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Dispersion of Axisymmetric Longitudinal Waves in A Bi-Material Compound Solid Cylinder Made of Viscoelastic Materials

S.D. Akbarov[1,2], T. Kocal[3], T. Kepceler[1]

Abstract: The paper studies the dispersion of axisymmetric longitudinal waves in the bi-material compound circular cylinder made of linear viscoelastic materials. The investigations are carried out within the scope of the piecewise homogeneous body model by utilizing the exact equations of linear viscoelasto-dynamics. The corresponding dispersion equation is derived for an arbitrary type of hereditary operator and the algorithm is developed for its numerical solution. Concrete numerical results are obtained for the case where the relations of the constituents of the cylinder are described through fractional exponential operators. The influence of the viscosity of the materials of the compound cylinder on the wave dispersion is studied through the rheological parameters which indicate the characteristic creep time and long-term values of the elastic constants of these materials. Dispersion curves are presented for certain selected dispersive and non-dispersive attenuation cases under various values of the problem parameters and the influence of the aforementioned rheological parameters on these curves is discussed. As a result of the numerical investigations, in particular, it is established that in the case where the rheological parameters of the components of the compound cylinder are the same, the viscosity of the layers' materials causes the axisymmetric wave propagation velocity to decrease.

Keywords: Characteristic creep time; Viscoelastic material; Wave dispersion; Bi-material compound solid cylinder; Wave attenuation.

[1] Department of Mechanical Engineering, Yildiz Technical University, Yildiz Campus, 34349, Besiktas, Istanbul, Turkey; E-mail: akbarov@yildiz.edu.tr (S.D. Akbarov); kepceler@yildiz.edu.tr (T. Kepceler)

[2] Institute of Mathematics and Mechanics of the National Academy of Sciences of Azerbaijan, 37041, Baku, Azerbaijan

[3] Department of Marine Engineering Operations, Yildiz Campus, 34349 Besiktas, Istanbul, Turkey, E-mail: tkocal@yildiz.edu.tr

1 Introduction

The study of time-harmonic wave dispersion and attenuation in viscoelastic materials and in elements of constructions made from these materials is required not only by theoretical, but also by application needs. The nondestructive inspection of tubes and pipes which are used in the infrastructure of many industries such as gas, oil, and water transport can be taken as an example of such applications. This is because, in many cases, these tubes are coated with viscoelastic polymer coatings for corrosion protection and therefore, under nondestructive testing of the tubes with guided waves, it is necessary to know the attenuation and dispersion rules of the waves propagating therein. Moreover, the study of the propagation of guided waves in viscoelastic materials and constructions made of those, used in viscoelastic systems for attenuation of vibrations and waves caused by an earthquake or with various types of sound sources, is an another example of the application of this study.

These and many other application fields using the results of the studies of wave propagation in viscoelastic bodies necessitate investigation of related problems from both the theoretical and experimental aspects. Nevertheless, up to now, investigations related to wave propagation in structural elements made from viscoelastic materials have not been as numerous as studies which have been made for the same structural elements made from purely elastic materials. We consider a brief review of these investigations the first of which was made in the papers by Weiss (1959); Tamm and Weiss (1961) which relate to Lamb wave propagation in an isotropic viscoelastic layer with stress-free surfaces. In these papers, non-dispersive attenuation is considered, i.e. it is assumed that the elastic constants are complex and independent of frequency. Lamb wave propagation in a plate from viscoelastic materials with small losses and frequency-dependent elastic moduli is investigated in a paper by Coquin (1964) in which an approximate method is also proposed for this investigation. Chervinko and Shevchenko (1986) investigated the influence of low-compressibility materials with real Poisson's ratio and frequency dependent complex shear moduli on the propagation of Lamb waves.

Lamb wave propagation in elastic plates coated with viscoelastic materials was also studied by Simonetti (2004) in which the effect of damped coatings on the dispersion characteristics of waves in these plates was also analyzed. The results reviewed above were noted and/or detailed in the monograph by Rose (2004).

Wolosewick and Raynor (1967) considered axisymmetric non-stationary torsional wave propagation in the semi-infinite circular cylinder for the case where arbitrary radial axially symmetric tangential shear stress distribution harmonic in time acts on the end of the cylinder.

Axisymmetric longitudinal guided wave dispersion and attenuation in a metal e-lastic hollow cylinder coated with a polymer viscoelastic layer was studied by Barshin- ger and Rose (2004). The viscoelasticity of the coated layer was taken into consideration through attenuation coefficients of the longitudinal and shear waves in the corresponding viscoelastic materials. These coefficients are determined experimentally for the frequencies in the order 1–5 MHz and are used for determination of the corresponding complex moduli. Consequently, using these complex moduli, the wave dispersion and attenuation dispersion in the bi-layered hollow cylinder was investigated.

It follows from the foregoing review that the investigations on the dispersion of guided waves in the plates or cylinders made from viscoelastic materials were carried out mainly in the following cases: the complex modulus of viscoelastic materials is taken as frequency independent; the viscoelasticity of the materials is described by the simplest models such as the Maxwell and Kelvin-Voigt models; and the expression for the complex elasticity modulus is obtained experimentally for concrete polymer materials. In other words, it follows from the works considered above, that the corresponding investigations on wave dispersion and attenuation were not connected with the more complicated and real models for viscoelastic materials and a few of the numerical results obtained in these works do not illustrate the character of the influence of the rheological parameters of the viscoelastic materials on this dispersion. In this sense, the first attempt was made in a paper by Akbarov and Kepceler (2015) in which the torsional wave dispersion in the sandwich hollow cylinder made from linear viscoelastic materials was studied. It was assumed that the mechanical relations of the layers' materials of the cylinder are given through the fractional exponential operators by Rabotnov (1980) and the numerical results obtained for the wave dispersion and attenuation dispersion are connected with the rheological parameters which enter these operators.

Note that the fractional exponential operators by Rabotnov (1980) have many advantages for describing the hereditary viscoelastic properties of many polymer materials and epoxy-based composites with continuous fibers and layers. For instance, these operators allow us to describe, with the very high accuracy required, the initial parts of the experimentally and theoretically constructed creep and relaxation graphs and their asymptotic values. Moreover, these operators have many simple rules for complicated mathematical transformations, for example, the Fourier and Laplace transformations were also used in the paper by Akbarov and Kepceler (2015). The results obtained in this paper were also detailed in the monograph by Akbarov (2015). Besides all the aforementioned advantages, through variation of the rheological parameters contained within these operators, many possible cases can be considered which relate to the dynamics of the viscoelastic materials. Note that such variation of the rheological parameters was already used in the paper by Akbarov and Kepceler (2015).

In the present paper, taking into consideration the foregoing discussions and significance of the related studies we continue the investigations begun by Akbarov and Kepceler (2015) and attempt to investigate the axisymmetric longitudinal wave dispersion in a bi-material compound circular solid cylinder made of viscoelastic materials, the rheological relations for which are given through the fractional exponential operators by Rabotnov (1980). The exact field equations and relations of the linear theory of viscoelasticity are used and it is assumed that perfect contact conditions take place on the interface surface between the inner solid and outer hollow cylinders.

2 Formulation of the problem

We consider the bi-material compound solid circular cylinder shown in Fig. 1 and assume that the radius of the cross-section circle of the inner solid cylinder is R and the thickness of the outer covering hollow cylinder is h. The values related to the inner solid and external hollow cylinders will be denoted by the upper indices (2) and (1), respectively.

We assume that the materials of the constituents are isotropic, homogeneous and hereditary-viscoelastic. We use the cylindrical system of coordinates $Or\theta z$ (Fig. 1) for determination of the position of the points of the system under consideration. Moreover, we assume that the cylinders have infinite length in the direction of the Oz axis.

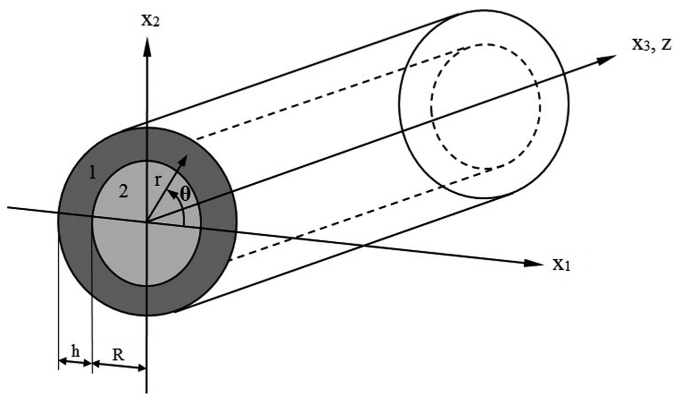


Figure 1: The geometry of the bi-material circular compound cylinder.

Thus, within the scope of the piecewise homogeneous body model let us investigate the axisymmetric longitudinal wave propagation along the Oz axis in the considered compound cylinder with the use of the equations of motion of the linear theory for viscoelastic bodies.

We write the governing field equations and mechanical relations for the case under consideration.

Equations of motion:

$$\frac{\partial T_{rr}^{(n)}}{\partial r}+\frac{\partial T_{rz}^{(n)}}{\partial z}+\frac{1}{r}(T_{rr}^{(n)}-T_{\theta\theta}^{(n)})=\rho^{(n)}\frac{\partial^2 u_r^{(n)}}{\partial t^2},$$
$$\frac{\partial T_{rz}^{(n)}}{\partial r}+\frac{\partial T_{zz}^{(n)}}{\partial z}+\frac{1}{r}T_{rz}^{(n)}=\rho^{(n)}\frac{\partial^2 u_z^{(n)}}{\partial t^2}. \quad (1)$$

Constitutive relations:

$$T_{(ii)}^{(n)}=\lambda^{(n)*}\theta^{(n)}+2\mu^{(n)*}\varepsilon_{(ii)}^{(n)},\ (ii)=rr, zz, \theta\theta,$$
$$T_{rz}^{(n)}=2\mu^{(n)*}\varepsilon_{rz}^{(n)},\ \theta^{(n)}=\varepsilon_{rr}^{(n)}+\varepsilon_{\theta\theta}^{(n)}+\varepsilon_{zz}^{(n)}, \quad (2)$$

where $\lambda^{(n)*}$ and $\mu^{(n)*}$ are the following viscoelastic operators:

$$\begin{Bmatrix}\lambda^{(n)*}\\ \mu^{(n)*}\end{Bmatrix}\varphi(t)=\begin{Bmatrix}\lambda_0^{(n)}\\ \mu_0^{(n)}\end{Bmatrix}\varphi(t)+\int_0^t\begin{Bmatrix}\lambda_1^{(n)}\\ \mu_1^{(n)}\end{Bmatrix}(t-\tau)\varphi(\tau)d\tau. \quad (3)$$

In Eq. (3), $\lambda_0^{(n)}$ and $\mu_0^{(n)}$ are the instantaneous values of Lame's constants as $t\to 0$, and $\lambda_1^{(n)}(t)$ and $\mu_1^{(n)}(t)$ are the corresponding kernel functions describing the hereditary properties of the materials of the constituents.

Strain-displacement relations:

$$\varepsilon_{rr}^{(n)}=\frac{\partial u_r^{(n)}}{\partial r},\ \varepsilon_{rz}^{(n)}=\frac{1}{2}\left(\frac{\partial u_r^{(n)}}{\partial z}+\frac{\partial u_z^{(n)}}{\partial r}\right),\ \varepsilon_{\theta\theta}^{(n)}=\frac{u_r^{(n)}}{r},\ \varepsilon_{zz}^{(n)}=\frac{\partial u_z^{(n)}}{\partial z}. \quad (4)$$

The relations (1)–(4) are the complete system of equations of the theory of linear viscoelasticity for isotropic bodies; conventional notation is used.

Consider also formulation of the boundary and contact conditions. According to Fig. 1 we can write the boundary and contact conditions.

$$T_{rr}^{(2)}\Big|_{r=R}=T_{rr}^{(1)}\Big|_{r=R},\ T_{rz}^{(2)}\Big|_{r=R}=T_{rz}^{(1)}\Big|_{r=R},$$
$$u_r^{(2)}\Big|_{r=R}=u_r^{(1)}\Big|_{r=R},\ u_z^{(2)}\Big|_{r=R}=u_z^{(1)}\Big|_{r=R},$$
$$T_{rr}^{(1)}\Big|_{r=R(1+h/R)}=0,\ T_{rz}^{(1)}\Big|_{r=R(1+h/R)}=0. \quad (5)$$

This completes the formulation of the problem on the axisymmetric longitudinal wave dispersion in the bi-material compound solid cylinder made of viscoelastic materials with arbitrary kernel functions $\lambda_1^{(n)}(t)$ and $\mu_1^{(n)}(t)$ which enter the constitutive relations (3).

3 Method of solution

First, we represent the displacements and strains as:

$$u_r^{(n)} = v_r^{(n)}(r)e^{i(kz-\omega t)},\ u_z^{(n)} = v_z^{(n)}(r)e^{i(kz-\omega t)},\ \theta^{(n)} = \upsilon^{(n)}(r)e^{i(kz-\omega t)},$$
$$\varepsilon_{(ii)}^{(n)} = \gamma_{(ii)}^{(n)}(r)e^{i(kz-\omega t)},\ (ii) = rr;\theta\theta;zz;rz \tag{6}$$

where k is the wave number and ω is the circular frequency, and

$$\gamma_{rr}^{(n)}=\frac{dv_r^{(n)}(r)}{dr},\ \gamma_{\theta\theta}^{(n)}=\frac{v_r^{(n)}(r)}{r},\ \gamma_{zz}^{(n)}=\frac{dv_z^{(n)}(r)}{dz},\ \gamma_{rz}^{(n)}=\frac{1}{2}\left(\frac{dv_r^{(n)}(r)}{dz}+\frac{dv_z^{(n)}(r)}{dr}\right), \tag{7}$$

and we use the relation

$$\int_0^t f_1(t-\tau)f_2(\tau)d\tau \approx \int_{-\infty}^t f_1(t-\tau)f_2(\tau)d\tau, \tag{8}$$

in the mechanical relations (2) and (3). Thus, taking the relations (6)–(8) into account in Eqs. (2) and (3), we can write the following relations:

$$T_{(ii)}^{(n)} = \lambda_0^{(n)}\vartheta^{(n)}(r)e^{i(kz-\omega t)} + e^{ikz}\vartheta^{(n)}(r)\int_{-\infty}^t \lambda_1^{(n)}(t-\tau)e^{-i\omega\tau}d\tau$$
$$+2\mu_0^{(n)}\gamma_{(ii)}^{(n)}(r)e^{i(kz-\omega t)} + e^{ikz}\gamma_{(ii)}^{(n)}(r)\int_{-\infty}^t \mu_1^{(n)}(t-\tau)e^{-i\omega\tau}d\tau. \tag{9}$$

Using the transformation $t-\tau = s$ we can make the following manipulations for the integrals which enter into Eq. (9):

$$\int_{-\infty}^t \begin{Bmatrix}\lambda_1^{(n)}\\ \mu_1^{(n)}\end{Bmatrix}(t-\tau)e^{-i\omega\tau}d\tau = -\int_{\infty}^0 \begin{Bmatrix}\lambda_1^{(n)}\\ \mu_1^{(n)}\end{Bmatrix}(t-\tau)e^{-i\omega t}e^{i\omega s}ds$$
$$= e^{-i\omega t}\int_0^\infty \begin{Bmatrix}\lambda_1^{(n)}\\ \mu_1^{(n)}\end{Bmatrix}(t-\tau)e^{i\omega s}ds = e^{-i\omega t}\left(\begin{Bmatrix}\lambda_{1c}^{(n)}\\ \mu_{1c}^{(n)}\end{Bmatrix} + i\begin{Bmatrix}\lambda_{1s}^{(n)}\\ \mu_{1s}^{(n)}\end{Bmatrix}\right) \tag{10}$$

where

$$\begin{Bmatrix}\lambda_{1c}^{(n)}\\ \mu_{1c}^{(n)}\end{Bmatrix} = \int_0^\infty \begin{Bmatrix}\lambda_1^{(n)}\\ \mu_1^{(n)}\end{Bmatrix}(s)\cos(\omega s)ds,\ \begin{Bmatrix}\lambda_{1s}^{(n)}\\ \mu_{1s}^{(n)}\end{Bmatrix} = \int_0^\infty \begin{Bmatrix}\lambda_1^{(n)}\\ \mu_1^{(n)}\end{Bmatrix}(s)\sin(\omega s)ds. \tag{11}$$

Taking the relations (8)–(11) into account we can write the following expressions for the stresses:

$$T_{(ii)}^{(n)} = \Lambda^{(n)} \vartheta^{(n)}(r) e^{i(kz-\omega t)} + 2M^{(n)} \gamma_{(ii)}^{(n)}(r) e^{i(kz-\omega t)} = \sigma_{(ii)}^{(n)}(r) e^{i(kz-\omega t)},$$
$$T_{rz}^{(n)} = 2M^{(n)} \gamma_{rz}^{(n)}(r) e^{i(kz-\omega t)} = \sigma_{rz}^{(n)}(r) e^{i(kz-\omega t)}, \tag{12}$$

where

$$\Lambda^{(n)} = \lambda_0^{(n)} + \lambda_{1c}^{(n)} + i\lambda_{1s}^{(n)}, \;\; M^{(n)} = \mu_0^{(n)} + \mu_{1c}^{(n)} + i\mu_{1s}^{(n)}. \tag{13}$$

Thus, we obtain the complex constants $\Lambda^{(n)}$ and $M^{(n)}$ (13), the real and imaginary parts of which are determined through the expressions (11) and (13). This means that the complete system of field Eqs. (1), (2), (4), (12) and (13) for the viscoelastic system, can also be obtained from that given for the purely elastic system by replacing the elastic constants $\lambda_0^{(n)}$ and $\mu_0^{(n)}$ with the complex constants $\Lambda^{(n)}$ and $M^{(n)}$ respectively. In other words, the foregoing mathematical calculations confirm the dynamic correspondence principle (see Fung (1965)) for the problem under consideration and the solution method used here coincides with this principle.

The real parts of the complex constants, i.e. $\mathrm{Re}\,\Lambda^{(n)}(\omega)$ and $\mathrm{Re}\,M^{(n)}(\omega)$, are called the storage moduli, while the imaginary parts, $\mathrm{Im}\,\Lambda^{(n)}(\omega)$ and $\mathrm{Im}\,M^{(n)}(\omega)$, are called the loss moduli. The ratios $\mathrm{Im}\,\Lambda^{(n)}(\omega)/\mathrm{Re}\,\Lambda^{(n)}(\omega)$ and $\mathrm{Im}\,M^{(n)}(\omega)/\mathrm{Re}\,M^{(n)}(\omega)$ determine the phase shifting between the strains and stresses.

Thus, substituting the expression (12) into the equation of motion (1) and taking the relation (6) into consideration we obtain the following equations of motion in terms of the displacement amplitudes.

$$m_1^{(n)} \frac{d^2 v_r^{(n)}}{d(kr)^2} + m_2^{(n)} \frac{d}{d(kr)} \left(\frac{v_r^{(n)}}{kr} \right) + i(m_2^{(n)} + m_3^{(n)}) \frac{dv_z^{(n)}}{d(kr)} - m_3^{(n)} v_r^{(n)}$$
$$+ \frac{1}{kr}(m_1^{(n)} - m_2^{(n)}) \frac{dv_r^{(n)}}{dr} + (m_2^{(n)} - m_1^{(n)}) \frac{v_r^{(n)}}{(kr)^2} = -\frac{\omega^2}{k^2} \rho^{(n)} v_r^{(n)},$$
$$im_3^{(n)} \frac{dv_r^{(n)}}{d(kr)} + m_3^{(n)} \frac{d^2 v_z^{(n)}}{d(kr)^2} + i\frac{1}{kr} m_3^{(n)} v_r^{(n)} + \frac{1}{kr} m_3^{(n)} \frac{dv_z^{(n)}}{d(kr)} + ikm_2^{(n)} \frac{dv_r^{(n)}}{d(kr)}$$
$$+ im_2^{(n)} \frac{v_r^{(n)}}{kr} - m_1^{(n)} v_z^{(n)} = -\frac{\omega^2}{k^2} \rho^{(n)} v_z^{(n)}, \tag{14}$$

where

$$m_1^{(n)} = \Lambda^{(n)} + 2M^{(n)}, \;\; m_2^{(n)} = \Lambda^{(n)}, \;\; m_3^{(n)} = M^{(n)}. \tag{15}$$

According to the foregoing transformations and expressions in (6) and (12), the boundary and contact conditions in (5) can be rewritten as follows:

$$\sigma_{rr}^{(2)}\Big|_{r=R}=\sigma_{rr}^{(1)}\Big|_{r=R},\ \ \sigma_{rz}^{(2)}\Big|_{r=R}=\sigma_{rz}^{(1)}\Big|_{r=R},$$
$$v_r^{(2)}\Big|_{r=R}=v_r^{(1)}\Big|_{r=R},\ \ v_z^{(2)}\Big|_{r=R}=v_z^{(1)}\Big|_{r=R},$$
$$\sigma_{rr}^{(1)}\Big|_{r=R(1+h/R)}=0,\ \ \sigma_{rz}^{(1)}\Big|_{r=R(1+h/R)}=0. \tag{16}$$

For the solution to the problem formulated through Eqs. (14), (6), (7) and (12), with the boundary and contact conditions in (16), according to Guz (2004), we employ the following representation for the displacement amplitudes:

$$v_r^{(n)}=-ik\frac{\partial}{\partial r}X^{(n)},\ \ v_z^{(n)}=\frac{1}{m_2^{(n)}+m_3^{(n)}}\left(m_1^{(n)}\Delta_1-k^2m_3^{(n)}+\omega^2\rho^{(n)}\right)X^{(n)},$$
$$\Delta_1=\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}, \tag{17}$$

where $X^{(n)}$ satisfies the equation

$$\left[\left(\Delta_1-k^2(\zeta_2^{(n)})^2\right)\left(\Delta_1-k^2(\zeta_3^{(n)})^2\right)\right]X^{(n)}=0. \tag{18}$$

In (18), $\zeta_2^{(n)}$ and $\zeta_3^{(n)}$ are determined from the following equations:

$$(\Lambda^{(n)}+2M^{(n)})M^{(n)}\left(\zeta'^{(n)}\right)^4-k^2\left(\zeta'^{(n)}\right)^2\left[(\Lambda^{(n)}+2M^{(n)})\left(\rho^{(n)}\left(\frac{\omega}{k}\right)^2\right.\right.$$
$$\left.\left.-(\Lambda^{(n)}+2M^{(n)})\right)+M^{(n)}\left(\rho^{(m)}\left(\frac{\omega}{k}\right)^2-M^{(n)}\right)+\left(\Lambda^{(n)}+M^{(n)}\right)^2\right]$$
$$+k^4\left(\rho^{(n)}\left(\frac{\omega}{k}\right)^2-(\Lambda^{(n)}+2M^{(n)})\right)\left(\rho^{(n)}\left(\frac{\omega}{k}\right)^2-M^{(n)}\right)=0, \tag{19}$$

where ω/k is the complex phase velocity of the wave propagation.

Thus, we determine the following expression for the function $X^{(n)}$ from Eqs. (18) and (19).

$$X^{(1)}=A_1^{(1)}J_0(\zeta_2^{(1)}kr)+A_2^{(1)}J_0(\zeta_3^{(1))}kr)+B_1^{(1)}Y_0(\zeta_2^{(1)}kr)+B_2^{(1)}Y_0(\zeta_3^{(1)}kr),$$
$$X^{(2)}=A_1^{(2)}J_0(\zeta_2^{(2)}kr)+A_2^{(2)}J_0(\zeta_3^{(2)}kr), \tag{20}$$

where $J_0(x)$ and $Y_0(x)$ are Bessel functions of the first and second kinds with zeroth order, respectively.

Using the expression (20) and Eqs. (17), (13), (6) and (7), we obtain the following dispersion equation from the conditions in (16):

$$\det\|\beta_{nm}\| = 0, \;\; n;m = 1,2,\ldots,6, \tag{21}$$

where

$$\beta_{11}\left(\zeta_2^{(2)},\chi_2^{(2)}\right) = \left(\Lambda^{(2)}(\omega)+2M^{(2)}(\omega)\right)\left(-\left(\zeta_2^{(2)}\right)^2\frac{1}{2}\left(J_2\left(\chi_2^{(2)}\right)-J_0\left(\chi_2^{(2)}\right)\right)\right)$$
$$+\frac{\Lambda^{(2)}(\omega)}{\eta}\zeta_2^{(2)}J_1\left(\chi_2^{(2)}\right)+\frac{\Lambda^{(2)}(\omega)}{2}\times\left(\beta_1^{(2)}\left(\zeta_2^{(2)}\right)^2\left(J_2\left(\chi_2^{(2)}\right)\right.\right.$$
$$\left.\left.-J_0\left(\chi_2^{(2)}\right)\right)-\frac{\zeta_2^{(2)}}{\eta}J_1\left(\chi_2^{(2)}\right)-\beta_2^{(2)}J_0\left(\chi_2^{(2)}\right)\right)$$

$$\beta_{21}\left(\zeta_2^{(2)},\chi_2^{(2)}\right) = -M^{(2)}(\omega)\zeta_2^{(2)}J_1\left(\chi_2^{(2)}\right)+\frac{M^{(2)}(\omega)}{4}\left(\beta_1^{(2)}\left(\left(\zeta_2^{(2)}\right)^3\left(3J_1\left(\chi_2^{(2)}\right)\right.\right.\right.$$
$$\left.-J_3\left(\chi_2^{(2)}\right)\right)+\frac{\zeta_2^{(2)}}{\eta^2}J_1\left(\chi_2^{(2)}\right)+\frac{\left(\zeta_2^{(2)}\right)^2}{2\eta}\left(J_2\left(\chi_2^{(2)}\right)\right.$$
$$\left.\left.\left.-J_0\left(\chi_2^{(2)}\right)\right)+\beta_2^{(2)}\zeta_2^{(2)}J_1\left(\chi_2^{(2)}\right)\right)\right)$$

$$\beta_{31}\left(\zeta_2^{(2)},\chi_2^{(2)}\right) = -\zeta_2^{(2)}J_1\left(\chi_2^{(2)}\right)$$

$$\beta_{41}\left(\zeta_2^{(2)},\chi_2^{(2)}\right) = \left(-\beta_1^{(2)}\left(\zeta_2^{(2)}\right)^2-\beta_2^{(2)}\right)J_0\left(\chi_2^{(2)}\right)$$

$$\beta_{13}\left(\zeta_2^{(1)},\chi_2^{(1)}\right) = \left(\Lambda^{(1)}(\omega)+2M^{(1)}(\omega)\right)\left(-\left(\zeta_2^{(1)}\right)^2\frac{1}{2}\left(J_2\left(\chi_2^{(1)}\right)-J_0\left(\chi_2^{(1)}\right)\right)\right)$$
$$+\frac{\Lambda^{(1)}(\omega)}{\eta}\zeta_2^{(1)}J_1\left(\chi_2^{(1)}\right)+\frac{\Lambda^{(1)}(\omega)}{2}\times\left(\beta_1^{(1)}\left(\zeta_2^{(1)}\right)^2\left(J_2\left(\chi_2^{(1)}\right)\right.\right.$$
$$\left.\left.-J_0\left(\chi_2^{(1)}\right)\right)-\frac{\zeta_2^{(1)}}{\eta}J_1\left(\chi_2^{(1)}\right)-\beta_2^{(1)}J_0\left(\chi_2^{(1)}\right)\right)$$

$$\beta_{15}\left(\zeta_2^{(1)},\chi_2^{(1)}\right) = \left(\Lambda^{(1)}(\omega)+2M^{(1)}(\omega)\right)\left(-\left(\zeta_2^{(1)}\right)^2\frac{1}{2}\left(Y_2\left(\chi_2^{(1)}\right)-Y_0\left(\chi_2^{(1)}\right)\right)\right)$$
$$+\frac{\Lambda^{(1)}(\omega)}{\eta}\zeta_2^{(1)}Y_1\left(\chi_2^{(1)}\right)+\frac{\Lambda^{(1)}(\omega)}{2}\times\left(\beta_1^{(1)}\left(\zeta_2^{(1)}\right)^2\left(Y_2\left(\chi_2^{(1)}\right)\right.\right.$$
$$\left.\left.-Y_0\left(\chi_2^{(1)}\right)\right)-\frac{\zeta_2^{(1)}}{\eta}Y_1\left(\chi_2^{(1)}\right)-\beta_2^{(1)}Y_0\left(\chi_2^{(1)}\right)\right)$$

$$\beta_{23}\left(\zeta_2^{(1)},\chi_2^{(1)}\right) = -M^{(1)}(\omega)\zeta_2^{(1)}J_1\left(\chi_2^{(1)}\right)+\frac{M^{(1)}(\omega)}{4}\left(\beta_1^{(1)}\left(\left(\zeta_2^{(1)}\right)^3\left(3J_1\left(\chi_2^{(1)}\right)\right.\right.\right.$$

$$-J_3\left(\chi_2^{(1)}\right)\Big)+\frac{\zeta_2^{(1)}}{\eta^2}J_1\left(\chi_2^{(1)}\right)+\frac{\left(\zeta_2^{(1)}\right)^2}{2\eta}\left(J_2\left(\chi_2^{(1)}\right)\right.$$
$$\left.-J_0\left(\chi_2^{(1)}\right)\right)+\beta_2^{(1)}\zeta_2^{(1)}J_1\left(\chi_2^{(1)}\right)\Big)$$

$$\beta_{25}\left(\zeta_2^{(1)},\chi_2^{(1)}\right)=-M^{(1)}(\omega)\zeta_2^{(1)}Y_1\left(\chi_2^{(1)}\right)+\frac{M^{(1)}(\omega)}{4}\left(\beta_1^{(1)}\left(\left(\zeta_2^{(1)}\right)^3\left(3Y_1\left(\chi_2^{(1)}\right)\right.\right.\right.$$
$$-Y_3\left(\chi_2^{(1)}\right)\Big)+\frac{\zeta_2^{(1)}}{\eta^2}Y_1\left(\chi_2^{(1)}\right)+\frac{\left(\zeta_2^{(1)}\right)^2}{\eta 2}\left(Y_2\left(\chi_2^{(1)}\right)\right.$$
$$\left.-Y_0\left(\chi_2^{(1)}\right)\right)+\beta_2^{(1)}\zeta_2^{(1)}Y_1\left(\chi_2^{(1)}\right)\Big)$$

$$\beta_{14}=\beta_{13}\left(\zeta_3^{(1)},\chi_3^{(1)}\right),\ \beta_{16}=\beta_{15}\left(\zeta_3^{(1)},\chi_3^{(1)}\right),\ \beta_{24}=\beta_{23}\left(\zeta_3^{(1)},\chi_3^{(1)}\right),$$
$$\beta_{26}=\beta_{25}\left(\zeta_3^{(1)},\chi_3^{(1)}\right),\ \beta_{33}\left(\zeta_2^{(1)},\chi_2^{(1)}\right)=-\zeta_2^{(1)}J_1\left(\chi_2^{(1)}\right),$$

$$\beta_{43}\left(\zeta_2^{(1)},\chi_2^{(1)}\right)=\left(-\beta_1^{(1)}\left(\zeta_2^{(1)}\right)^2-\beta_2^{(1)}\right)J_0\left(\chi_2^{(1)}\right),$$

$$\beta_{35}\left(\zeta_2^{(1)},\chi_2^{(1)}\right)=-\zeta_2^{(1)}Y_1\left(\chi_2^{(1)}\right),$$

$$\beta_{45}\left(\zeta_2^{(1)},\chi_2^{(1)}\right)=\left(-\beta_1^{(1)}\left(\zeta_2^{(1)}\right)^2-\beta_2^{(1)}\right)Y_0\left(\chi_2^{(1)}\right),\ \beta_{34}=\beta_{33}\left(\zeta_3^{(1)},\chi_3^{(1)}\right),$$
$$\beta_{36}=\beta_{35}\left(\zeta_3^{(1)},\chi_3^{(1)}\right),\ \beta_{44}=\beta_{43}\left(\zeta_3^{(1)},\chi_3^{(1)}\right),\ \beta_{46}=\beta_{45}\left(\zeta_3^{(1)},\chi_3^{(1)}\right),$$

$$\beta_{53}\left(\zeta_2^{(1)},\chi_{2h}^{(1)}\right)=\left(\Lambda^{(1)}(\omega)+2M^{(1)}(\omega)\right)\left(-\left(\zeta_2^{(1)}\right)^2\frac{1}{2}\left(J_2\left(\chi_{2h}^{(1)}\right)-J_0\left(\chi_{2h}^{(1)}\right)\right)\right)$$
$$+\frac{\Lambda^{(1)}(\omega)}{\eta}\zeta_2^{(1)}J_1\left(\chi_{2h}^{(1)}\right)+\frac{\Lambda^{(1)}(\omega)}{2}\times\left(\beta_1^{(1)}\left(\zeta_2^{(1)}\right)^2\left(J_2\left(\chi_{2h}^{(1)}\right)\right.\right.$$
$$\left.\left.-J_0\left(\chi_{2h}^{(1)}\right)\right)-\frac{s_2^{(1)}}{\eta}J_1\left(\chi_{2h}^{(1)}\right)-\beta_2^{(1)}J_0\left(\chi_{2h}^{(1)}\right)\right)$$

$$\beta_{63}\left(\zeta_2^{(1)},\chi_{2h}^{(1)}\right)=-M^{(1)}(\omega)\zeta_2^{(1)}J_1\left(\chi_{2h}^{(1)}\right)+\frac{M^{(1)}(\omega)}{4}\left(\beta_1^{(1)}\left(\left(\zeta_2^{(1)}\right)^3\left(3J_1\left(\chi_{2h}^{(1)}\right)\right.\right.\right.$$
$$-J_3\left(\chi_{2h}^{(1)}\right)\Big)+\frac{\zeta_2^{(1)}}{\eta^2}J_1\left(\chi_{2h}^{(1)}\right)+\frac{\left(\zeta_2^{(1)}\right)^2}{2\eta}\left(J_2\left(\chi_{2h}^{(1)}\right)-J_0\left(\chi_{2h}^{(1)}\right)\right)$$
$$+\beta_2^{(1)}\zeta_2^{(1)}J_1\left(\chi_{2h}^{(1)}\right)\Big)$$

$$\beta_{55}\left(\zeta_2^{(1)},\chi_{2h}^{(1)}\right)=\left(\Lambda^{(1)}(\omega)+2M^{(1)}(\omega)\right)\left(-\left(\zeta_2^{(1)}\right)^2\frac{1}{2}\left(Y_2\left(\chi_{2h}^{(1)}\right)-Y_0\left(\chi_{2h}^{(1)}\right)\right)\right)$$

$$
+\frac{\Lambda^{(1)}(\omega)}{\eta}\zeta_2^{(1)}Y_1\left(\chi_{2h}^{(1)}\right)+\frac{\Lambda^{(1)}(\omega)}{2}\times\left(\beta_1^{(1)}\left(\zeta_2^{(1)}\right)^2\left(Y_2\left(\chi_{2h}^{(1)}\right)\right.\right.
$$

$$
\left.\left.-Y_0\left(\chi_{2h}^{(1)}\right)\right)-\frac{\zeta_2^{(1)}}{\eta}Y_1\left(\chi_{2h}^{(1)}\right)-\beta_2^{(1)}Y_0\left(\chi_{2h}^{(1)}\right)\right)
$$

$$
\beta_{65}\left(\zeta_2^{(1)},\chi_{2h}^{(1)}\right)=-M^{(1)}(\omega)\zeta_2^{(1)}Y_1\left(\chi_{2h}^{(1)}\right)+\frac{M^{(1)}(\omega)}{4}\left(\beta_1^{(1)}\left(\left(\zeta_2^{(1)}\right)^3\left(3Y_1\left(\chi_{2h}^{(1)}\right)\right.\right.\right.
$$

$$
\left.-Y_3\left(\chi_{2h}^{(1)}\right)\right)+\frac{\zeta_2^{(1)}}{\eta^2}Y_1\left(\chi_{2h}^{(1)}\right)+\frac{\left(\zeta_2^{(1)}\right)^2}{2\eta}\left(Y_2\left(\chi_{2h}^{(1)}\right)\right.
$$

$$
\left.\left.\left.-Y_0\left(\chi_{2h}^{(1)}\right)\right)+\beta_2^{(1)}\zeta_2^{(1)}Y_1\left(\chi_{2h}^{(1)}\right)\right)\right)
$$

$$
\beta_{n4}=\beta_{n3}\left(\zeta_3^{(1)},\chi_{3h}^{(1)}\right),\ \ \beta_{n6}=\beta_{n5}\left(\zeta_3^{(1)},\chi_{3h}^{(1)}\right),
$$

$$
\beta_{n1}=\beta_{n2}=0,\ \ n=5,6. \tag{22}
$$

In relation (22), $J_n(x)$ and $Y_n(x)$ are Bessel functions of the first and second kinds, respectively. Moreover in (22) the following notation is used:

$$
\chi_2^{(n)}=kR\zeta_2^{(n)},\ \ \chi_3^{(n)}=kR\zeta_3^{(n)},\ \ n=1,2,
$$

$$
\chi_{2h}^{(1)}=kR\left(1+\frac{h}{R}\right)\zeta_2^{(1)},\ \ \chi_{3h}^{(1)}=kR\left(1+\frac{h}{R}\right)\zeta_3^{(1)}
$$

$$
\beta_1^{(n)}=\frac{(\Lambda^{(n)}(\omega)+2M^{(n)}(\omega))}{(\Lambda^{(n)}(\omega)+M^{(n)}(\omega))}
$$

$$
\beta_2^{(n)}=\frac{M^{(n)}(\omega)}{(\Lambda^{(n)}(\omega)+M^{(n)}(\omega))}-\rho^{(n)}\left(\frac{\omega}{k}\right)^2(\Lambda^{(n)}(\omega)+M^{(n)}(\omega))^{-1}. \tag{23}
$$

Thus, the dispersion equations obtained for the considered wave propagation problems have been derived in the form (21)–(23).

In the case where $\lambda_{1c}^{(n)}=\lambda_{1s}^{(n)}=\mu_{1c}^{(n)}=\mu_{1s}^{(n)}=0$ in (13), i.e. in the case where $\Lambda^{(n)}=\lambda_0^{(n)}$ and $M^{(n)}=\mu_0^{(n)}$ the foregoing dispersion equation transforms into the corresponding one obtained for the wave dispersion in the purely elastic case which is detailed for instance in the papers by Akbarov and Guliev (2009), Akbarov and Ipek (2010) and Akbarov (2013) and in the monograph by Akbarov (2015).

4 Numerical results and discussions

4.1 Selection of the operators in (3) and dimensionless rheological parameters

According to the problem formulation, we must take the complex wave number k

which can be presented as follows:

$$k = k_1 + ik_2 = k_1(1+i\beta), \ \ \beta = \frac{k_2}{k_1}, \tag{24}$$

where k_2 (or parameter β in (24)), i.e. the imaginary part of the wave number k, defines the attenuation of the wave amplitude under consideration and β is called the coefficient of the attenuation.

We determine the phase velocity of the studied waves through the expression

$$c = \frac{\omega}{k_1} \tag{25}$$

and introduce the notation $c_{20}^{(n)} = \sqrt{\mu_0^{(n)}/\rho^{(n)}}$.

We use below the arguments

$$\frac{c}{c_{20}^{(2)}}, \ \ k_1R, \text{ and } \frac{h}{R}. \tag{26}$$

Thus, to solve the dispersion equation (21) it is necessary to give the values of $\lambda_{1c}^{(n)}$,$\lambda_{1s}^{(n)}$, $\mu_{1c}^{(n)}$ and $\mu_{1s}^{(n)}$ which are determined by the expressions in (11) through the kernel functions $\mu_1^{(n)}(t)$ and $\lambda_1^{(n)}(t)$ of the operators in (3). We recall that these operators are the viscoelastic properties of the materials of the cylinder's layers. Consequently, for determination of the quantities $\lambda_{1c}^{(n)}$, $\lambda_{1s}^{(n)}$, $\mu_{1c}^{(n)}$ and $\mu_{1s}^{(n)}$, it is necessary to give explicit expression for the functions $\mu_1^{(n)}(t)$ and $\lambda_1^{(n)}(t)$.

As in the papers by Akbarov (2014) and Akbarov and Kepceler (2015), here we also assume that the viscoelasticity of the materials of the cylinder's layers is described by Rabotnov's (1980) fractional exponential operator, i.e. we assume that

$$\mu^{(n)*}\varphi(t) = \mu_0^{(n)}\left[\varphi(t) - \frac{3\beta_0^{(n)}}{2(1+\nu_0^{(n)})}\Pi_{\alpha^{(n)}}^{(n)*}\left(-\frac{3\beta_0^{(n)}}{2(1+\nu_0^{(n)})} - \beta_\infty^{(n)}\right)\varphi(t)\right],$$

$$\lambda^{(n)*}\varphi(t) = \lambda_0^{(n)}\left[\varphi(t) + \frac{\beta_0^{(n)}}{(1+\nu_0^{(n)})}\Pi_{\alpha^{(n)}}^{(n)*}\left(-\frac{3\beta_0^{(n)}}{2(1+\nu_0^{(n)})} - \beta_\infty^{(n)}\right)\varphi(t)\right],$$

$$E^{(n)*}\varphi(t) = E_0^{(n)}\left[\varphi(t) - \beta_0^{(n)}\Pi_{\alpha^{(n)}}^{(n)*}\left(-\beta_0^{(n)} - \beta_\infty^{(n)}\right)\varphi(t)\right],$$

$$\nu^{(n)*}\varphi(t) = \nu_0^{(n)}\left[\varphi(t) + \frac{1-2\nu_0^{(n)}}{2\nu_0^{(n)}}\beta_0^{(n)}\Pi_{\alpha^{(n)}}^{(n)*}\left(-\beta_0^{(n)} - \beta_\infty^{(n)}\right)\varphi(t)\right], \tag{27}$$

where

$$\Pi_{\alpha^{(n)}}^{(n)*}(x^{(n)})\varphi(t) = \int_0^t \Pi_{\alpha^{(n)}}^{(n)}(x^{(n)}, t-\tau)\varphi(\tau)d\tau,$$

$$\Pi^{(n)}_{\alpha^{(n)}}(x^{(n)},t)=t^{-\alpha^{(n)}}\sum_{p=0}^{\infty}\frac{(x^{(n)})^{p}t^{p(1-\alpha^{(n)})}}{\Gamma((1+p)(1-\alpha^{(n)}))},\ \ 0\leq\alpha^{(n)}<1. \tag{28}$$

In (28) $\Gamma(x)$ is the gamma function. Moreover, the constants $\alpha^{(n)}$, $\beta_0^{(n)}$ and $\beta_\infty^{(n)}$ in (27) and (28) are the rheological parameters of the *n-th* component of the cylinder.

According to expressions in (27), we can write that

$$(\lambda^{(n)*}+\frac{2}{3}\mu^{(n)*})\varphi(t)=(\lambda_0^{(n)}+\frac{2}{3}\mu_0^{(n)})\varphi(t). \tag{29}$$

As $(\lambda_0^{(n)}+\frac{2}{3}\mu_0^{(n)})$ is the modulus of volume expansion (denote it by $K_0^{(n)}$), we can conclude that the selection of the operators in (27) corresponds to the case where the volumetric expansion of the materials of the layers is purely elastic. Introducing the notation:

$$\begin{aligned}
&T^{(n)}=T^{(n)}_{rr}+T^{(n)}_{\theta\theta}+T^{(n)}_{zz},\ \ D^{(n)}_{rr}=T^{(n)}_{rr}-T^{(n)},\\
&D^{(n)}_{\theta\theta}=T^{(n)}_{\theta\theta}-T^{(n)},\ \ D^{(n)}_{zz}=T^{(n)}_{zz}-T^{(n)},\\
&D^{(n)}_{rz}=T^{(n)}_{rz},\ \ s^{(n)}_{rr}=\varepsilon^{(n)}_{rr}-\theta^{(n)}/3,\ \ s^{(n)}_{\theta\theta}=\varepsilon^{(n)}_{\theta\theta}-\theta^{(n)}/3,\\
&s^{(n)}_{zz}=\varepsilon^{(n)}_{zz}-\theta^{(n)}/3,\ \ s^{(n)}_{rz}=\varepsilon^{(n)}_{rz},
\end{aligned} \tag{30}$$

the constitutive relations in (2) can be rewritten as follows

$$T^{(n)}(t)=K_0^{(n)}\theta^{(n)}(t),\ \ D^{(n)}_{(ii)}=2\mu^{(n)*}s^{(n)}_{(ii)},\ \ (ii)=rr;\theta\theta;zz;rz \tag{31}$$

As usual in the literature, $D^{(n)}_{(ii)}$ ($s^{(n)}_{(ii)}$) is called the component of the deviatoric stresses (strains).

Consequently, it follows from (31) that in the case under consideration, the operator $\mu^{(n)*}$ is sufficient to describe the viscoelasticity of the materials of the layers.

According to Rabotnov (1980); Adojfsson et al. (2005); Sawicki and Padovan (1999), and other works listed therein and according to the Laplace transformation $\bar{f}(s)=\int_0^\infty f(t)e^{-st}dt$ of the functions $\Pi_\alpha(x,t)$ (28) and $\Pi_{1\alpha}(x,t)=\int_0^t \Pi_\alpha(x,t-\tau)$ $d\tau$ which are

$$\bar{\Pi}_\alpha(x,s)=\frac{1}{s^{1-\alpha}-x},\ \ \bar{\Pi}_{1\alpha}(x,s)=\frac{1}{s(s^{1-\alpha}-x)}, \tag{32}$$

In the papers by Akbarov (2014); Akbarov and Kepceler (2015) the mechanical meaning of the rheological parameters $\alpha^{(n)}$, $\beta_0^{(n)}$ and $\beta_\infty^{(n)}$ was detailed. It was concluded that the dimensionless rheological parameter $\alpha^{(n)}$ characterizes the mechanical behavior of the viscoelastic material around the initial state of the deformation, i.e. in the vicinity of $t = 0$. Moreover, it was concluded that the dimension of the rheological parameter $\beta_\infty^{(n)}$ coincides with the dimension of the rheological parameter $\beta_0^{(n)}$ and is $T^{\alpha^{(n)}-1}$, where T is the time dimension. And, according to (27) and (28), the following expressions are obtained for the long term values of the elastic constants:

$$\begin{aligned}
\lambda_\infty^{(n)} &= \lim_{t\to\infty} \lambda^{(n)*} = \lambda_0^{(n)} \left(1 + \frac{1}{1+\nu_0^{(n)}} \frac{1}{(3/(2(1-\nu_0^{(n)})) + d^{(n)})}\right), \\
\mu_\infty^{(n)} &= \lim_{t\to\infty} \mu^{(n)*} = \mu_0^{(n)} \left(1 - \frac{3}{2(1+\nu_0^{(n)})} \frac{1}{(3/(2(1-\nu_0^{(n)})) + d^{(n)})}\right), \\
E_\infty^{(n)} &= \lim_{t\to\infty} E^{(n)*} = E_0^{(n)} \left(1 - \frac{1}{1+d^{(n)}}\right), \\
\nu_\infty^{(n)} &= \lim_{t\to\infty} \nu^{(n)*} = \nu_0^{(n)} \left(1 + \frac{1-2\nu_0^{(n)}}{2\nu_0^{(n)}} \frac{1}{1+d^{(n)}}\right),
\end{aligned} \tag{33}$$

where

$$d^{(n)} = \frac{\beta_\infty^{(n)}}{\beta_0^{(n)}}. \tag{34}$$

is used. The expressions (33) and (34) show that the constant $d^{(n)}$ characterizes the long-term values of the elastic constants.

Consider the expressions for $\mu_c^{(n)}$ and $\mu_s^{(n)}$ for the selected fractional exponential operator (27) and (28). Using the relations (11) and (27) these expressions can be written as follows:

$$\begin{aligned}
\mu_c^{(n)} &= \mu_0^{(n)} \left[1 - \frac{3}{2(1+\nu_0^{(n)})} \left(d^{(n)} + \beta_{01}^{(n)}\right)^{-1} \Pi_{\alpha^{(n)}c}^{(n)}(-\beta_{01}^{(n)} - \beta_\infty^{(n)}, k_1 Rc)\right], \\
\mu_s^{(n)} &= -\mu_0^{(n)} \frac{3}{2(1+\nu_0^{(n)})} \left(d^{(n)} + \beta_{01}^{(n)}\right)^{-1} \Pi_{\alpha^{(n)}s}^{(n)}(-\beta_{01}^{(n)} - \beta_\infty^{(n)}, k_1 Rc),
\end{aligned} \tag{35}$$

where

$$\beta_{01}^{(n)} = \frac{3\beta_0^{(n)}}{2(1+\nu_0^{(n)})}. \tag{36}$$

We recall that the ratio $\mu_s^{(n)}/\mu_c^{(n)}$ is the loss tangent, i.e. $\tan\eta^{(n)} = \mu_s^{(n)}/\mu_c^{(n)}$, where the angle $\eta^{(n)}$ can be interpreted as providing the phase angle by which the deviatoric strain lags behind the deviatoric stress in steady-state harmonic oscillation in the viscoelastic materials under consideration.

Thus, substituting $(i\omega)$ for s in the Laplace transformation (32) of the core function (28) of the fractional exponential operator (27) and doing some mathematical manipulations, we obtain

$$\Pi^{(n)}_{\alpha^{(n)}c}(-\beta^{(n)}_{01}-\beta^{(n)}_{\infty},k_1Rc)=\frac{(\xi^{(n)})^2+\xi^{(n)}\sin\frac{\pi\alpha^{(n)}}{2}}{(\xi^{(n)})^2+2\xi^{(n)}\sin\frac{\pi\alpha^{(n)}}{2}+1},$$
$$\Pi^{(n)}_{\alpha^{(n)}s}(-\beta^{(n)}_{01}-\beta^{(n)}_{\infty},k_1Rc)=\frac{\xi^{(n)}\cos\frac{\pi\alpha^{(n)}}{2}}{(\xi^{(n)})^2+2\xi^{(n)}\sin\frac{\pi\alpha^{(n)}}{2}+1}. \quad (37)$$

where

$$\xi^{(n)}=(Q^{(n)}\Omega)^{\alpha^{(n)}-1},\ Q^{(n)}=\frac{c^{(n)}_{20}}{R(\beta^{(n)}_{01}+\beta^{(n)}_{\infty})^{\frac{1}{1-\alpha^{(n)}}}},\ \Omega=k_1R\frac{c}{c^{(2)}_{20}}. \quad (38)$$

It follows from the foregoing discussions on the rheological parameters $\beta^{(n)}_{\infty}$ and $\beta^{(n)}_{0}$, that $Q^{(n)}$ and $\xi^{(n)}$ in (37) and (38) are dimensionless parameters. Moreover, it follows from (35), (37) and (38), and from the numerical analyses made in the papers by Akbarov (2014) and Akbarov and Kepceler (2015) that $\mu^{(n)}_{s}/\mu^{(n)}_{0}\to 0$ as $\xi^{(n)}\to 0$ or as $\xi^{(n)}\to\infty$, but the absolute values of $\mu^{(n)}_{c}(\Pi^{(n)}_{\alpha^{(n)}c})$ decrease (increase) monotonically with $\xi^{(n)}$ and $\mu^{(n)}_{c}/\mu^{(n)}_{0}\to 1$ $(\mu^{(n)}_{c}/\mu^{(n)}_{0}\to\mu^{(n)}_{\infty}/\mu^{(n)}_{0})$ as $\xi^{(n)}\to 0(\xi^{(n)}\to\infty)$.

Now we attempt to give mechanical sense to the parameter $Q^{(n)}$, although such meaning has been already discussed in the monograph by Akbarov (2015) and in the papers by Akbarov (2014); Akbarov and Kepceler (2015). Thus, according to these references we introduce the notation

$$t^{(n)}_{c}=\left(\beta^{(n)}_{01}+\beta^{(n)}_{0}\right)^{\frac{-1}{(1+\alpha^{(n)})}} \quad (39)$$

and call it the characteristic creep time for the n-th layer's material. According to (38) and (39), we obtain

$$Q^{(n)}=t^{(n)}_{c}c^{(2)}_{20}/R. \quad (40)$$

It follows from (40) that for fixed $c^{(2)}_{20}/R$ an increase (a decrease) in the values of $Q^{(n)}$ means an increase (a decrease) in the values of the characteristic creep time

$t_c^{(n)}$. Therefore we call the parameter $Q^{(n)}$ the dimensionless characteristic creep time.

The other dimensionless rheological parameter which was introduced above is the parameter $d^{(n)}$ (34). This parameter enters into the expressions (33) and (35) and characterizes the long-term values of the mechanical properties, i.e., for instance, the values of $\mu_\infty^{(n)}$ are determined by expression (33) and $\mu_\infty^{(n)} < \mu_0^{(n)}$. Nevertheless, the magnitude of $\mu_\infty^{(n)}$ increases with $d^{(n)}$. To be more precise, the following relation occurs

$$\mu_\infty^{(n)} \to \mu_0^{(n)}, \;\; \mu_s^{(n)} \to 0 \text{ as } d^{(n)} \to \infty. \tag{41}$$

Consequently, the wave dispersion curves in the considered viscoelastic system must approach the corresponding ones obtained for the same purely elastic system with the constant $d^{(n)}$.

At the same time, according to the expressions (37) and (38), we can write the following limit cases:

$$\xi^{(n)} \to \infty; \;\; \Pi^{(n)}_{\alpha^{(n)}c}(-\beta_1^{(n)} - \beta_\infty^{(n)}, k_1Rc) \to 1; \;\; \Pi^{(n)}_{\alpha^{(n)}s}(-\beta_1^{(n)} - \beta_\infty^{(n)}, k_1Rc) \to 0$$
$$\text{as } (Q^{(n)}\Omega) \to 0 \text{ or as } k_1R \to 0, \tag{42}$$
$$\xi^{(n)} \to 0; \;\; \Pi^{(n)}_{\alpha^{(n)}c}(-\beta_1^{(n)} - \beta_\infty^{(n)}, k_1Rc) \to 0; \;\; \Pi^{(n)}_{\alpha^{(n)}s}(-\beta_1^{(n)} - \beta_\infty^{(n)}, k_1Rc) \to 0$$
$$\text{as } (Q^{(n)}\Omega) \to \infty \text{ or as } k_1R \to \infty. \tag{43}$$

It follows from the relation (42) that in the cases where $(Q^{(n)}\Omega) \ll 1$, the behavior of the viscoelastic system must be very close to the corresponding purely elastic system with long-term values of the elastic constants. As well, it follows from the relation (43) that in the cases where $(Q^{(n)}\Omega) \gg 1$, the behavior of the viscoelastic system must be very close to that of the corresponding purely elastic system with instantaneous values of the elastic constants at $t = 0$.

Thus, according to the foregoing discussions, we can conclude that the influence of the viscosity of the viscoelastic materials under consideration on the wave propagation velocity dispersion (i.e. on the dependence between $c/c_{20}^{(2)}$ and k_1R) and on the wave attenuation dispersion (i.e. on the dependence between the attenuation coefficient β (24) and k_1R) can be characterized through the parameters $Q^{(n)}$ and $d^{(n)}$. It must be taken into account that an increase in the values of the parameters $Q^{(n)}$ and $d^{(n)}$ will correspond to a decrease in the viscous part of all the viscoelastic deformations of the constituents. Note that the influence of the other rheological parameter $\alpha^{(n)}$ on the viscous part of the viscoelastic deformations can be taken into account through the parameter $Q^{(n)}$ (38).

This completes consideration of the selection of the dimensionless rheological parameters through which we will study the influence of the viscoelasticity properties of the layers' materials on the axisymmetric longitudinal wave dispersion.

4.2 On the algorithm for numerical solution to the dispersion equation (21)

As the components β_{ij} in (22) are complex, the values of the determinant obtained in (21) are also complex. Therefore the dispersion equation (21) can be reduced to the following one

$$\left|\det\left\|\beta_{ij}\right\|\right| = 0. \tag{44}$$

Here $\left|\det\left\|\beta_{ij}\right\|\right|$ means the modulus of the complex number $\det\left\|\beta_{ij}\right\|$. Consequently for construction of the dispersion curves it is necessary to solve numerically the equation (44) for the selected problem parameters. In this solution procedure, the values of all the problem parameters (except c, k_1R and β) are selected in advance. Consequently, the equation (44) has three unknowns: c, k_1R and β which must be determined from this equation. Note that in the corresponding purely elastic problems the dispersion equation contains only two unknowns: c and k_1R. The values of c are determined for each possible selected value of k_1R through the solution to this equation. Moreover, in the purely elastic case this solution procedure is carried out by employing the well-known "bi-section" method which is based on the sign change of the dispersion determinant. A more detailed description of the solution algorithm of the dispersion equations related to the purely elastic problems is given in the monograph by Akbarov (2015) and in papers such as Akbarov and Guliev (2009); Akbarov and Ipek (2010). However, in the case under consideration we have not changed the sign of the dispersion determinant, i.e. $\left|\det\left\|\beta_{ij}\right\|\right| \geq 0$ and this determinant, as noted above, contains three unknowns. Consequently, for the solution to the dispersion equation (44) we cannot employ the aforementioned algorithm based on the "bi-section" method. Therefore, for the solution to the dispersion equation (44) we use the algorithm which is based on direct calculation of the values of the moduli of the dispersion determinant $\left|\det\left\|\beta_{ij}\right\|\right|$ and determination of the sought roots from the criterion $\left|\det\left\|\beta_{ij}\right\|\right| \leq 10^{-12}$. It should be noted that under employing this algorithm it is necessary to give in advance a certain value of one of the unknowns c, k_1R or β. For instance, in the paper by Barshinger and Rose (2004) the admissible values for the wave propagation velocity c are given in advance and the values of the attenuation coefficient β are determined for each selected value of k_1R. It is also possible to give a value to the attenuation coefficient β and then to determine the phase velocity c for each selected value of k_1R. This latter case was considered in the paper by Akbarov and Kepceler (2015) and we will also follow this approach in the present paper.

Thus, according to Ewing, Jazdetzky, and Press (1957); Kolsky (1963), as in the paper by Akbarov and Kepceler (2015), we assume that

$$\beta = \frac{1}{2}\frac{\mu_{1s}^{(1)}(\omega)}{\mu_0^{(1)} + \mu_{1c}^{(1)}(\omega)}, \tag{45}$$

or

$$\beta = \frac{1}{2}\frac{\mu_{1s}^{(2)}(\omega)}{\mu_0^{(2)} + \mu_{1c}^{(2)}(\omega)}. \tag{46}$$

It should be noted that the cases (45) and (46) which are given for the attenuation coefficient relate to the dispersive attenuation case. At the same time, the non-dispersive attenuation case can also be considered under which the values selected for k_2R (or β) in (24) do not depend on the wave frequency ω.

This completes consideration of the numerical solution algorithm to the dispersion equation (48).

4.3 On the low and high wavenumber limit values on the wave propagation velocity

First, we note that the results presented in the present subsection occur in the case where the attenuation of the waves under consideration is dispersive and the coefficient of attenuation satisfies the following conditions:

$$\beta \to 0 \text{ as } k_1R \to 0 \text{ and } \beta \to 0 \text{ as } k_1R \to \infty. \tag{47}$$

For instance, the conditions in (47) satisfy the cases where the relation (45) or (46) takes place.

Thus, according to the discussions made in subsection 4.1 and according to the monograph by Akbarov (2015), we can write the following low wavenumber limit values for the wave propagation velocity in the bi-layered circular solid cylinder made of viscoelastic materials.

$$\frac{c}{c_{20}^{(2)}} = \sqrt{\frac{\mu_\infty^{(2)}}{\mu_0^{(2)}}}\left(\frac{e_\infty^{(2)}\eta^{(2)} + e_\infty^{(1)}\eta^{(1)}\dfrac{\mu_\infty^{(1)}}{\mu_\infty^{(2)}}}{\eta^{(2)} + \eta^{(1)}\dfrac{\rho^{(1)}}{\rho^{(2)}}}\right)^{\frac{1}{2}} \quad \text{as } k_1R \to 0, \tag{48}$$

where

$$e_\infty^{(n)} = 2\left(1 + \frac{\lambda_\infty^{(n)}}{2(\lambda_\infty^{(n)} + \mu_\infty^{(n)})}\right), \;\; \eta^{(2)} = \left(1 + \frac{h}{R}\right)^{-2},$$

$$\eta^{(1)} = \left(2\frac{h}{R} + \left(\frac{h}{R}\right)^2\right)\left(1+\frac{h}{R}\right)^{-2}. \tag{49}$$

In a similar manner, according to the discussions made in subsection 4.1 and according to the monograph by Akbarov (2015), we can write the following high wavenumber limit values for the case under consideration:

$$\frac{c}{c_{20}^{(2)}} = \min\left\{\frac{c_R^{(1)}}{c_{20}^{(2)}}, 1, \frac{c_S}{c_{20}^{(2)}}\right\}, \tag{50}$$

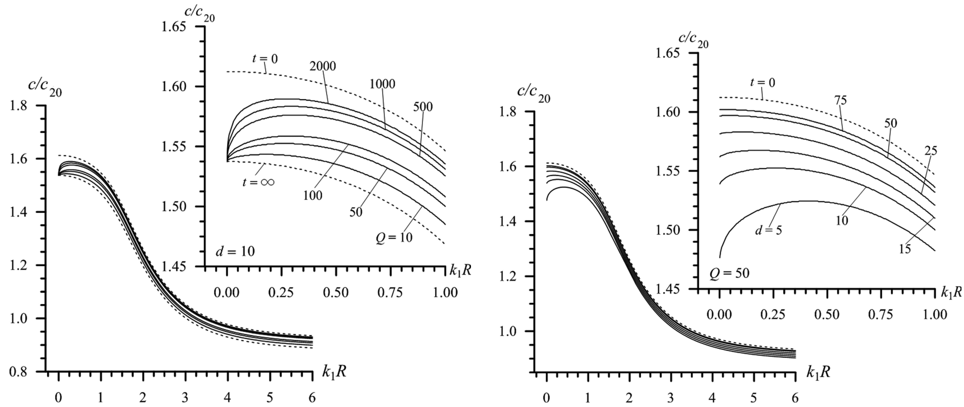
where $c_R^{(n)}$ is the Rayleigh wave propagation velocity of the n-th material for the instantaneous values of the elastic constants of this material and c_S is the Stoneley wave propagation velocity of the selected pair of materials of the layers as well as for the instantaneous values of the elastic constants of these materials.

It should be noted that the expressions (48)–(50) occur not only for the fractional exponential operators given in (27) and (28), but also for arbitrary possible operators describing the viscoelasticity of the constituents' materials of the cylinder. However, the existence of the expressions (48)–(50) requires satisfaction of the conditions in (47).

4.4 Numerical results on dispersion and dispersive attenuation curves

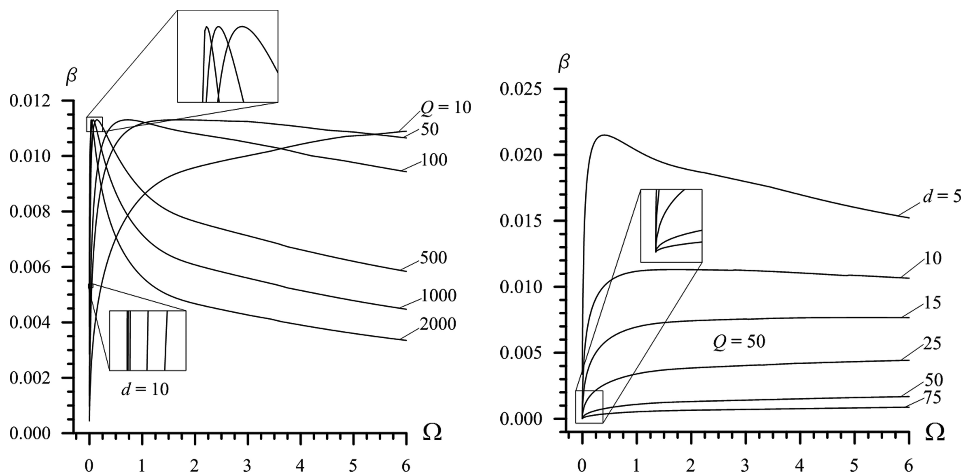
First, we consider dispersion curves obtained for the homogeneous solid circular cylinder in the case where $\nu_0 = 0.3$ (instantaneous value of the Poisson coefficient) and $\alpha = 0.5$. These curves are constructed for the first lowest mode as a result of the solution to the corresponding dispersion equation which can be easily derived from the foregoing expressions and equations. Corresponding results for the dispersive attenuation case where the coefficient of the attenuation β is determined through the expression (45) (or (46)) are given in Fig. 2. Note that the curves illustrated in this figure are constructed for various values of the parameter Q under a fixed value of the parameter d $(= 10)$ (Fig. 2(a)) and for various values of the parameter d under a fixed value of the parameter $Q\,(= 50)$ (Fig. 2(b)). Corresponding dispersion curves for the coefficient of the attenuation β, i.e. the graphs of the dependents between β and the parameter Ω (38) are given in Fig. 3(a) for the case where $d = 10$ and in Fig. 3(b) for the case where $Q = 50$. Note that the curves given in Fig. 3 occur for all cases which will be considered below and therefore we do not return to analysis of the attenuation dispersion, because the considered type attenuation depends only on the rheological parameters of the cylinder's materials.

If we assume that $\eta^{(1)} = 0$ and $\eta^{(2)} = 1$, then we obtain from (48) and (49) that the low wavenumber limit value of the wave propagation velocity is determined from



(a) (b)

Figure 2: Dispersion curves constructed for the solid homogeneous cylinder under various values of the parameter Q under a fixed value of the parameter d ($= 10$) (a) and for various values of the parameter d under a fixed value of the parameter Q ($= 50$) (b).



(a) (b)

Figure 3: The graphs of the dependence between the coefficient of the attenuation β and the dimensionless frequency Ω (38) constructed for various values of the parameter Q under a fixed value of the parameter d ($= 10$) (a) and for various values of the parameter d under a fixed value of the parameter Q ($= 50$) (b).

the expression

$$\frac{c}{c_{20}} = \sqrt{\frac{\mu_\infty}{\mu_0} e_\infty}, \;\; e_\infty = 2\left(1 + \frac{\lambda_\infty}{2(\lambda_\infty + \mu_\infty)}\right) \text{ as } k_1 R \to 0 \tag{51}$$

Taking the relations $E_\infty = \mu_\infty e_\infty$ and $c_{20} = \sqrt{\mu_0/\rho}$ into consideration, we obtain from (51) that

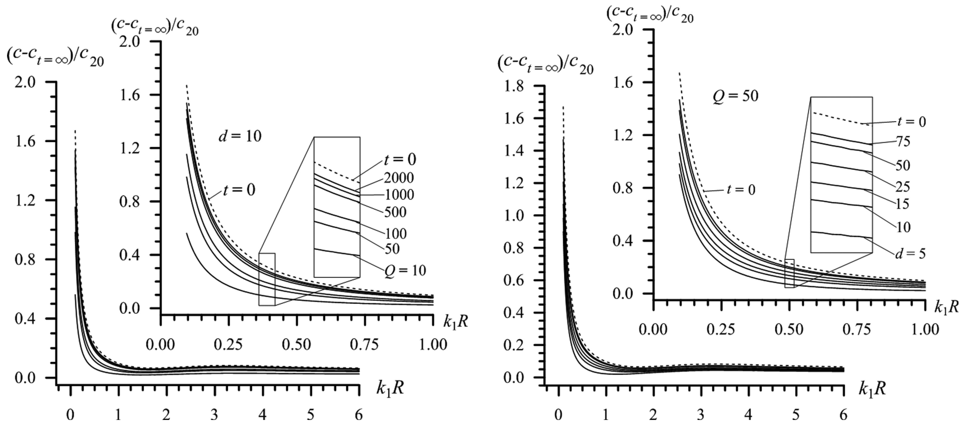
$$c = \sqrt{E_\infty/\rho} \text{ as } k_1 R \to 0, \tag{52}$$

where $\sqrt{E_\infty/\rho}$ is the long-term bar velocity for the cylinder under consideration.

According to well-known mechanical considerations and to the expression (50), we can conclude that the high wavenumber limit value of the wave propagation velocity c is c_R, which is the Rayleigh wave propagation velocity in the cylinder material with instantaneous values of the elastic constants. Thus, we turn to consideration of the numerical results given in Fig. 2, which show that they are limited with the velocities obtained for the instantaneous values (i.e. at $t = 0$) of the elastic constants (upper limit) and with the velocities obtained for the long-term values (i.e. at $t = \infty$) of elastic constants (lower limit).

It follows from Fig. 2 that the viscoelasticity of the cylinder material causes a decrease in the values of the wave propagation velocity. According to the foregoing discussions, it can be predicted that the low wavenumber limit of the wave propagation velocity depends only on the rheological parameter d. Consequently, the low wavenumber limit values of the wave propagation velocity must be the same for various values of the rheological parameter Q under a fixed d and this limit is equal to the long-term bar velocity of the cylinder material for which the value E_∞ in (52) is determined through the expression (33) for the selected value of d. Note that this prediction is proven by the results given in Fig. 2(a). This prediction is also proven with the results illustrated in Fig. 2(b), according to which, the low wavenumber limit values of the wave propagation velocity increase with the rheological parameter d. At the same time, the results given in Fig. 2 show that the propagation velocity of the axisymmetric wave in the cylinder approaches its corresponding "instantaneous values" with the rheological parameters d and Q. Observation of the results allows us to also conclude that, in the case under consideration the effect of the viscoelasticity of the cylinder material on the wave propagation velocity becomes significant under relatively small values of the dimensionless wavenumber $k_1 R$.

Consider also the numerical results related to the second mode. For clarity we consider the graphs between $(c - c|_{t=\infty})/c_{20}$ and $k_1 R$ through which the influence of the rheological parameters of the cylinder material on the wave propagation velocity in the second mode can be estimated. These graphs are given in Fig. 4(a) (in Fig. 4(b)) which are constructed for various values of the rheological parameter Q (parameter d) under a fixed value of d $(= 10)$ (of Q $(= 50)$). It follows from these graphs that a decrease in the values of the parameter Q as well as in the values of the parameter d also causes a decrease in the values of the wave propagation velocity in the second mode.



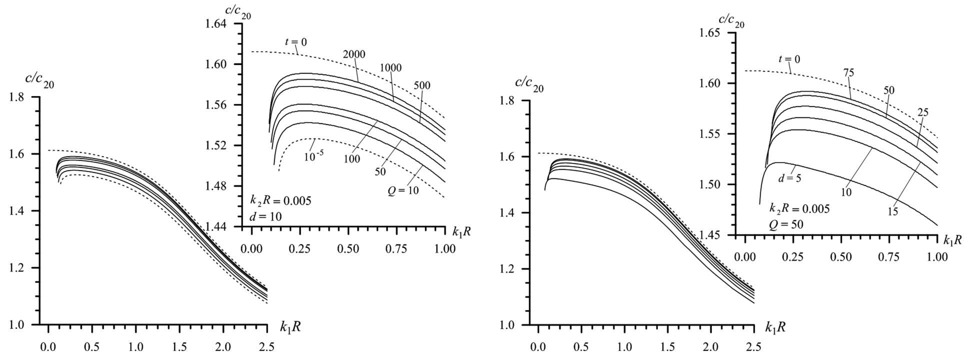
(a) (b)

Figure 4: The graphs of the dependence between $(c - c|_{t=\infty})/c_{20}$ and k_1R obtained for the homogeneous solid cylinder in the second mode under various values of the parameter Q, under a fixed value of the parameter d $(= 10)$ (a) and for various values of the parameter d under a fixed value of the parameter Q $(= 50)$ (b).

We recall that the foregoing results are obtained for the attenuation dispersion case for which the attenuation is determined through the expression (45) (or (46)). Now we consider the results obtained for the first lowest mode in the non-dispersive attenuation case and assume that $k_2R = 0.005$. The graphs illustrating the dispersion curves related to this case are given in Fig. 5(a) (in Fig. 5(b)), which are constructed for various values of the rheological parameter Q (parameter d) under a fixed value of d $(= 10)$ (under a fixed value of Q $(= 50)$). It follows from the graphs that in the non-dispersive attenuation case, the cut off values of the dimensionless wavenumber k_1R (denoted by $(k_1R)_{c.f.}$) (or cut off frequency (denoted by $\omega_{c.f.}$) determined by expression $\omega_{c.f.} = (k_1R)_{c.f.} \times c|_{k_1R=(k_1R)_{c.f.}}$), appear. Also it follows from the graphs that, as in the dispersive attenuation case, a decrease in the values of the rheological parameters Q and d causes a decrease in the wave propagation velocity. Moreover, the results show that the values of $(k_1R)_{c.f.}$ increase (decrease) with Q (with d).

The results given in Fig. 6 illustrate how the values of k_2R effect the values of $(k_1R)_{c.f.}$ and accordingly, it can be concluded that $(k_1R)_{c.f.}$ decreases with decreasing k_2R.

Now we consider the results related to the bi-material compound solid cylinder and assume that $\nu_0^{(1)} = \nu_0^{(2)} = 0.3$ and $\alpha^{(1)} = \alpha^{(2)} = 0.5$. Consider the cases where



(a) (b)

Figure 5: Dispersion curves constructed in the non-dispersive attenuation case for the homogeneous solid cylinder under various values of the parameter Q for a fixed value of the parameter d (= 10) (a) and for various values of the parameter d for a fixed value of the parameter Q (= 50) (b) in the case where $k_2R = 0.005$.

$\mu_0^{(2)}/\mu_0^{(1)} = 0.5$ and $\mu_0^{(2)}/\mu_0^{(1)} = 2$. First, we analyze the results obtained in the case where the viscoelasticity properties of the cylinder's layers are the same, i.e. first, we analyze the results obtained in the case where $Q^{(1)} = Q^{(2)} (= Q)$ and $d^{(1)} = d^{(2)} (= d)$ and denote it as the V.V. case. We recall that, unless otherwise specified, the results discussed below are obtained within the scope of the attenuation relation (45) or (46). Here we consider mainly the results obtained for the first lowest (fundamental) mode. At the same time, we will also consider some results related to the second mode.

Thus, we consider the graphs given in Figs. 7, 8 and 9 (in Figs. 10, 11 and 12) which are constructed in the cases where $h/R = 0.1$, 0.3 and 0.5, respectively, under $\mu_0^{(2)}/\mu_0^{(1)} = 0.5$ (under $\mu_0^{(2)}/\mu_0^{(1)} = 2$). Note that in these figures the graphs grouped by the letter a illustrate the influence of the parameter Q on the dispersion curves under a fixed value of the parameter d (i.e. under $d = 10$) and the graphs grouped by the letter b illustrate the influence of the parameter d on the dispersion curves under a fixed value of the parameter Q (i.e. under $Q = 50$).

According to the discussions made in subsections 4.1 and 4.3, and according to the discussions made above for the solid homogeneous cylinder, we can predict that the wave propagation velocity in the compound solid cylinder obtained for all the selected values of the parameter Q under a fixed value of the parameter d must have the same low wavenumber limit as $k_1R \to 0$ and the values of this limit coincide with those determined by the expression (48). Moreover, according to the foregoing discussions it can be predicted that the limit values of the wave propaga-

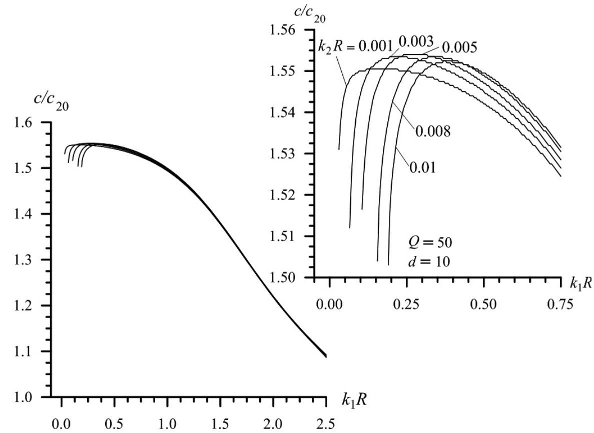


Figure 6: The influence of k_2R on the cut off values of $(k_1R)_{c.f.}$ obtained for the non-dispersive attenuation case for the homogeneous solid cylinder under $Q = 50$ and $d = 10$.

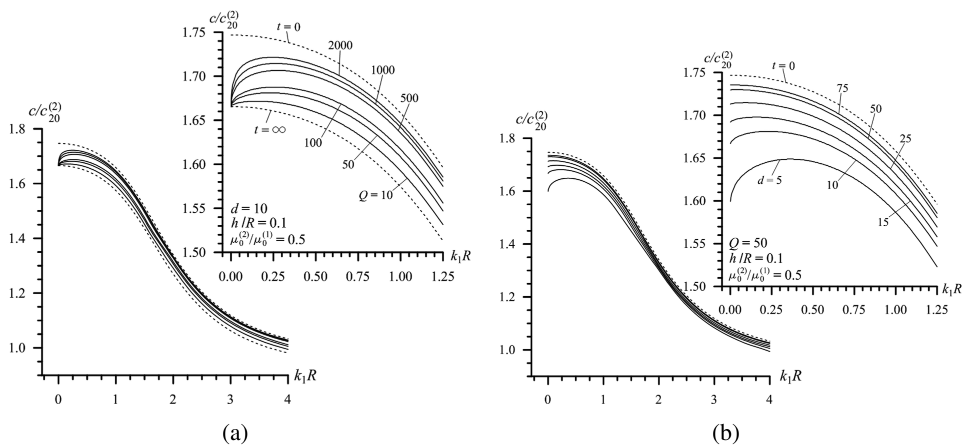
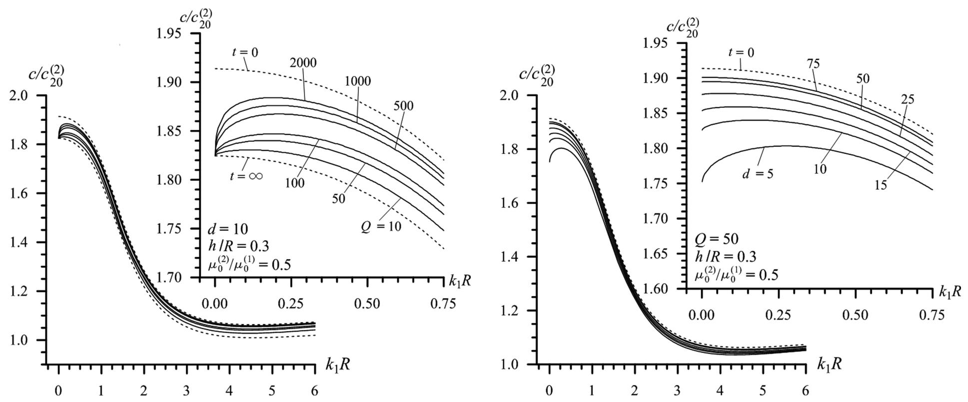


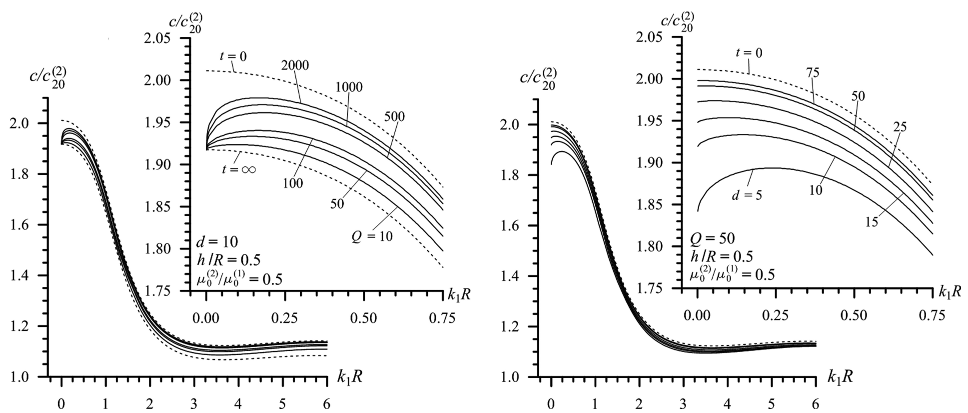
Figure 7: Dispersion curves obtained for various values of the parameter Q under a fixed value of the parameter d (= 10) (a) and for various values of the parameter d under a fixed value of the parameter Q (= 50) (b) in the V.V. case under $\mu_0^{(2)}/\mu_0^{(1)} = 0.5$ and $h/R = 0.1$.

tion velocity in the compound cylinder also depend on the rheological parameter d and do not depend on the rheological parameter Q. These predictions are proven with the results illustrated in Figs. 7–12 and it follows from these results that the dispersion curves obtained under fixed values of the parameter d are limited to the corresponding dispersion curves obtained for the purely elastic cases under instantaneous values of the elastic constants (upper limits), i.e. under $t = 0$, and under



(a) (b)

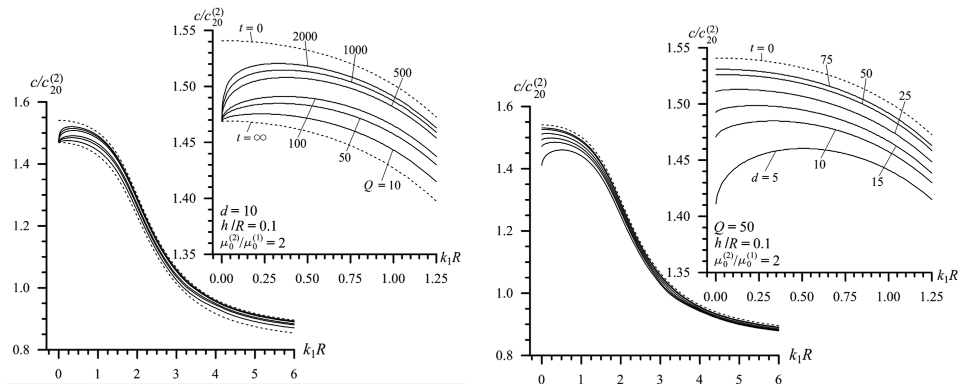
Figure 8: Dispersion curves obtained in the case considered in Fig. 7 under $h/R = 0.3$.



(a) (b)

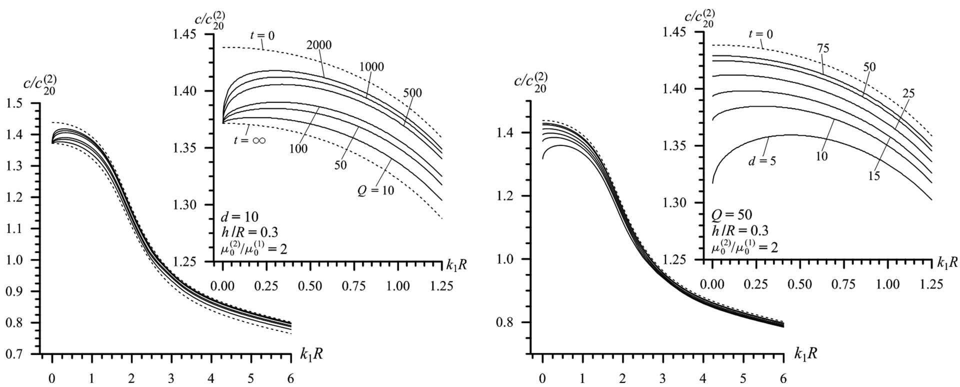
Figure 9: Dispersion curves obtained in the case considered in Fig. 7 under $h^{(1)}/R = 0.5$.

long-term values of the elastic constants (lower limits), i.e. under $t = \infty$. Note that in the figures, the graphs related to the limit cases are given by dashed lines and coincide with the corresponding ones given in the monograph by Akbarov (2015) and in the paper by Akbarov and Ipek (2012). Consequently, the results illustrated in Figs. 7–12, not only give new information about the influence of the rheological parameters on the dispersion curves, but also illustrate the reliability of these results and the reliability of the calculation algorithm by which they were obtained. Moreover, the results show that for each value of the rheological parameters the



(a) (b)

Figure 10: Dispersion curves obtained in the case considered in Fig. 7 under $\mu_0^{(2)}/\mu_0^{(1)} = 2$ and $h/R = 0.1$.

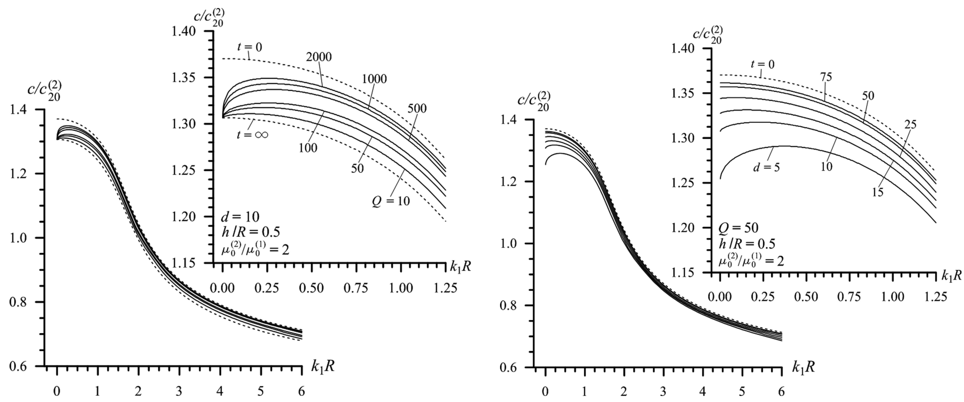


(a) (b)

Figure 11: Dispersion curves obtained in the case considered in Fig. 10 under $h/R = 0.3$.

wave propagation velocity approaches the corresponding one which relates to the elastic cases as $k_1R \to \infty$, i.e. the relation (50) holds. According to the results obtained, we can conclude, as can be predicted, that as a result of the increase in the values of the rheological parameters d and Q (see the graphs given in Figs. 7–12) the dispersion curves come near to the corresponding dispersion curves obtained for the purely elastic case constructed under $t = 0$.

Analyses of the foregoing results show that for the considered change range of the problem parameters, the influence of the viscoelasticity parameters d and Q on the wave propagation velocity is significant in the cases where $k_1R \leq 1.5$. More-



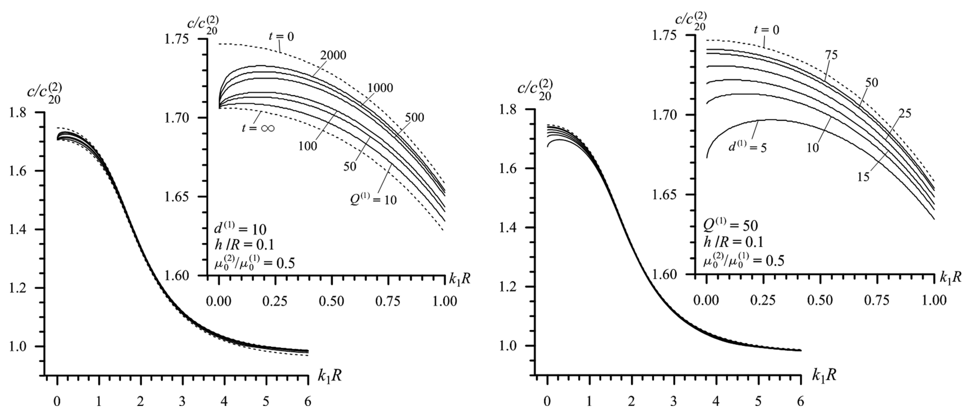
(a) (b)

Figure 12: Dispersion curves obtained in the case considered in Fig. 10 under $h/R = 0.5$.

over, analyses of the foregoing results show that in the case where $\mu_0^{(2)}/\mu_0^{(1)} = 0.5$ ($\mu_0^{(2)}/\mu_0^{(1)} = 2$) an increase in the values of h/R causes an increase (a decrease) in the values of the wave propagation velocity $c/c_{20}^{(2)}$. This is because in the case where $\mu_0^{(2)}/\mu_0^{(1)} = 0.5$ ($\mu_0^{(2)}/\mu_0^{(1)} = 2$) an increase in the values of h/R means that the volumetric concentration of stiffer (softer) material in the compound cylinder increases and, according to this provision, the wave propagation velocity increases. Moreover, according to this provision, this explains the fact that the wave propagation velocity obtained for a homogeneous solid cylinder, as illustrated in Fig. 2, is less (greater) than the corresponding one obtained in the case where $\mu_0^{(2)}/\mu_0^{(1)} = 0.5$ ($\mu_0^{(2)}/\mu_0^{(1)} = 2$) for the compound cylinder as illustrated in Figs. 7–12.

We note the following important fact which also follows from the analysis of the foregoing results and from the results which will be considered below. This fact relates to the character of the dependence between $c/c_{20}^{(2)}$ and k_1R in the cases where $0 < k_1R < 1.5$. So that, if the materials of the cylinders are purely elastic then this dependence is monotonic, i.e. the values of $c/c_{20}^{(2)}$ increase monotonically with decreasing k_1R. However, if the materials of the cylinders are viscoelastic, then this dependence may have a non-monotonic character. The non-monotonic character is observed for all values of Q under a fixed d and also for the relatively small values of the parameter d under a fixed Q. It follows from the appearance of the aforementioned non-monotonic part in the dispersion curves that there exists such a value of k_1R (denoted by $(k_1R)^*$) under which $dc/d(k_1R) = 0$. This equation means that at $k_1R = (k_1R)^*$ the phase velocity is equal to the group velocity and around $k_1R = (k_1R)^*$ the backward waves may appear (see, Akbarov (2015)). Consequently, the viscoelasticity of the cylinder's materials acts not only quantitatively on the dispersion curves of the axisymmetric waves in the cylinders, but also quantitatively.

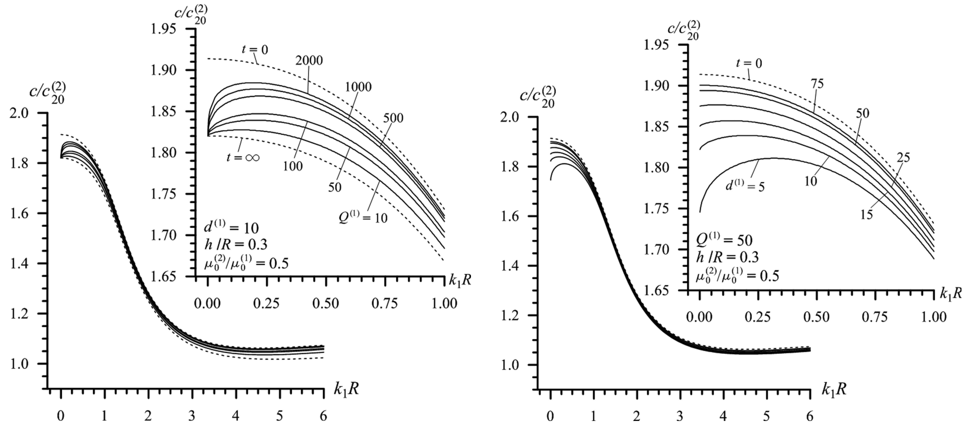
We recall that the foregoing results are obtained in the cases where both the materials of the inner and outer layers of the hollow cylinder are viscoelastic, i.e. under the V.V. case. Now we consider the results obtained in the case where the material of the inner solid cylinder is purely elastic, but the material of the outer hollow cylinder is viscoelastic and denote this case as the V.E. case. Assume that the attenuation coefficient β for this case is given through the relation (45). Thus, we analyze the related dispersion curves which are given in Figs. 13–15 (Figs. 16–18) and obtained for $\mu_0^{(2)}/\mu_0^{(1)} = 0.5$ ($\mu_0^{(2)}/\mu_0^{(1)} = 2$) in the cases where $h/R = 0.1, 0.3$ and 0.5, respectively. As above, in these figures the graphs grouped by the letter a are constructed for a fixed value of the parameter $d^{(1)}$ (= 10) for various values of the parameter $Q^{(1)}$. However the graphs grouped by the letter b are constructed for a fixed value of the parameter $Q^{(1)}$ (= 50) for various values of the parameter $d^{(1)}$.



(a) (b)

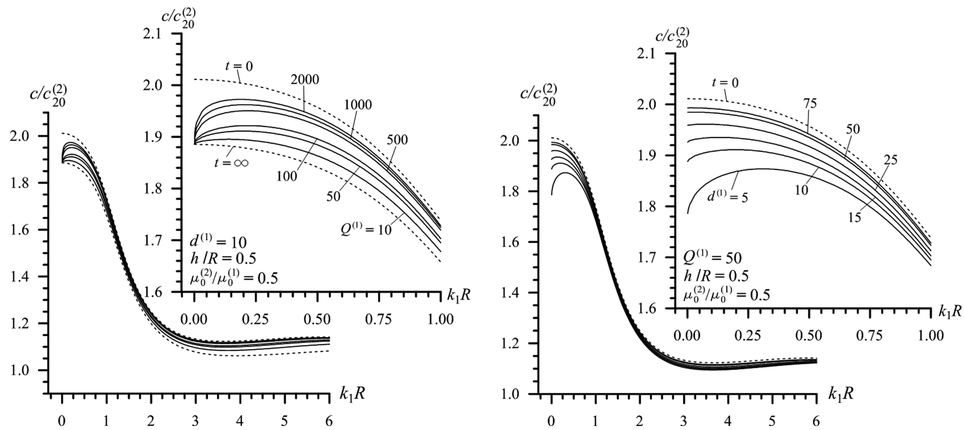
Figure 13: Dispersion curves obtained for various values of the parameter $Q^{(1)}$ under a fixed value of the parameter $d^{(1)}$ (= 10) (a) and for various values of the parameter $d^{(1)}$ under a fixed value of the parameter $Q^{(1)}$ (= 50) (b) in the V.E. case under $\mu_0^{(2)}/\mu_0^{(1)} = 0.5$ and $h/R = 0.1$.

We introduce the notation $c_{v.v.}$ and $c_{v.e.}$ to indicate the wave propagation velocity in the V.V. and V.E. cases. Comparison of the results given in Figs. 13, 14 and 15 (Figs. 16–18) with the corresponding ones given in Figs. 7, 8 and 9 (Figs. 10–12) shows that $c_{v.e.} > c_{v.v.}$. Moreover, this comparison shows that the character of the dispersion curves and the character of the influence of the problem parameters on



(a) (b)

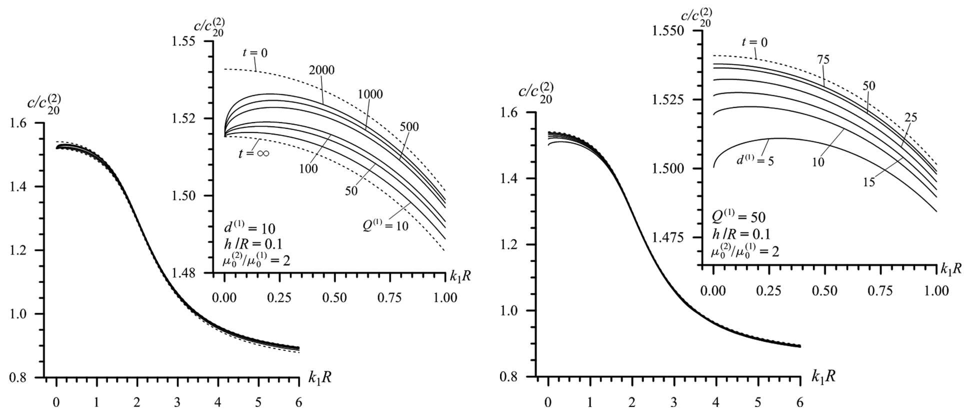
Figure 14: Dispersion curves obtained in the case considered in Fig. 13 under $h/R = 0.3$.



(a) (b)

Figure 15: Dispersion curves obtained in the case considered in Fig. 13 under $h/R = 0.5$.

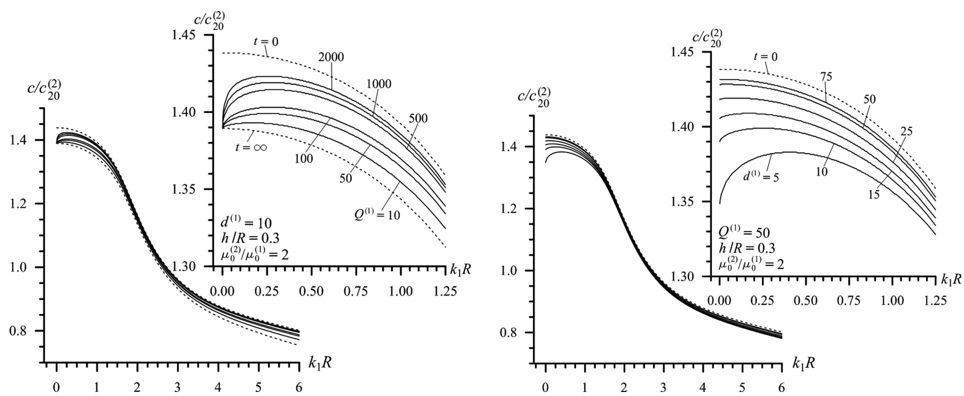
these curves in the V.E. case are similar to those observed in the V.V. case. For instance, in both cases for all selected values of the problem parameters, the wave propagation velocity is less (greater) than the corresponding one obtained for the purely elastic case with instantaneous (long-term) values of the elastic constants. However, as we will consider below, in the case where the material of the outer cylinder is purely elastic, but the material of the inner solid cylinder is viscoelastic (denote this case as the E.V. case) the aforementioned type of limitation of the wave



(a) (b)

Figure 16: Dispersion curves obtained in the case considered in Fig. 13 under $\mu_0^{(2)}/\mu_0^{(1)} = 2$ and $h/R = 0.1$.

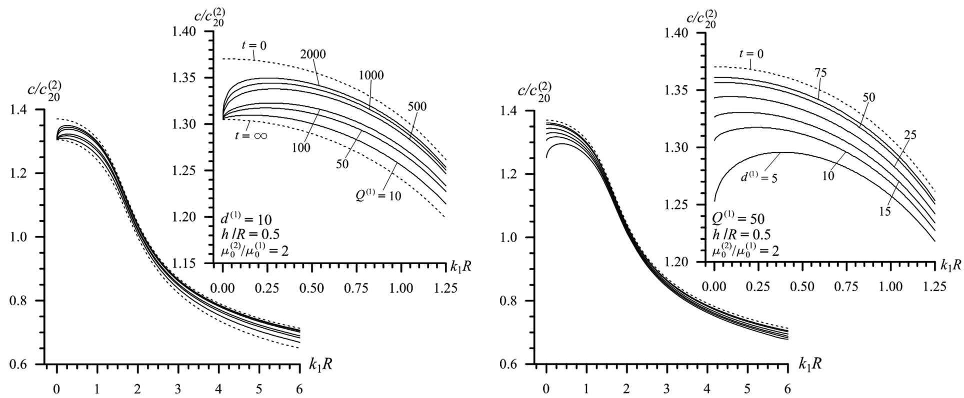
propagation velocity may be violated.



(a) (b)

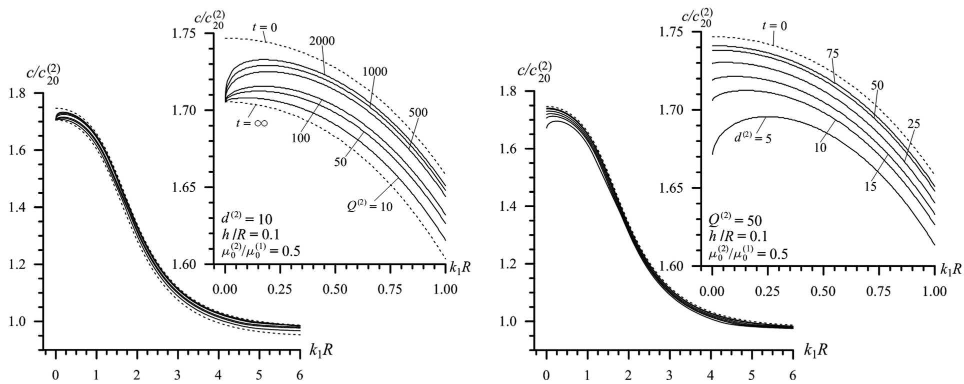
Figure 17: Dispersion curves obtained in the case considered in Fig. 16 under $h/R = 0.3$.

Thus, we consider the dispersion curves related to the E.V. case and given in Figs. 19, 20, 21 and 22 (Figs. 23, 24, 25 and 26) constructed for $\mu_0^{(2)}/\mu_0^{(1)} = 0.5$ ($\mu_0^{(2)}/\mu_0^{(1)} = 2$) in the cases where $h/R = 0.1$, 0.3, 0.5 and 0.7 ($h/R = 0.1$, 0.3, 0.7 and 1), respectively. Analysis of these results shows that for the relatively small values of h/R (for instance, under $h/R = 0.1$ in the case where $\mu_0^{(2)}/\mu_0^{(1)} = 0.5$ and under $h/R \leq 0.3$ in the case where $\mu_0^{(2)}/\mu_0^{(1)} = 2$) the upper and lower limitations of



(a) (b)

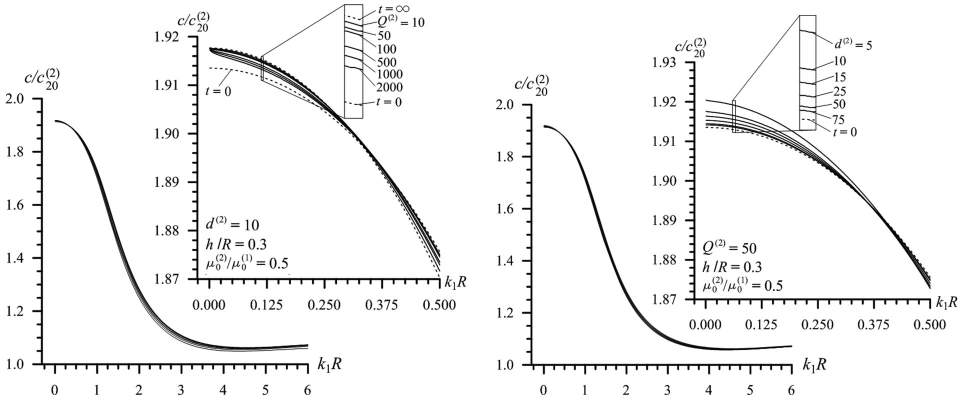
Figure 18: Dispersion curves obtained in the case considered in Fig. 16 under $h/R = 0.5$.



(a) (b)

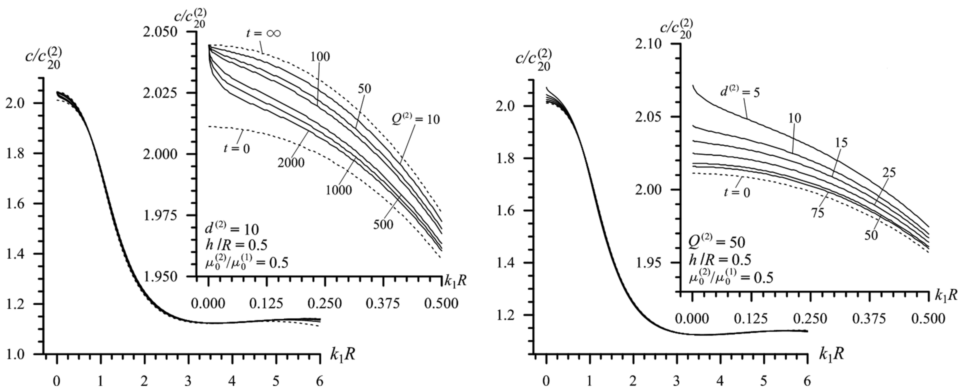
Figure 19: Dispersion curves obtained for various values of the parameter $Q^{(2)}$ under a fixed value of the parameter $d^{(2)}$ $(= 10)$ (a) and for various values of the parameter $d^{(2)}$ under a fixed value of the parameter $Q^{(2)}$ $(= 50)$ (b) in the E.V. case under $\mu_0^{(2)}/\mu_0^{(1)} = 0.5$ and $h/R = 0.1$.

the dispersion curves are the same type as in the V.V. and V.E. cases. However, for the relatively greater values of h/R (for instance, under $h/R \geq 0.3$ in the case where $\mu_0^{(2)}/\mu_0^{(1)} = 0.5$ and under $h/R \geq 0.7$ in the case where $\mu_0^{(2)}/\mu_0^{(1)} = 2$) the foregoing limitation of the dispersion curves is violated, i.e. in the cases where $0 < k_1R \leq 1.0$ the dispersion curves obtained for the E.V. case have an upper (lower) limit and this upper (lower) limit is the dispersion curve constructed for the purely elastic case with long-term (instantaneous) values of the elastic constants. Consequently,



(a) (b)

Figure 20: Dispersion curves obtained in the case considered in Fig. 19 under $h/R = 0.3$.

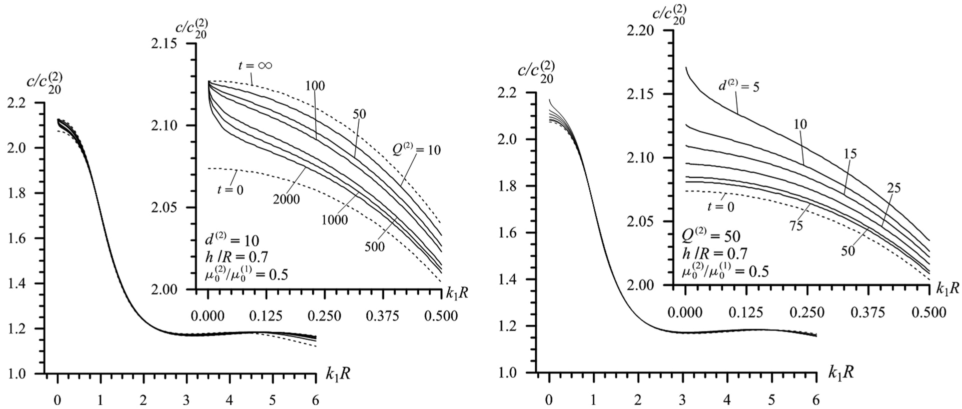


(a) (b)

Figure 21: Dispersion curves obtained in the case considered in Fig. 19 under $h/R = 0.5$.

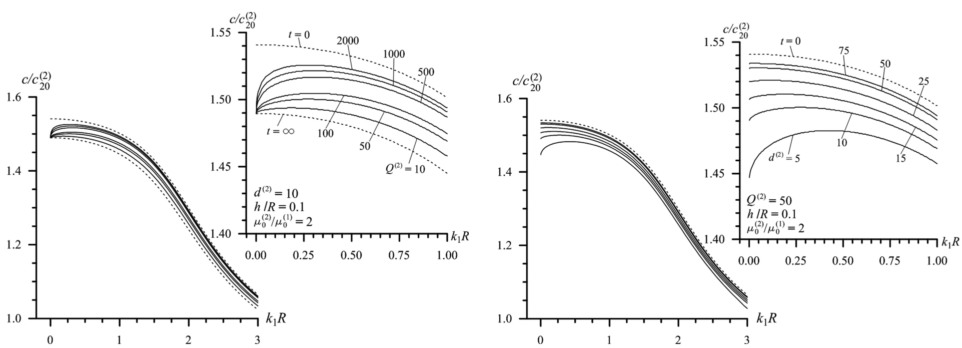
in the E.V. case under satisfaction of certain conditions, the viscoelasticity of the inner solid cylinder may increase the longitudinal axisymmetric wave propagation velocity in the compound solid cylinder under consideration. In these cases, the wave propagation velocity increases with a decrease in the parameters d and Q.

We recall that all the results discussed above relate to the dispersive attenuation case. Now we consider the dispersion curves obtained for the non-dispersive attenuation case and for this we select the V.E. case and assume that $k_2R = 0.005$, $h/R = 0.3$ and $\mu_0^{(2)}/\mu_0^{(1)} = 0.5$. These curves are given in Fig. 27 which are constructed for various values of the parameter $Q^{(1)}$ under a fixed $d^{(1)}$ $(= 10)$



(a) (b)

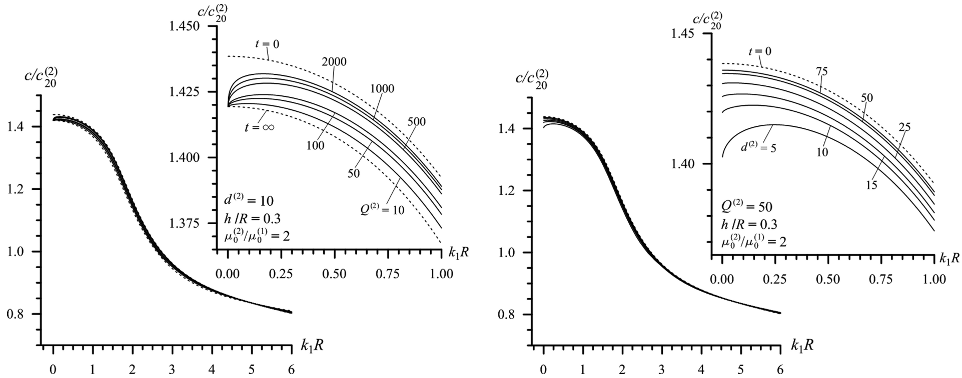
Figure 22: Dispersion curves obtained in the case considered in Fig. 19 under $h/R = 0.7$.



(a) (b)

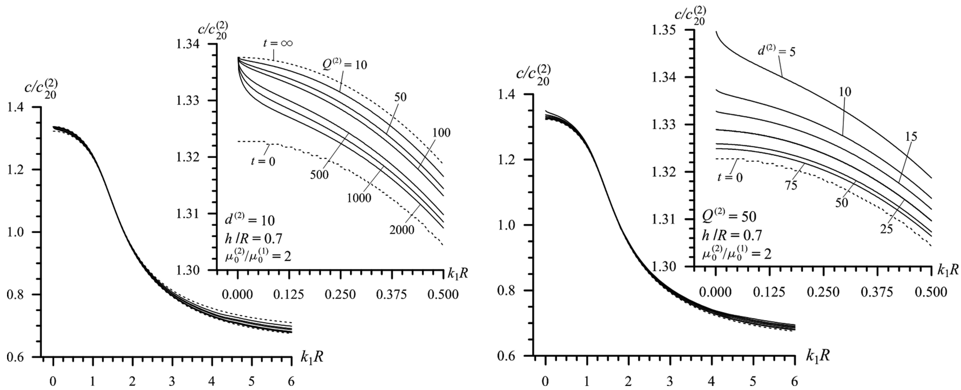
Figure 23: Dispersion curves obtained in the case considered in Fig. 19 under $\mu_0^{(2)}/\mu_0^{(1)} = 2$ and $h/R = 0.1$.

(Fig. 27(a)) and for various values of the parameter $d^{(1)}$ under a fixed $Q^{(1)}$ (= 50) (Fig. 27(b)). It follows from these results that, as in the dispersive attenuation case, in the non-dispersive attenuation case the viscoelasticity of the external cylinder material causes a decrease in the values of the wave propagation velocity. However, in the non-dispersive attenuation case, the cut off values of k_1R (denoted by $(k_1R)_{c.f.}$) (or cut off frequency (denoted by $\omega_{c.f.}$) determined by expression $\omega_{c.f.} = (k_1R)_{c.f.} \times c|_{k_1R=(k_1R)_{c.f.}}$) appear. The values of $(k_1R)_{c.f.}$ decrease (increase) with $d^{(1)}$ (with $Q^{(1)}$). Moreover, the values of $(k_1R)_{c.f.}$ depend on the values of k_2R. This dependence is illustrated with the graphs given in Fig. 28, according to which,



(a) (b)

Figure 24: Dispersion curves obtained in the case considered in Fig. 23 under $h/R = 0.3$.



(a) (b)

Figure 25: Dispersion curves obtained in the case considered in Fig. 23 under $h/R = 0.7$.

as can be predicted, the values of $(k_1R)_{c.f.}$ decrease with decreasing k_2R.

Consider also some results related to the second mode and for this purpose we again select the V.E. case and assume that the attenuation is dispersive and $h/R = 0.3$ and $\mu_0^{(2)}/\mu_0^{(1)} = 0.5$. For clarity of the illustration we consider the graphs of the dependence between $(c - c|_{t=\infty})/c_{20}$ and k_1R instead of the corresponding dispersion curves. Graphs of this dependence are given in Fig. 29 which are constructed for various values of the parameter $Q^{(1)}$ under fixed $d^{(1)}$ $(= 10)$ (Fig. 29(a)) and for various values of the parameter $d^{(1)}$ under fixed $Q^{(1)}$ $(= 50)$ (Fig. 29(b)). It follows from these graphs that the viscoelasticity of the outer hollow cylinder material of

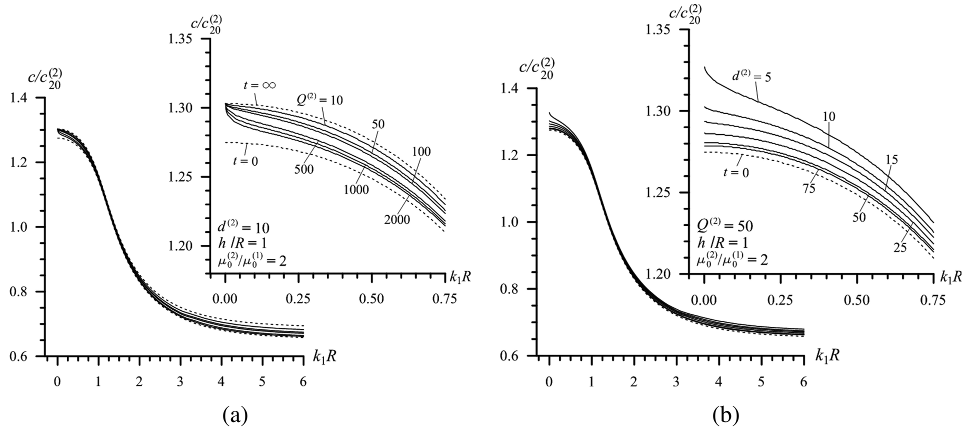


Figure 26: Dispersion curves obtained in the case considered in Fig. 23 under $h/R = 1.0$.

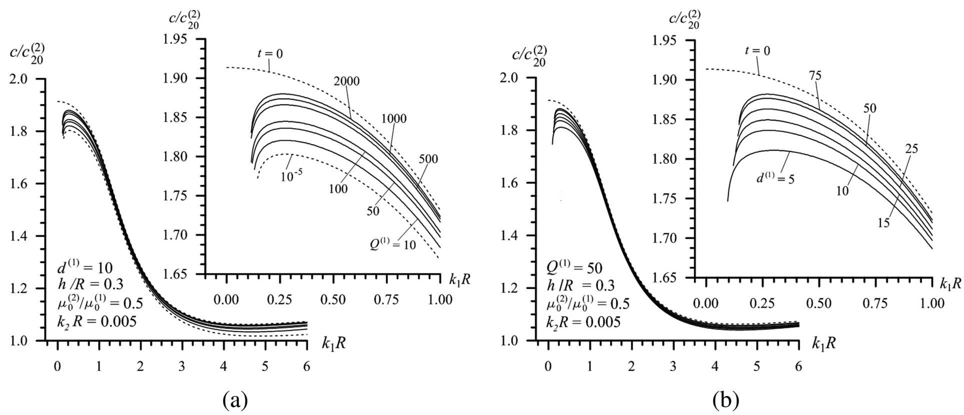


Figure 27: Dispersion curves obtained in the non-dispersive attenuation case under $k_2R = 0.005$ in the V.E. case for various values of the parameter $Q^{(1)}$ under a fixed value of the parameter $d^{(1)}$ $(= 10)$ (a) and for various values of the parameter d under a fixed value of the parameter Q $(= 50)$ (b) in the V.E. case under $\mu_0^{(2)}/\mu_0^{(1)} = 0.5$ and $h/R = 0.3$.

the compound solid cylinder also causes a decrease in the wave propagation velocity in the second mode. More careful study of the influence of the viscoelasticity of the cylinder's materials on the dispersion of the longitudinal axisymmetric wave propagation in the second and subsequent modes requires separate investigations which will be made in future investigations by the authors.

This completes the analysis of the numerical results considered in the present paper.

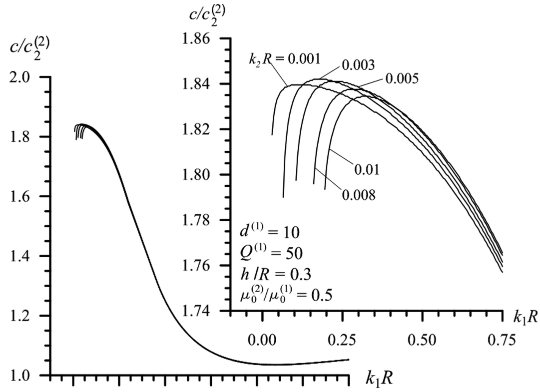
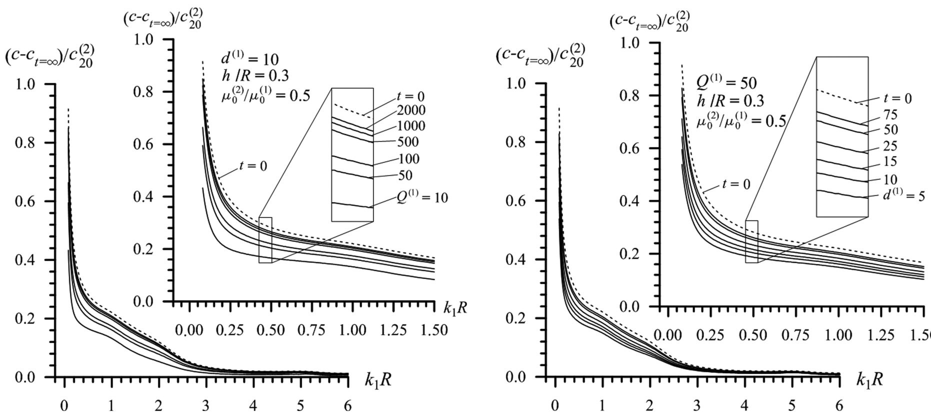


Figure 28: The influence of the "attenuation order" k_2R on the cut off values of k_1R, i.e. on the values of $(k_1R)_{c.f.}$ in the case considered in Fig. 27.



(a) (b)

Figure 29: The influence of the parameter $Q^{(1)}$ under a fixed value of the parameter $d^{(1)}$ ($= 10$) (a) and of the parameter $d^{(1)}$ under a fixed value of the parameter $Q^{(1)}$ ($= 50$) (b) on the wave propagation velocity in the second mode in the V.E. case under the dispersive attenuation case and under $\mu_0^{(2)}/\mu_0^{(1)} = 0.5$ and $h/R = 0.3$.

5 Conclusions

Thus, in the present paper, within the scope of the exact equations of motion of the theory of linear viscoelasticity, the axisymmetric longitudinal wave propagation in the bi-material circular compound cylinder made of viscoelastic materials has been investigated. The corresponding dispersion equation is obtained for arbitrary hereditary type viscoelastic operators. For concrete numerical investigations, the viscoelasticity of the materials is described by the fractional exponential operators by Rabotnov (1980). The dimensionless rheological parameters characterizing the characteristic creep time (denoted by Q) and the long-term values of the elastic constants (denoted by d) are introduced, and through these parameters the viscoelasticity of the constituents' materials on the dispersion curves is studied. An increase in the values of these parameters means a decrease in the viscosity properties of the related material. The main attention is focused on the results related to the dispersive attenuation case but nevertheless some examples related to the nondispersive attenuation case are also considered. In the dispersive attenuation case it is assumed that the attenuation coefficient is given in advance through the relation (45) or (46). The expressions for the low and high wavenumber limit values of the wave propagation velocity are derived. Related numerical results (dispersion curves) are presented and discussed for the case where the instantaneous shear modulus of the outer hollow cylinder material is twice that of the inner solid cylinder, as well as for the case where the instantaneous shear modulus of the inner solid cylinder material is twice that of the outer hollow cylinder. Moreover, the numerical results obtained for the case where the viscoelasticity properties of the cylinders' materials are the same (denoted by the "V.V. case"); for the case where the material of the outer cylinder is viscoelastic, but the material of the inner cylinder is purely elastic (denoted by the "V.E. case"); and for the case where the material of the outer cylinder material is purely elastic, but the material of the inner cylinder material is viscoelastic (denoted by the "E.V. case"), are considered separately. Numerical investigations are made mainly for the first (fundamental) lowest mode and some examples of the dispersion curves related to the second mode are also considered. According to these numerical results, we can draw the following main conclusions:

In the V.V. and V.E. cases in the considered attenuation dispersion case, the viscoelasticity of the constituents' materials of the cylinder causes the axisymmetric longitudinal wave propagation velocity to decrease. The magnitude of this decrease increases with a decrease in the aforementioned dimensionless rheological parameters;

The dispersion curves obtained for the V.V. and V.E. cases are limited by the dispersion curves obtained for the purely elastic case with instantaneous values of the elastic constants (upper limit) and by those obtained for the purely elastic case with long-term values of the elastic constants (lower limit);

A significant effect of the viscosity of the layers' materials on the wave propagation velocity appears in the case where $k_1 R \leq 1.5$;

For relatively small values of h/R the foregoing results also occur for the E.V. case. However, for relatively greater values of h/R in the E.V. case, the viscoelasticity of the inner cylinder material causes an increase in the values of the wave propagation velocity and the magnitude of the increase grows with a decrease in the rheological parameters related to the inner solid cylinder material;

For all the cases, the low (high) wavenumber limit values of the wave propagation velocity depends on the long-term (instantaneous) values of the elastic constants of the layers' material;

In the non-dispersive attenuation case, the cut off values of k_1R (denoted by $(k_1R)c.f.$) arise and the values of $(k_1R)c.f.$ increase (decrease) with the parameter d (with the parameter Q). Moreover, the values of $(k_1R)c.f.$ increase with the non-dispersive "attenuation order" k_2R;

In the V.E. case, under selected types of dispersive attenuation, the viscoelasticity of the constituents' materials of the cylinder also causes the wave propagation velocity of the second mode to decrease. Moreover, the viscoelasticity of the layers' materials causes the cut off values of k_1R for the second mode to increase.

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