0.3 are similar. Moreover, there also is the transition phenomenon in the second and third modes.

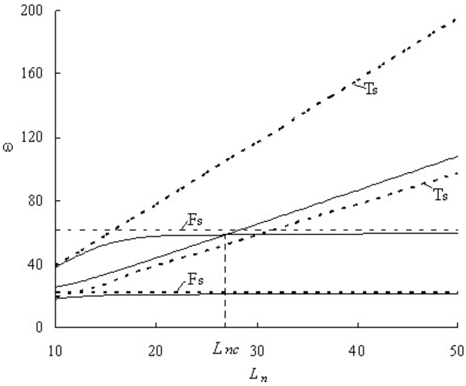

Figure 2: Influence of the slenderness ratio L_n on the first three dimensionless natural frequencies ω of the clamped-clamped uniform beam [$\eta = 1.24$, $m = b_n = b_t = I_t = c = 1$; solid lines: curved beam with curved angle $\theta_0 = 75°$; dashed lines: straight beam curved angle $\theta_0 = 0°$; F_s: flexural mode of straight beam; T_s: torsional mode of straight beam].

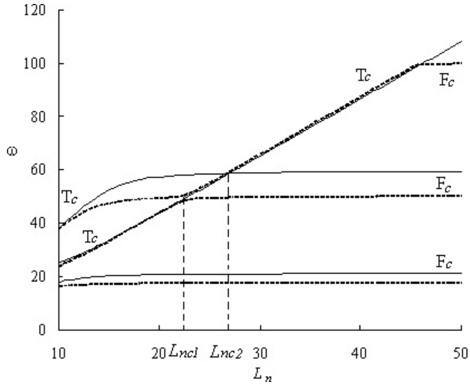

Figure 3: Influence of the slenderness ratio on the first three dimensionless natural frequencies ω of the clamped-clamped tapered beam [$\theta_0 = 75°$, $\beta = 0$; solid line: tapered ratio $\varepsilon = 0$; dashed line: $\varepsilon = 0.3$; F_c: predominant flexural mode of curved beam; T_c: predominant torsional mode of curved beam].

For the system with $\varepsilon = 0$, the critical value $L_{nc1} \approx 22.3$ and for $\varepsilon = 0.3$, the critical value $L_{nc2} \approx 26.9$. There exists the shift of the critical value of the mode transition due to the tapered ratio ε. Moreover, the predominant flexural frequencies of beam with $\varepsilon = 0.3$ are obviously lower than those with $\varepsilon = 0$. But the effect of the tapered ratio ε on the predominant torsional frequencies is negligible.

Figure 4 demonstrates the influence of the taper ratio on the first three dimensionless natural frequencies ω of a clamped-clamped curved beam. It is found that if the tapered ratio ε is large, the first three modes of vibration are dominated by the flexural mode. On the contrary, the third mode of vibration is dominated by the torsional mode for small tapered ratio ε. Moreover, if the tapered ratio ε is increased from zero over a critical one, there exists the mode transition phenomenon from the torsional mode to the flexural mode for the third mode. In addition, the larger the tapered ratio ε is, the lower the natural frequency of flexural mode is.

Figure 5 demonstrates the influence of the slenderness ratio L_n on the first three natural frequencies $\omega$of the clamped-clamped beam with different curvature gradient β. It is found that the $\omega - L_n$ curves with $\beta = -0.7$ and 0.2 are similar. Moreover, there exists also the transmit phenomenon in the second and third modes. For

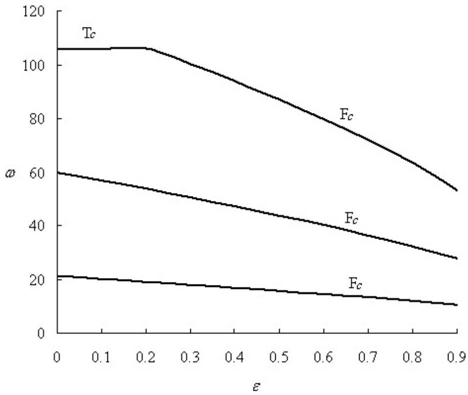

Figure 4: Influence of the taper ratio on the first three dimensionless natural frequencies ω of a curved beam with clamped-clamped ends [$\theta_0 = 60°$, $L_n = 50$, $\beta = 0.2$; F_c: predominant flexural mode; T_c: predominant torsional mode].

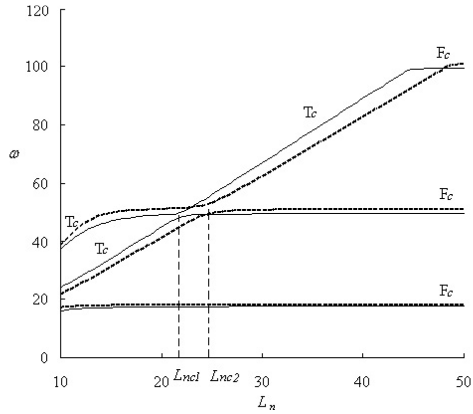

Figure 5: Influence of the slenderness ratio and curvature gradient β on the first three dimensionless natural frequencies ω of the clamped-clamped beam [$\theta_0 = 75°$, $\varepsilon = 0.3$; solid line: $\beta = 0.2$; dashed line: $\beta = -0.7$; F_c: predominant flexural mode; T_c: predominant torsional mode].

the system with $\beta = -0.7$, the critical value $L_{nc2} \approx 24.46$ and the critical value $L_{nc1} \approx 21.62$ for $\beta = 0.2$. These are slightly different by observation. In addition, the frequencies of beams with $\beta = -0.7$ and 0.2 are very close. It is revealed that the effect of the curvature gradient β on the natural frequency is slight. Further, Figure 6 demonstrates the influence of the curvature gradient β on the first three dimensionless natural frequencies $\omega$of curved beams with clamped-clamped ends and different taper ratio ε. It is found that the effect of the curvature gradient β on the natural frequencies is negligible.

Figure 7 demonstrates the influence of the arc angle θ_0 on the first four natural frequencies ω of the clamped-clamped curved beam with different taper ratio ε. It is found that the torsional natural frequencies of the straight non-uniform beams with tapered ratio $\varepsilon = 0.2$ and 0.3 are very close. The effect of the curved angle θ_0 on the flexural frequencies is slight. However, its effect on the torsional frequency is significant. Moreover, if the curved angle θ_0 is increased from zero over a critical one, the transition phenomenon from the torsional mode to the flexural mode for the third frequency. Similarly, there is the transition phenomenon from the flexural mode to the torsional mode for the fourth frequency.

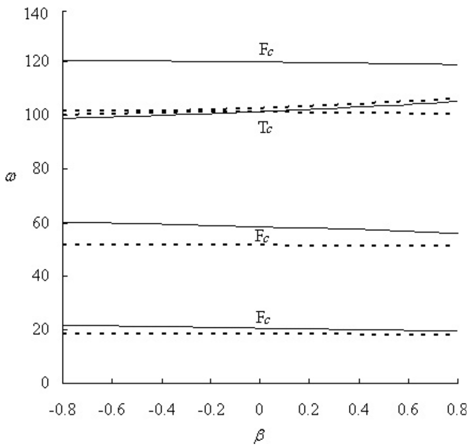

Figure 6: Influence of the curvature gradient β on the first three dimensionless natural frequencies ω of curved beams with clamped-clamped ends and different taper ratio ε [$L_n = 50$, $\theta_0 = 45°$; solid lines: $\varepsilon = 0$; dotted lines: $\varepsilon = 0.3$; F_c: predominant flexural mode; T_c: predominant torsional mode].

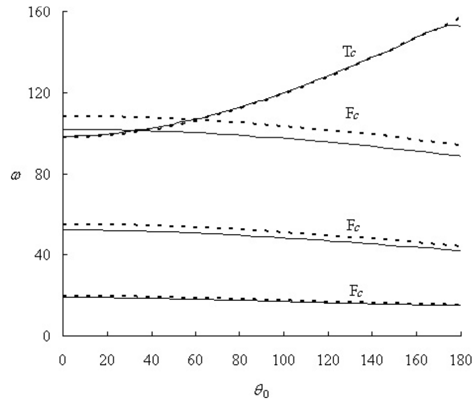

Figure 7: Influence of the arc angle on the first four natural frequencies ω of a curved beam with clamped-clamped ends and different taper ratio ε [$\eta = 1.24$, $L_n = 50$, $\beta = 0.2$; solid lines: $\varepsilon = 0.3$; dotted lines: $\varepsilon = 0.2$; F_c: predominant flexural mode; T_c: predominant torsional mode].

7 Conclusions

In this paper, the exact solutions for the free out-of-plane vibrations of a curved non-uniform beam with variable curvature are presented. In the coupled flexural and torsional vibrations of the general system, the mode transition phenomenon is clearly investigated. The major findings of the present study are summarized as follows:

(1) When the slenderness ratio L_n is increased over the critical value L_{nc}, the transition between the higher torsional and flexural modes happens.

(2) The torsional frequencies are almost proportional to the slenderness ratio. However, the effect of the slenderness ratio on the flexural frequencies is slight.

(3) The larger the tapered ratio ε is, the lower the natural frequency of flexural mode is. But its effect on the natural frequency of torsional mode is negligible.

(4) The effect of the curvature gradient β on the natural frequencies is negligible.

(5) The effect of the arc angle θ_0 on the torsional frequency is significant. However, its effect on the flexural frequencies is slight.

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