Copyright © 2017 Tech Science Press CMC, vol.53, no.2, pp.129-156, 2017

Effect of Rotation on the Propagation of Waves in Hollow Poroelastic Circular Cylinder with Magnetic Field

A.M. Farhan[1,2]

Abstract：Employing Biot's theory of wave propagation in liquid saturated porous media, the effect of rotation and magnetic field on wave propagation in a hollow poroelastic circular of infinite extent are investigated. An exact closed form solution is presented. General frequency equations for propagation of poroelastic cylinder are obtained when the boundaries are stress free. The frequencies are calculated for poroelastic cylinder for different values of magnetic field and rotation. Numerical results are given and illustrated graphically. The results indicate that the effect of rotation, and magnetic field are very pronounced. Such a model would be useful in large-scale parametric studies of mechanical response.

Keywords: Wave propagation, rotation, magnetic field, poroelastic medium, natural frequency.

1 Introduction

The study of wave propagation over a continuous media is of practical importa nce in the field of engineering, medicine and bio-engineering. [Abd-Alla, et al. (2016)] investigated the reflection of Plane Waves from studied the electro-ma gneto-thermoelastic Half-space with a Dual-Phase-Lag Model. [Ahmed and Abd-Alla (2002)] studied the electromechanical wave propagation in a cylindrical po roelastic bone with cavity. [Abd-Alla, et al. (2011)] investigated the wave propa gation modeling in cylindrical human long wet bones with cavity. [Abd-Alla an d Abo-Dahab (2013)] discussed the effect of magnetic field on poroelastic bone model for internal remodeling. [Abo-Dahab, et al. (2014)] investigated the effe ct of rotation on wave propagation in hollow poroelastic circular cylinder. [Abd -Alla and Yahya (2013)] studied the wave propagation in a cylindrical human l ong wet bone [Biot (1955)] studied the theory of elasticity and consolidation f or a porous anisotropic solid. [Biot (1956)] studied the theory of propagation o f elastic waves in a fluid-saturated porous solid. [Brynk, et al. (2011)] investiga ted the experimental poromechanics of trabecular bone strength: role of Terzagh

[1]Physics Department, Faculty of Science Jazan University-K.S.A.

[2]Physics Department, Faculty of Science, Zagazig University, Zagazig,

i's effective stress and of tissue level stress fluctuations. [Cardoso and Cowin (2012)] discussed the role of structural anisotropy of biological tissues in poroelastic wave propagation. [Cui, et al. (1997)] studied the poroelastic solutions of an inclined borehole. Transactions. [Cowin (1999)] studied the bone poroelasticity. [El-Naggar, et al. (2001)] investigated the analytical solution of electro-mechanical wave propagation in long bones. [Gilbert, et al. (2012)] investigated a quantitative ultrasound model of the bone with blood as the interstitial fluid. [Love (1944)] studied a theoretical on the mathematical theorey of elasticity. [Matuszyk and Demkowicz (2014)] found the solution of coupled poroelastic/acoustic/elastic wave propagation problems using automatic hp-adaptivity . [Misra and Samanta (1984)] studied the wave propagation in tubular bones. [Mathieu, et al. (2012)] investigated the influence of healing time on the ultrasonic response of the bone-implant interface. [Marin, et al. (2015)] discussed the structural continuous dependence in micropolar porous bodies. [Marin (2010)] studied the harmonic vibrations in thermoelasticity of microstretch materials. [Marin, M. (1997)] found the weak solutions in elasticity of dipolar bodies with voids. [Morin and Hellmich (2014)] investigated a multiscale poro-micromechanical approach to wave propagation and attenuation in bone. [Nguyen, et al. (2010)] studied the poroelastic behaviour of cortical bone under harmonic axial loading: A finite element study at the osteonal scale. [Papathanasopoulou, et al. (2002)] investigated a poroelastic bone model for internal remodeling. [Potsika, et al. (2014)] discussed the application of an effective medium theory for modeling ultrasound wave propagation in healing long bones. [Qin, et al. (2005)] studied the thermoelectroelastic solutions for surface bone remodeling under axial and transverse loads. [Shah (2011)] investigated the flexural wave propagation in coated poroelastic cylinders with reference to fretting fatigue. [SHARMA and Marin. M. (2013)] investigated the effect of distinct conductive and thermodynamic temperatures on the reflection of plane waves in micropolar elastic half-space [Yoon and Katz (1976)] studied the ultrasonic wave propagation in human cortical bone—II. Measurements of elastic properties and microhardness. [Wen (2010)] studied the Meshless local Petrov–Galerkin (MLPG) method for wave propagation in 3D poroelastic solids.

In the present, the wave propagation in a cylindrical poroelastic medium with cavity is studied. The frequency equation for poroelastic medium is obtained. From measurements of the density, angular velocity, and bone thickness, the coefficients of the poroelastic medium may be evaluated. The frequencies are calculated for poroelastic medium is obtained for various values of rotation and magnetic field are given in graphs. The propagation of flexural waves in an infinite cylindrical element which is porous in nature is considered and numerical results are carried out. The results indicate that the effect of magnetic field and rotation are very pronounced. [Parnell, et al. (2012)] studied the analytical methods to determine the effective mesoscopic and macroscopic elastic properties of cortical bone.

2 Formulation of the problem

The stresses τ_{ij} and the liquid pressure τ [Shah, A. (2011)] are

$$\tau_{rr} = c_{11}\frac{\partial u_r}{\partial r} + c_{12}r^{-1}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right) + c_{13}\frac{\partial u_z}{\partial z} + M\left[\frac{\partial v_r}{\partial r} + r^{-1}\left(v_r + \frac{\partial v_\theta}{\partial \theta}\right) + \frac{\partial v_z}{\partial z}\right],$$

$$\tau_{\theta\theta} = c_{12}\frac{\partial u_r}{\partial r} + c_{11}r^{-1}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right) + c_{13}\frac{\partial u_z}{\partial z} + M\left[\frac{\partial v_r}{\partial r} + r^{-1}\left(v_r + \frac{\partial v_\theta}{\partial \theta}\right) + \frac{\partial v_z}{\partial z}\right],$$

$$\tau_{zz} = c_{13}\left[\frac{\partial u_r}{\partial r} + r^{-1}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right)\right] + c_{33}\frac{\partial u_z}{\partial z} + Q\left[\frac{\partial v_r}{\partial r} + r^{-1}\left(v_r + \frac{\partial v_\theta}{\partial \theta}\right) + \frac{\partial v_z}{\partial z}\right],$$

$$\tau_{rz} = c_{44}\left[\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right],$$

$$\tau_{r\theta} = c_{66}\left[\frac{\partial u_\theta}{\partial r} + r^{-1}\left(\frac{\partial u_r}{\partial \theta} - u_\theta\right)\right] \quad \tau_{\theta z} = c_{44}\left[\frac{\partial u_\theta}{\partial z} + r^{-1}\frac{\partial u_z}{\partial \theta}\right], \tag{1}$$

$$\tau = M\left[\frac{\partial u_r}{\partial r} + r^{-1}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right)\right] + Q\frac{\partial v_z}{\partial z} + R\left[\frac{\partial v_r}{\partial r} + r^{-1}\left(v_r + \frac{\partial v_\theta}{\partial \theta}\right) + \frac{\partial v_z}{\partial z}\right], \tag{2}$$

The magnetic stress is

$$\sigma_{rr} = \mu_e H_0^2\left(\frac{\partial u_r}{\partial r} + \frac{1}{r}u_r + \frac{\partial u_z}{\partial z}\right) \tag{2a}$$

where τ_{ij} is the average stress of solid, τ is the average stress of fluid per unit of mass, and σ_{rr} is the magnetic stress with elastic constants c_{ij}, M, Q, R and $c_{66} = \frac{1}{2}(c_{11} - c_{12})$.

The equation of the flow [Papathanasopoulou, et al. (2002)] is

$$b_{rr}^{-1}\nabla^2\tau + b_{zz}^{-1}\tau_{zz} = \frac{\partial(\varepsilon - \tau)}{\partial t}, \tag{3}$$

where $b_{rr} = \frac{(\mu f^2)}{k_{rr}}, b_{zz} = \frac{(\mu f^2)}{k_{zz}}$, ∇^2 is Laplacian operator in polar coordinates, μ is the viscosity, f is the porosity and k_{rr}, k_{zz} are the permeability of the medium. The average displacements of solid and velocity of fluid phases are taken as u_i and v_i, respectively.

The strains are expressed as

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{4}$$

and dilation of the phases as $e = u_{i,j}$ and $\varepsilon = v_{i,i}$.

In general, the stress-strain relation for a piezoelectric body can be written in the following way in matrix notation:

$$\tau_m = C_{mn}S_n - e_{mk}E_k, \quad 1 \le m, n \le 6, \ \ 1 \le k \le 3, \tag{5}$$

where e_{mk} and E_k are, respectively, the piezoelectric strain constants and the component of the electrical field.

The last term in Eq. (5) is ignored in Eq. (2) for simplifying the calculation. But this step can be justified by the results of [Yoon and Katz (1976)], who showed that the piezoelectric stiffening in bones in the ultrasonic wave propagation is negligibly small

The equations of motion are

$$\frac{\partial\tau_{rr}}{\partial r} + r^{-1}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\partial\tau_{rz}}{\partial z} + r^{-1}(\tau_{rr} - \tau_{\theta\theta}) + \mu_e H_0^2\left(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{1}{r^2}u_r + \frac{\partial^2 u_z}{\partial r\partial z}\right) = \rho\left(\frac{\partial^2 u_r}{\partial r^2} - \Omega^2 u_r\right)$$

$$\frac{\partial\tau_{r\theta}}{\partial r} + r^{-1}\frac{\partial\tau_{\theta\theta}}{\partial\theta} + \frac{\partial\tau_{\theta z}}{\partial z} + 2r^{-1}\tau_{\theta r} = \rho\frac{\partial^2 u_\theta}{\partial t^2}$$

$$\frac{\partial\tau_{rz}}{\partial r} + r^{-1}\frac{\partial\tau_{\theta z}}{\partial\theta} + \frac{\partial\tau_{zz}}{\partial z} + r^{-1}\tau_{rz} + \mu_e H_0^2\left(\frac{\partial^2 u_r}{\partial r\partial z} + \frac{\partial^2 u_z}{\partial z^2} - \frac{1}{r^2}\frac{\partial^2 u_z}{\partial\theta\partial z}\right) = \rho\left(\frac{\partial^2 u_r}{\partial r^2} - \Omega^2 u_z\right) \tag{6}$$

where, ρ is the density of the bone, $\vec{\Omega} = (0, \Omega, 0)$ is the rotation vector, H_0 is the magnetic field acts normal on the plane $(r - z)$ and t is the time.

Substituting from equations (1) into equations (6), we obtain

$$(c_{11}+\mu_e H_0^2)\left[\frac{\partial^2 u_r}{\partial r^2}+r^{-1}\frac{\partial u_r}{\partial r}-r^{-2}u_r\right]+r^{-2}\frac{1}{2}(c_{11}-c_{12})\frac{\partial^2 u_r}{\partial \theta^2}+(c_{44}$$

$$+\mu_e H_0^2)\frac{\partial^2 u_r}{\partial z^2}+r^{-1}\left(\frac{1}{2}(c_{12}+c_{11})\right)\frac{\partial^2 u_\theta}{\partial\theta\partial r}$$

$$-r^{-2}\left(\frac{3}{2}c_{11}-c_{12})\right)\frac{\partial u_\theta}{\partial\theta}$$

$$+\left(c_{13}+c_{44}+\mu_e H_0^2\right)\frac{\partial^2 u_z}{\partial r\partial z}$$

$$+M\left[\frac{\partial^2 v_r}{\partial r^2}+r^{-1}\frac{\partial v_r}{\partial r}-r^{-2}v_r-r^{-2}\frac{\partial^2 v_\theta}{\partial\theta^2}+r^{-1}\frac{\partial^2 v_\theta}{\partial\theta\partial r}+\frac{\partial^2 v_z}{\partial r\partial z}\right]$$

$$=\rho\left(\frac{\partial^2 u_r}{\partial t^2}-\Omega^2\right)$$

$$r^{-1}\left(\frac{1}{2}(c_{11}+c_{12})\right)\frac{\partial^2 u_r}{\partial\theta\partial r}+r^{-2}\left(\frac{3}{2}c_{11}-c_{12}\right)\frac{\partial u_r}{\partial\theta}$$

$$+\frac{1}{2}(c_{11}-c_{12})\left[\frac{\partial^2 u_\theta}{\partial r^2}+r^{-1}\frac{\partial u_\theta}{\partial r}-r^{-2}u_\theta\right]+r^{-2}c_{11}\frac{\partial^2 u_\theta}{\partial\theta^2}$$

$$+c_{44}\frac{\partial^2 u_\theta}{\partial z^2}+r^{-1}(c_{13}+c_{44})\frac{\partial^2 u_z}{\partial\theta\partial z}$$

$$+r^{-1}M\left[\frac{\partial^2 v_r}{\partial r\partial\theta}+r^{-1}\frac{\partial v_r}{\partial\theta}+r^{-1}\frac{\partial^2 v_\theta}{\partial\theta^2}+\frac{\partial^2 v_z}{\partial r\partial z}\right]=\rho\left(\frac{\partial^2 u_\theta}{\partial t^2}-\Omega^2\right),$$

$$\left(c_{13}+c_{44}+\mu_e H_0^2)\right)\frac{\partial^2 u_r}{\partial r\partial z}+r^{-1}\left(c_{13}+c_{44}+\mu_e H_0^2\right)\frac{\partial u_r}{\partial z}$$

$$+r^{-1}(c_{13}+c_{44})\frac{\partial^2 u_\theta}{\partial\theta\partial z}$$

$$+(c_{44}+\mu_e H_0^2)\left[\frac{\partial^2 u_r}{\partial r^2}+r^{-1}\frac{\partial u_z}{\partial r}-r^{-2}\frac{\partial^2 u_z}{\partial \theta^2}\right]+(c_{44}+\mu_e H_0^2)\frac{\partial^2 u_z}{\partial z^2}$$

$$+Q\left[\frac{\partial^2 v_r}{\partial r\partial z}+r^{-1}\frac{\partial v_r}{\partial z}+r^{-1}\frac{\partial^2 v_\theta}{\partial \theta\partial z}+\frac{\partial^2 v_z}{\partial z^2}\right]=\rho\left(\frac{\partial^2 u_z}{\partial t^2}-\Omega^2\right). \quad (7)$$

3 Solution of the problem

Let (r,θ,z) be the cylindrical polar coordinates. Consider a homogeneous, transversely isotropic, infinite hollow poroelastic cylinder with inner and outer radii a and b respectively having thickness $h=b-a$ whoso axis is in the direction of z ,

Let

$$u_r(r,\theta,z,t)=\left[\frac{\partial\Phi}{\partial r}+\frac{1}{r}\frac{\partial\Psi}{\partial\theta}\right]e^{i(kz-\omega t)}, \qquad v_r(r,\theta,z,t)=-\frac{\partial\eta}{\partial r}e^{i(kz-\omega t)},$$

$$u_\theta(r,\theta,z,t)=\left[\frac{1}{r}\frac{\partial\Phi}{\partial\theta}-\frac{\partial\Psi}{\partial r}\right]e^{i(kz-\omega t)}, \qquad v_\theta(r,\theta,z,t)=-\frac{1}{r}\frac{\partial\eta}{\partial\theta}e^{i(kz-\omega t)},$$

$$u_z(r,\theta,z,t)=\left[\frac{i}{h}w\right]e^{i(kz-\omega t)}, \qquad v_z(r,\theta,z,t)=-\mathrm{ik}\eta e^{i(kz-\omega t)}, \quad (8)$$

where, $u_r,u_\theta,u_z,v_r,v_\theta,v_z$ are mechanical displacements and velocities, ω is the angular frequency, k is the wave number and Φ,Ψ,w and η are functions of r,θ.

Substituting from Eqs. (1) into Eqs. (3), (6) and using Eqs. (7), the following equations are obtained:

$$\left((c_{11}+\mu_e H_0^2)\nabla^2+\rho\left(\omega^2-\Omega^2\right)-k^2(c_{44}-\mu_e H_0^2)\right)\Phi-\left(c_{13}+c_{44}+\mu_e H_0^2\right)\left(\frac{k\zeta}{h}\right)--M(\nabla^2-\mathrm{k}^2)\eta=0$$

$$\left((c_{44}+\mu_e H_0^2)\nabla^2+\rho\left(\omega^2-\Omega^2\right)-k^2 c_{33}\right)\left(\frac{\zeta}{h}\right)+\left(c_{13}+c_{44}+\mu_e H_0^2\right)k\nabla^2\Phi - kQ(\nabla^2-\mathrm{k}^2)\eta=0,$$

$$\left\{M\left[\frac{\nabla^4}{b_{rr}}-\frac{k^2\nabla^2}{b_{zz}}\right]+i\omega\nabla^2\right\}\Phi+\left\{\left(\frac{Q}{h}\right)\left[\frac{-k\nabla^2}{b_{rr}}+\frac{k^3}{b_{zz}}\right]-\frac{ik\omega}{h}\right\}\zeta$$
$$+\left\{R\left[\frac{-\nabla^2(\nabla^2-k^2)}{b_{rr}}+\frac{k^2(\nabla^2-k^2)}{b_{zz}}\right]i\omega(\nabla^2-k^2)\right\}\eta=0$$

$$\left(\frac{1}{2}\left(\left(c_{11}+\mu_e H_0^2\right)-c_{12}\right)\nabla^2+\rho\left(\omega^2-\Omega^2\right)-k^2(c_{44}-\mu_e H_0^2)\right)\psi=0 \quad (9)$$

By defining the dimensionless coordinate $x=\frac{r}{h}$ and $\varepsilon_1=kh$. the above equations are written in dimensionless parameter x and ε_1 as

$$\left((c_{11}+\mu_e H_0^2)\nabla^2+(ch)^2-\varepsilon 1^2\right)\Phi-\left(c_{13}+1\right)(\varepsilon 1w)-\bar{M}\xi=0,$$

$$\left(c_{13}+1\right)\varepsilon_1\nabla^2\Phi+\left(\nabla^2+(ch)^2-\varepsilon_1^2\, c_{33}\right)w-\varepsilon_1\bar{Q}\xi=0,$$

$$\nabla^2\left(\nabla^2-b\,\varepsilon_1^2+iD\right)\Phi-\varepsilon_1\left(Q'\,\nabla^2+bQ'\varepsilon_1^2-iD\right)\zeta+(-R'\nabla^2+\quad bR'\,\varepsilon_1^2$$
$$+iD)\xi=0,$$
$$\left(C_{66}\nabla^2+(ch)^2-\varepsilon_1^2\right)\Psi=0, \quad (10)$$

where

$$\nabla^2=\left(\frac{\partial^2}{\partial x^2}+\frac{1}{x}\frac{\partial}{\partial x}+\frac{1}{x^2}\frac{\partial^2}{\partial\theta^2}\right),$$

$$\xi=(\nabla^2-k^2)\eta,\quad D=\frac{\omega h^2 b_{rr}}{M},\quad Q'=\frac{Q}{M},\quad R'=\frac{R}{M},\quad \bar{M}=\frac{M}{c_{44}+\mu_e H_0^2},$$

$$\bar{Q}=\frac{Q}{c_{44}+\mu_e H_0^2},\quad c^2=\frac{\rho\left(\omega^2-\Omega^2\right)}{c_{44}+\mu_e H_0^2},\quad b=\frac{b_{rr}}{b_{zz}},\quad \bar{c}_{ij}=\frac{c_{ij}}{c_{44}}\ (i,j=1,2,3),$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}. \tag{12}$$

The reason for ξ being defined as above and not being solved for variable η is that the flow of fluid through the boundaries of bone does not take place during the study of the propagation of waves. However, η can be calculated if the flow on the boundaries are prescribed.

We can write the Eq. (10) in the determinant form:

$$\begin{vmatrix} ((c_{11}+\mu_e H_0^2)\nabla^2 + A) & -\overline{B} & -\overline{M} \\ B\nabla^2 & (\nabla^2 + C) & -\overline{Q}\varepsilon_1 \\ T_1 & T_2 & T_3 \end{vmatrix} (\phi, w, \xi) = 0 \tag{13}$$

where

$$T_1 = \nabla^2\left(\nabla^2 - b\,\varepsilon_1^2 + iD\right), \qquad T_2 = -\varepsilon_1\left(Q'\nabla^2 + bQ'\,\varepsilon_1^2 - iD\right),$$

$$T_3 = \left(-R'\nabla^2 + bR'\,\varepsilon_1^2 + iD\right), \quad A = (ch)^2 - \varepsilon_1^2,$$

$$\bar{B} = \left(1 + \bar{c}_{13}\right)\varepsilon_1 \qquad and \qquad C = (ch)^2 - \varepsilon_1^2\,\bar{c}_{13}.$$

Evaluating the determinant form, the following equations are obtained:

$$(\nabla^6 + P\nabla^4 + G\nabla^2 + H)(\Phi, \zeta, \xi) = 0, \tag{14}$$

where

$$\mathrm{P} = \Big(R'\,\varepsilon_1^2\,(\bar{c}_{11} + \mu_e H_0^2) + \mathrm{iD}(\bar{c}_{11} + \mu_e H_0^2) + \mathrm{C}R'\,(\bar{c}_{11} + \mu_e H_0^2) -$$

$$\overline{Q}\;Q'\,\varepsilon_1^2\,(\bar{c}_{11} + \mu_e H_0^2) + \mathrm{A}R' - -B^2\,R' + \mathrm{B}Q'\,\varepsilon_1 + \overline{M}\;\varepsilon_1^2 - \overline{Q}\varepsilon_1 B$$

$$+ i\overline{M}D + \overline{M}D\Big)/(-R'\,(\bar{c}_{11} + \mu_e H_0^2) + \bar{M}),$$

$$G = \Big(\mathrm{C} R' \varepsilon_1^2 (\bar{c}_{11} + \mu_e H_0^2) + i\,\mathrm{DC} (\bar{c}_{11} + \mu_e H_0^2) + Q' \varepsilon_1^4 \bar{\mathrm{Q}} (\bar{c}_{11} + \mu_e H_0^2)$$

$$- \mathrm{iD}\bar{\mathrm{Q}} \varepsilon_1^2 (\bar{c}_{11} + \mu_e H_0^2) + \mathrm{A} R' \varepsilon_1^2 + \mathrm{iDA} + \mathrm{A} R' \mathrm{C} - \mathrm{A} Q' \bar{Q} \varepsilon_1^2$$

$$+ \mathrm{BD} \varepsilon_1^3 - \mathrm{iD} \varepsilon_1 + C \varepsilon_1^2 - B \bar{Q} \varepsilon_1^3 + iDB \varepsilon 1 \bar{M} + iDC \bar{M} - \bar{M} C \varepsilon_1^2 \Big)$$

$$/(-\grave{R}' (\bar{c}_{11} + \mu_e H_0^2) + \bar{M})$$

$$\mathrm{H} = \frac{A\Big(\mathrm{C} R' \varepsilon_1^2 + iDC + \bar{Q} Q' \varepsilon_1^4 - \mathrm{iD} \bar{Q} \; \varepsilon 1^2 \Big)}{-R' (\bar{c}_{11} + \mu_e H_0^2) + \bar{M}}. \tag{15a}$$

The general solutions of equation (14) can be obtained by using Mathematica program in terms of the Bessel functions of the first and second kind J and Y respectively as

$$\Phi = \sum_{i=1}^{3} [A_i J_n(\alpha_i x) + B_i Y_n(\alpha_i x)] \cos(n\theta),$$

$$\zeta = \textstyle\sum_{i=1}^{3} d_i [A_i J_n(\alpha_i x) + B_i Y_n(\alpha_i x)] \cos(n\theta), \tag{15b}$$

$$\xi = \sum_{i=1}^{3} e_i [A_i J_n(\alpha_i x) + B_i Y_n(\alpha_i x)] \cos(n\theta),$$

where α_i^2 are the non-zero roots of the equation

$$\alpha^6 - P\alpha^4 + G\alpha^2 - H = 0, \tag{16a}$$

The roots of the equation (16) by using Mathematica program are

$$\alpha_1^2 = \frac{P}{3} - \frac{\sqrt[3]{2}(3G - P^2)}{3F1} + \frac{F1}{3\sqrt{32}},$$
$$\alpha_2^2 = \frac{P}{3} + \frac{(1+i\sqrt{3})(3G - P^2)}{\sqrt[3]{32}F1} - \frac{(1-i\sqrt{3})F1}{\sqrt[3]{62}}, \quad (16b)$$
$$\alpha_3^2 = \frac{P}{3} + \frac{(1-i\sqrt{3})(3G - P^2)}{\sqrt[3]{1024}F1} - \frac{(1-i\sqrt{3})F1}{\sqrt[3]{62}}.$$

where

$$F1 = (27H - 9GP + 2P^3 + 3\sqrt{3}\sqrt{4G^3 + 27H^2 - 18GHP + 4HP^3})^{\frac{1}{3}}$$

Solving equations (17) we obtain d_i and e_i

$$\left(\left(1 + \bar{c}_{13}\right)\varepsilon_1 d_i + \bar{M}e_i = \left(\bar{c}_{11} + \mu_e H_0^2\right)\alpha_i^2 - (ch)^2 - \varepsilon_1^2\right),$$

$$\left(-\alpha_i^2 + (ch)^2 - \varepsilon_1^2\,\bar{c}_{33}\right)d_i - \bar{Q}\varepsilon 1 e_i = \left(1 + \bar{c}_{13}\right)\varepsilon_1\alpha_i^2. \quad (17)$$

Solving Eq. (11) we have

$$\Psi = [A_4 J_n(\alpha_4 x) + B_4 Y_n(\alpha_4 x)]\sin(n\theta), \quad (18)$$

where

$$\alpha_4^2 = \frac{2\left((ch)^2 - \varepsilon_1^2\right)}{\left(\left(\bar{c}_{11} + \mu_e H_0^2\right) - \bar{c}_{13}\right)}$$

4 Frequency equation

The boundary conditions for traction free inner and outer surfaces of the hollow poroelastic cylinder are

$$\tau_{rr} + \sigma_{rr} = \tau_{rz} = \tau_{r\theta} = \tau = 0 \quad at\ r = \bar{a},$$

$$\tau_{rr} + \sigma_{rr} = \tau_{rz} = \tau_{r\theta} = \tau = 0 \quad at\ r = \bar{b} \quad (19)$$

where

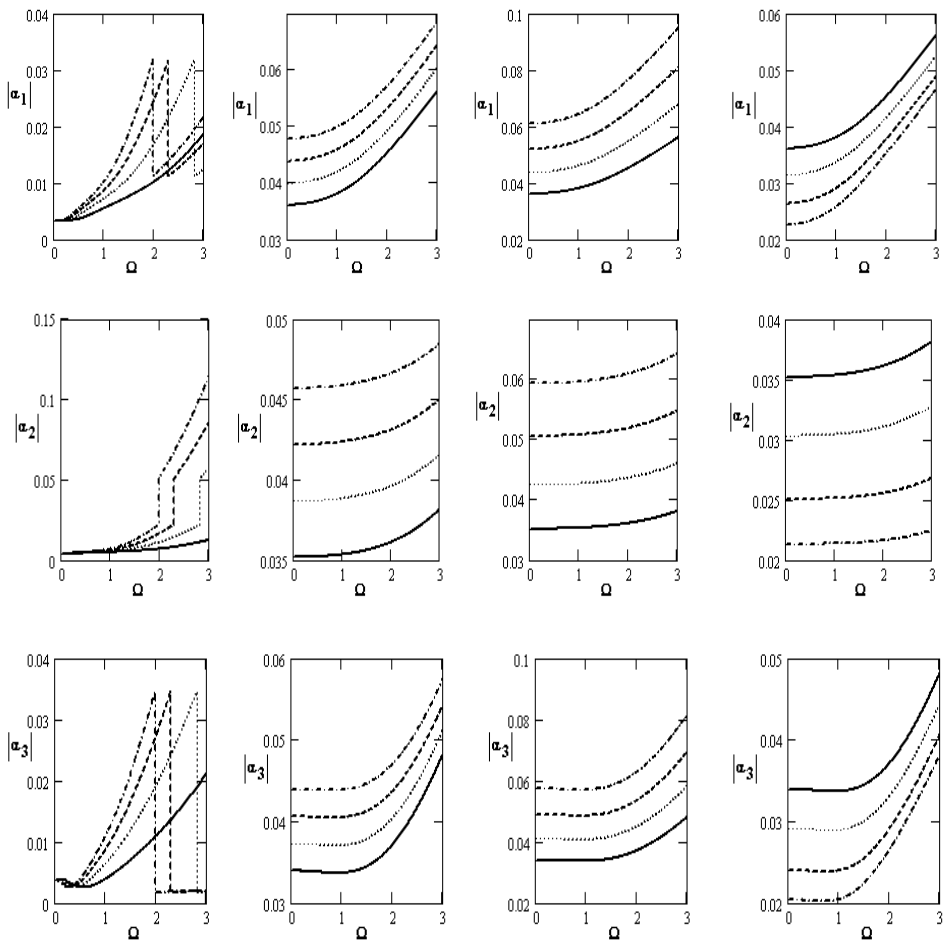
$\bar{a} = \frac{a}{h}, \quad \bar{b} = \frac{b}{h}$

Substituting from Eqs. (8), (15) and (18) into Eq. (19) and grouping the coefficients of $A_1, B_1,\ A_2, B_2, A_3,\ B_3$ and A_4, B_4 leads to a determinant which is the characteristic frequency equation:

$$|a_{ij}| = 0 \quad (i, j = 1,2,3, \ldots ,8), \tag{20}$$

where, the coefficients of a_{ij} are take the form in Appendix A in the paper end.

Equation (20) is called the characteristic frequency equation. The element a_{ij} is analytically expressed in terms of the elastic constants of the material. Eq. (20) is a transcendental equation of the frequency and wave number. The roots of Eq. (20) provide the dispersion curves of the guided modes. i.e. the wave number as a function of frequency



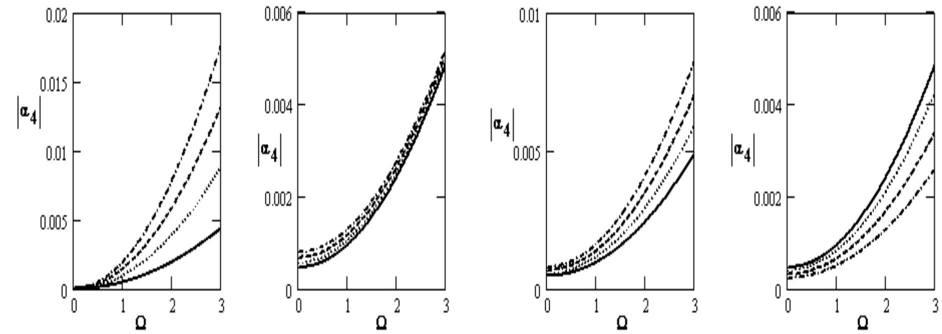
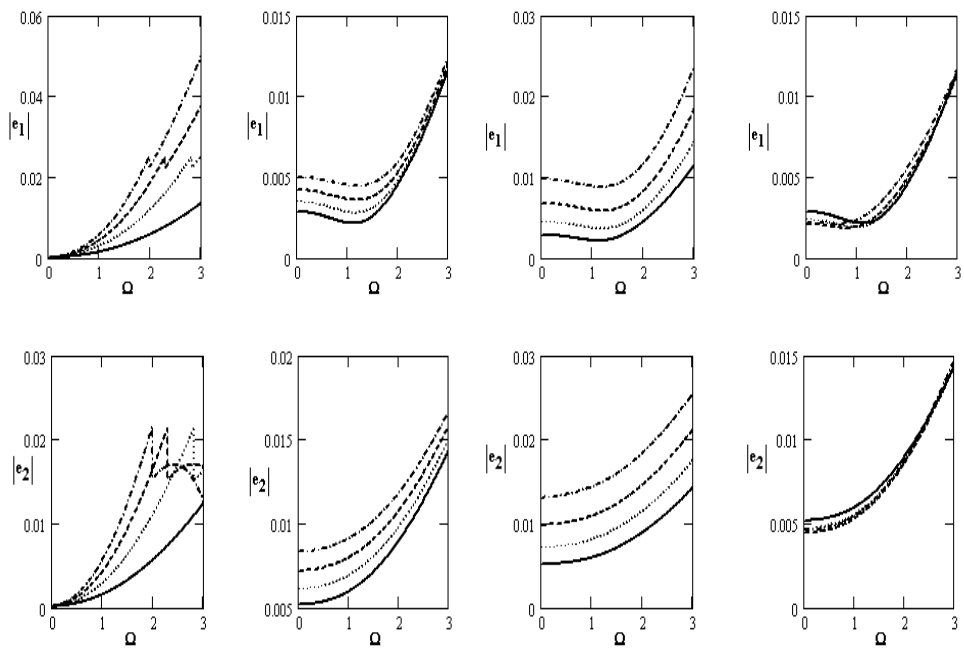


Figure 1: Variations of the roots $\left|\alpha_j\right|\ (j=1,2,3,4)$ with respect to the rotation Ω with the variation of ρ,ω,h and H

$\rho = 0.1$ _, 0.2 .., 0.3 – –, 0.4 _.

$\omega = 1$ _, 1.1 .., 1.2 – –, 1.3 _. $h = 0.1$ _, 0.11 .., 0.12 – –, 0.13

$H = 0.1$ __, 0.3 ..., 0.5 – –, 0.7 _._



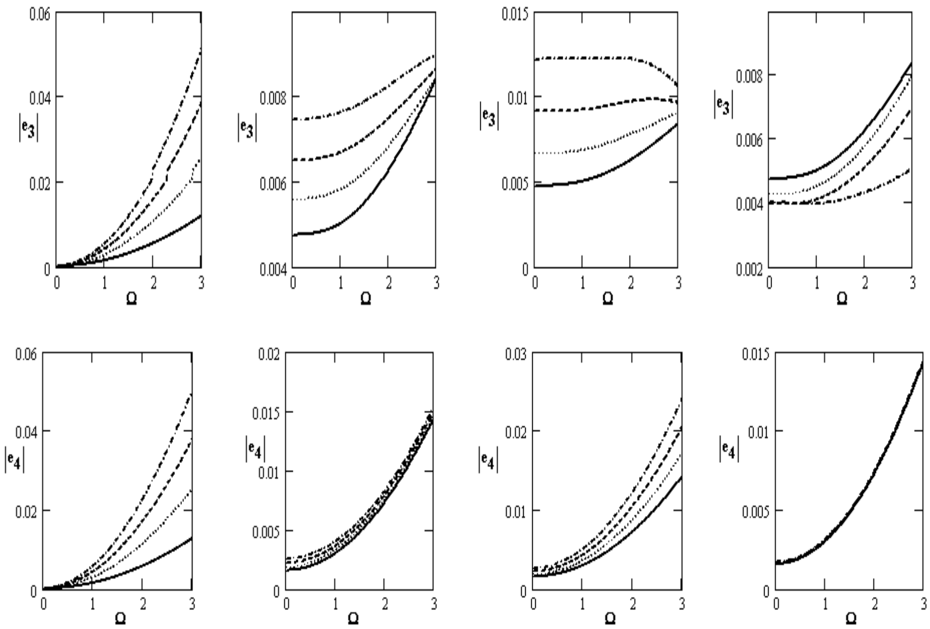
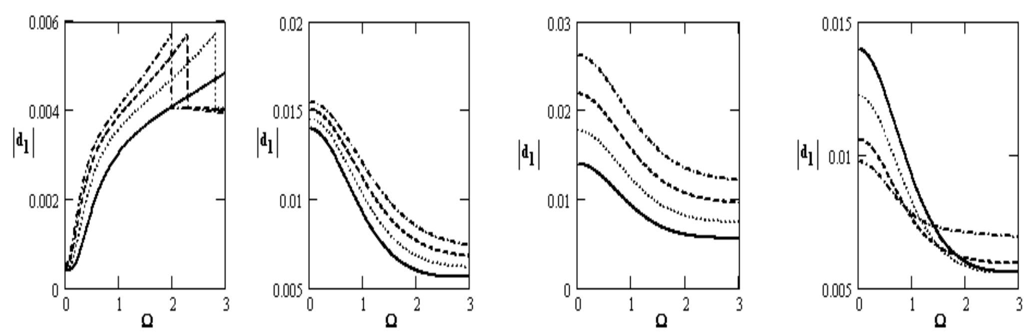


Figure 2: Variations of $\left|e_j\right|\ (j=1,2,3,4)$ with respect to the rotation Ω with the variation of ρ, ω, h and H

$\rho = 0.1$_, 0.2.., 0.3−−, 0.4_.

$\omega = 1$_, 1.1.., 1.2−−, 1.3_. $h = 0.1$_, 0.11.., 0.12−−, 0.13

$H = 0.1$_, 0.3.., 0.5−−, 0.7_.



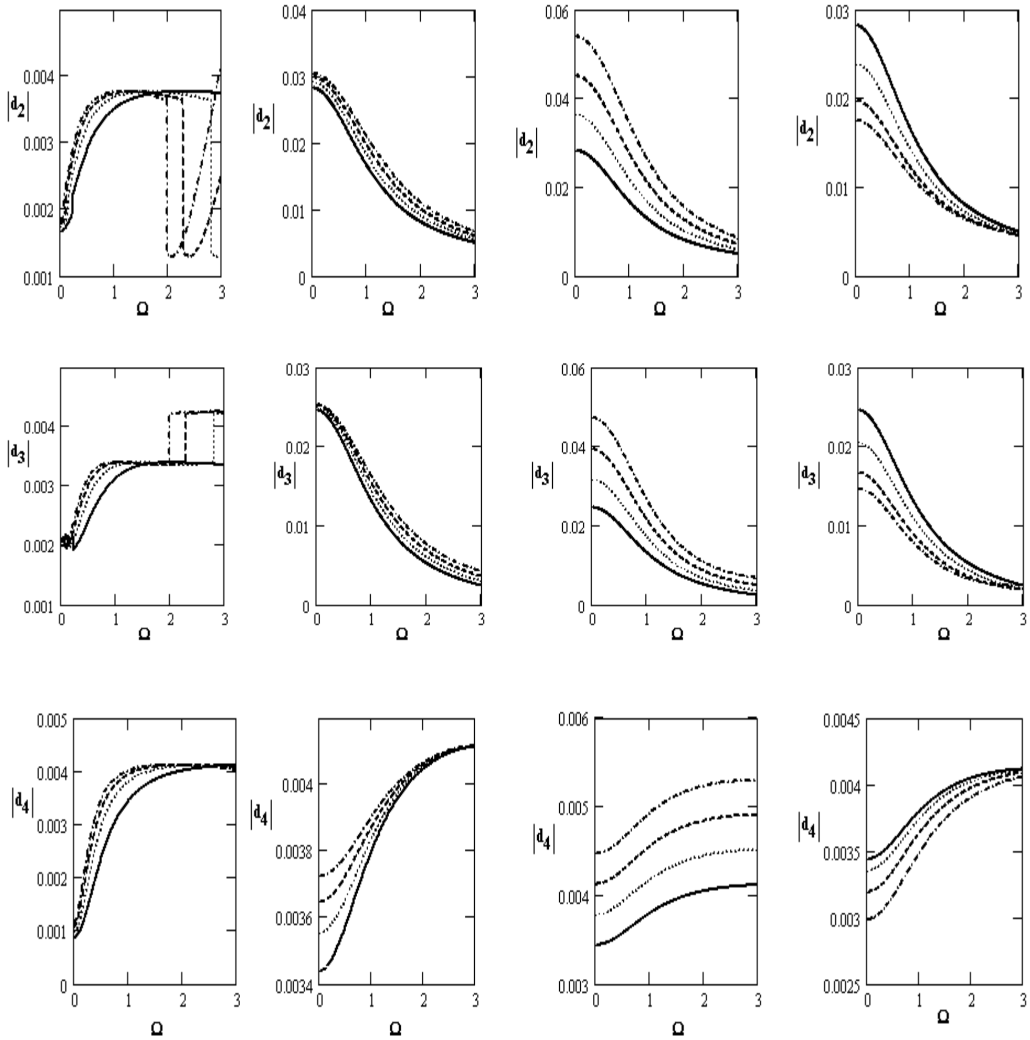


Figure 3: Variations of $\left|d_j\right|\ (j=1,2,3,4)$ with respect to the rotation Ω with the variation of ρ, ω, h and H

$\rho = 0.1$ _, 0.2 .., 0.3 − −, 0.4 _.

$\omega = 1$ _, 1.1 .., 1.2 − −, 1.3 _. $h = 0.1$ _, 0.11 .., 0.12 − −, 0.13
$H = 0.1$ __, 0.3 ..., 0.5 − −, 0.7 _ . _

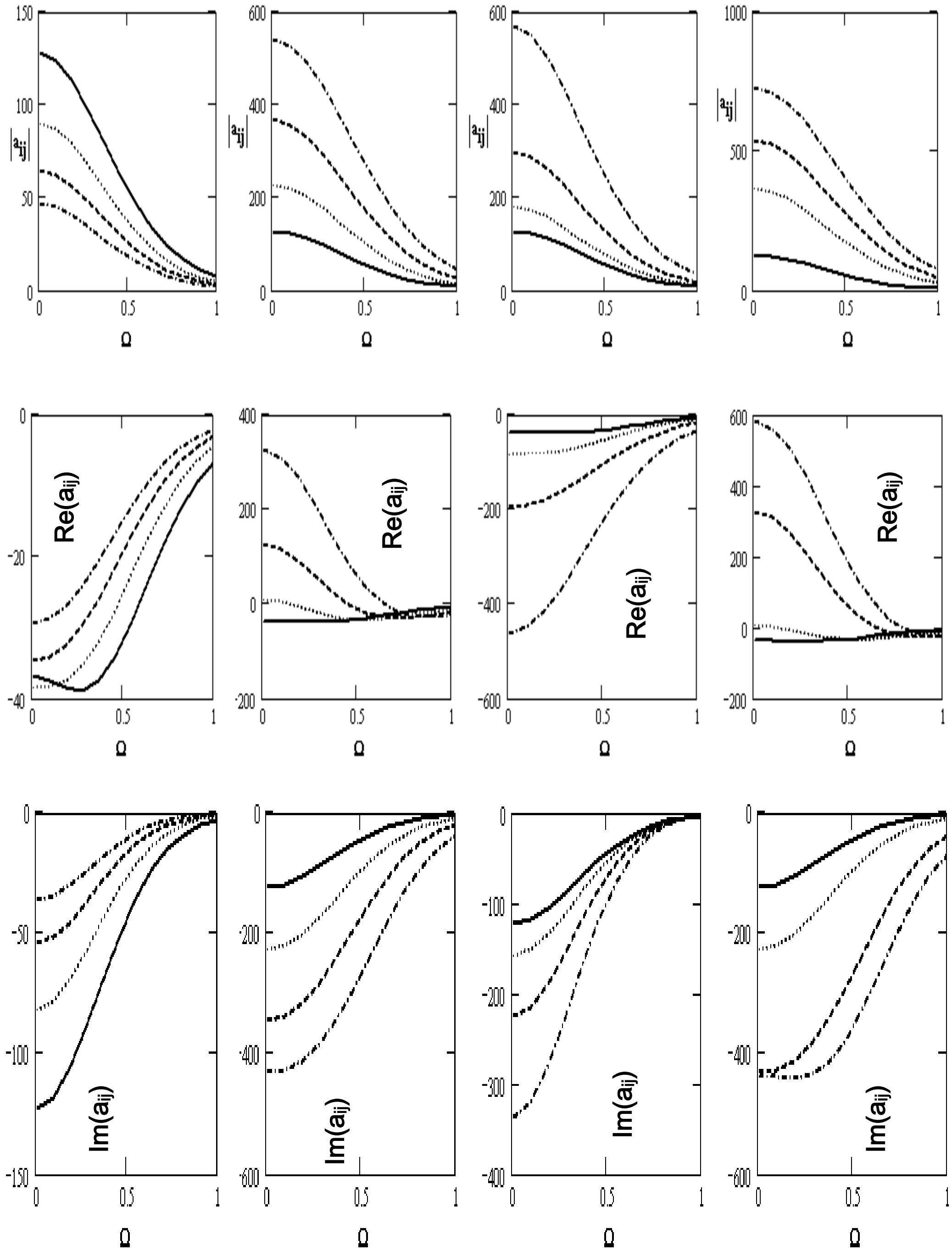


Figure 4: Variations of the determinant $\left|a_{ij}\right|, \mathrm{Re}(a_{ij}), \mathrm{Im}(a_{ij})\ (i, j = 1, 2, 3, 4)$ with respect to the rotation Ω with the variation of ρ, ω, h and H

$\rho = 0.1\_, 0.2.., 0.3--, 0.4\_.$

$\omega = 1\_, 1.1.., 1.2--, 1.3\_. \; h = 0.1\_, 0.11.., 0.12--, 0.13$

$H = 0.1\_\_, 0.3..., 0.5--, 0.7\_.\_$

5 Numerical results and discussion

The numerical results for the frequency equation are computed for the wet bone. Since the frequency equation is transcendental in nature, there are an infinite number of roots for the frequency equation. The results are evaluated in the range $0 < \varepsilon 1 < 4$ and $0 < ch < 4$ with the ratio of $\frac{b}{a} = 3$ and the thickness $b - a = h$. The values of the elastic constant of the bone are taken from [5] and the poroelastic constant is evaluated from the expression given by

$$Q = \frac{f(1 - f - \frac{\delta}{\chi})}{(\gamma + \delta + \frac{\delta^2}{\chi})}, \qquad R = \frac{f^2}{(\gamma + \delta + \frac{\delta^2}{\chi})},$$

where f is the porosity and γ, δ, χ are related by Young's modulus and the Poisson ratio, The expression for γ, δ, χ are given by

$$\chi = \frac{3(1 - 2\nu)}{E}, \quad \delta = 0.6\chi \quad and \quad \gamma = f(c - \delta)$$

where c is taken to be zero for the incompressibility for the fluid.

The porosity of the human bone in the age group 35-40 years is taken to be 0.24 [7]. To evaluate one more poroelastic constant it is assumed that $\frac{M}{Q} = \frac{c_{12}}{c_{13}}$ as the value M is not provided. Since the fluid in general is isotropic, it is taken that $b_{rr} = b_{zz}$. The density of the fluid in the porospace, permeability of the medium and mass density of the bone are taken from [17].

Table 1: The approximate geometry of the femur and the material constants which are used in the computations.

c_{11}	c_{12}	c_{13}	c_{33}	c_{44}	a	b
2.12	0.95	1.02	3.76	0.75	0.8	1.4

Fig.1 shows that the variations of the absolute value of the coefficients of $|\alpha_1|,|\alpha_2|,|\alpha_3|$ and $|\alpha_4|$ for poroelastic cylinder with respect to the rotation Ω, which it increases with increasing of rotation for different values of density ρ, the frequency ω, thickness h and magnetic field H_0, as well it increases with increasing of the density, frequency and magnetic field, except when effect the density, the coefficients of $|\alpha_1|,|\alpha_3|$ increase and decrease with increasing of the density.

Fig. 2 show that the variations of the coefficients of $|d_1|$, $|d_2|$, $|d_3|$ and $|d_4|$ for poroelastic cylinder with respect to the rotation Ω, which increases with increasing of rotation for different values of the frequency ω, thickness h and magnetic field H_0 except when effect the density it increases and decreases, as well it decreases with increasing of frequency, thickness and magnetic field except the coefficient $|d_4|$ increases with increasing of density, frequency and thickness, while the coefficients decrease with increasing of magnetic field.

Fig. 3 shows that the variations of the absolute of coefficients for poroelastic cylinder of $|e_1|,|e_2|,|e_3|$ and $|e_4|$ with respect to the rotation Ω, which it increases with increasing of the rotation for different values of density ρ, frequency ω, thickness and magnetic field H_0, while it increases with increasing of density, frequency and thickness except when effect of magnetic field, it has oscillatory behavior in the whole range of the Ω-axis for different values of magnetic field.

Fig. 4 shows that the variations of the scalar equation $|a_{ij}|$, wave velocity $\mathrm{Re}(|a_{ij}|)$ and attenuation coefficient $\mathrm{Im}(|a_{ij}|)$ with respect to the rotation Ω, for different values of density ρ, frequency ω, thickness h and magnetic field H_0 which it decreases with increasing of rotation. It is observed that the scalar equation increases with increasing of frequency, thickness, and magnetic field, while it decreases with increasing of density, wave velocity increases with increasing of density and rotation, as well it increases with increasing of frequency and magnetic field, while it decreases with increasing of rotation, as well it decreases with increasing of thickness and attenuation coefficient decreases with increasing of frequency, thickness, magnetic field and rotation, while it increases with increasing of rotation and density.

6 Conclusion

The investigation of propagation of wave in hollow poroelastic circular cylinder of infinite extent has led to the following conclusion:

(i) The frequency equation of free vibrations is independent of the nature of surface, rotation, magnetic field and presence of fluid in poroelastic media.

(ii) By comparing figures 1–4, it was found that the frequency equation, wave velocity, and attenuation coefficient have the same behavior in both media; but, with the passage of rotation, magnetic field, density, frequency and thickness, numerical values of frequency in the poroelastic cylinder are large in comparison due to the influences of rotation and magnetic field.

(iii) The frequency equation is obtained by considering the material as transversely isotropic in nature.

(iv)The results presented in this paper should prove to be useful for researchers in material science and designers of new materials and bones.

Appendix A:

$$a_{11} = \left[(\bar{c}_{11} + \mu_e H_0^2) \left(\left(\frac{n^2}{\alpha_1^2 \bar{a}^2} - \frac{n}{\alpha_1 \bar{a}} - 1 \right) J_n(\alpha_1 \bar{a}) + (\frac{1}{\alpha_1 \bar{a}}) J_{n+1}(\alpha_1 \bar{a}) \right) \right.$$

$$+ (\bar{c}_{12} + \mu_e H_0^2) \left(\left(\frac{n}{\alpha_1 \bar{a}} - \frac{n^2}{\bar{a}^2} \right) J_n(\alpha_1 \bar{a}) - \frac{1}{\bar{a}} J_{n+1}(\alpha_1 \bar{a}) \right)$$

$$- (\bar{c}_{13} + \mu_e H_0^2)(\varepsilon 1) d_1 J_n(\alpha_1 \bar{a})$$

$$\left. + \overline{M} \left(\left(\frac{n^2}{\alpha_1^2 \bar{a}^2} - \frac{n^2}{\bar{a}^2} - 1 \right) J_n(\alpha_1 \bar{a}) + (\frac{1}{\alpha_1 \bar{a}} - \frac{1}{\bar{a}}) J_{n+1}(\alpha_1 \bar{a}) \right) \right],$$

$$a_{12} = \Bigg[(\bar{c}_{11} + \mu_e H_0^2) \left(\left(\frac{n^2}{\alpha_2^2 \bar{a}^2} - \frac{n}{\alpha_2 \bar{a}} - 1 \right) J_n(\alpha_2 \bar{a}) + \left(\frac{1}{\alpha_2 \bar{a}}\right) J_{n+1}(\alpha_2 \bar{a}) \right)$$

$$+ (\bar{c}_{12} + \mu_e H_0^2) \left(\left(\frac{n}{\alpha_2 \bar{a}} - \frac{n^2}{\bar{a}^2} \right) J_n(\alpha_2 \bar{a}) - \frac{1}{\bar{a}} J_{n+1}(\alpha_2 \bar{a}) \right)$$

$$- (\bar{c}_{13} + \mu_e H_0^2)(\varepsilon) d_2 J_n(\alpha_2 \bar{a})$$

$$+ \overline{M} \left(\left(\frac{n^2}{\alpha_2^2 \bar{a}^2} - \frac{n^2}{\bar{a}^2} - 1 \right) J_n(\alpha_2 \bar{a}) \right.$$

$$\left. + \left(\frac{1}{\alpha_2 \bar{a}} - \frac{1}{\bar{a}} \right) J_{n+1}(\alpha_2 \bar{a}) \right) \Bigg],$$

$$a_{13} = \Bigg[(\bar{c}_{11} + \mu_e H_0^2) \left(\left(\frac{n^2}{\alpha_3^2 \bar{a}^2} - \frac{n}{\alpha_3 \bar{a}} - 1 \right) J_n(\alpha_3 \bar{a}) + \left(\frac{1}{\alpha_3 \bar{a}}\right) J_{n+1}(\alpha_3 \bar{a}) \right) +$$

$$(\bar{c}_{12} + \mu_e H_0^2) \left(\left(\frac{n}{\alpha_3 \bar{a}} - \frac{n^2}{\bar{a}^2} \right) J_n(\alpha_3 \bar{a}) - \frac{1}{\bar{a}} J_{n+1}(\alpha_3 \bar{a}) \right) - (\bar{c}_{13} + \mu_e H_0^2)$$

$$(\varepsilon 1) d_3 J_n(\alpha_3 \bar{a}) + \overline{M} \left(\left(\frac{n^2}{\alpha_3^2 \bar{a}^2} - \frac{n^2}{\bar{a}^2} - 1 \right) J_n(\alpha_3 \bar{a}) + \left(\frac{1}{\alpha_3 \bar{a}} - \frac{1}{\bar{a}} \right) J_{n+1}(\alpha_3 \bar{a}) \right) \Bigg],$$

$$a_{14} = \left(\frac{n}{\bar{a}}\right) (\bar{c}_{11} - \bar{c}_{12}) \left[- \alpha_4 J_{n+1}(\alpha_4 \bar{a}) + \frac{(n-1)}{\bar{a}} J_n(\alpha_4 \bar{a}) \right],$$

$$a_{15} = \Bigg[(\bar{c}_{11} + \mu_e H_0^2) \left(\left(\frac{n^2}{\alpha_1^2 \bar{a}^2} - \frac{n}{\alpha_1 \bar{a}} - 1 \right) Y_n(\alpha_1 \bar{a}) + \left(\frac{1}{\alpha_1 \bar{a}}\right) Y_{n+1}(\alpha_1 \bar{a}) \right) +$$

$$(\bar{c}_{12}+\mu_e H_0^2)\left(\left(\frac{n}{\alpha_1\bar{a}}-\frac{n^2}{\bar{a}^2}\right)Y_n(\alpha_1\bar{a})+\left(-\frac{1}{a}\right)Y_{n+1}(\alpha_1\bar{a})\right)-(\bar{c}_{13}+\mu_e H_0^2)$$

$$(\varepsilon 1)d_1 Y_n(\alpha_1\bar{a})+\overline{M}\left(\left(\frac{n^2}{\alpha_1^2\bar{a}^2}-\frac{n^2}{\bar{a}^2}-1\right)Y_n(\alpha_1\bar{a})+\left(\frac{1}{\alpha_1\bar{a}}-\frac{1}{\bar{a}}\right)Y_{n+1}(\alpha_1\bar{a})\right)\Bigg],$$

$$a_{16}=\Bigg[(\bar{c}_{11}+\mu_e H_0^2)\left(\left(\frac{n^2}{\alpha_2^2\bar{a}^2}-\frac{n}{\alpha_2\bar{a}}-1\right)Y_n(\alpha_2\bar{a})+\left(\frac{1}{\alpha_2\bar{a}}\right)Y_{n+1}(\alpha_2\bar{a})\right)+$$

$$(\bar{c}_{12}+\mu_e H_0^2)\left(\left(\frac{n}{\alpha_2\bar{a}}-\frac{n^2}{\bar{a}^2}\right)Y_n(\alpha_2\bar{a})-\frac{1}{\bar{a}}Y_{n+1}(\alpha_2\bar{a})\right)-(\bar{c}_{13}+\mu_e H_0^2)$$

$$(\varepsilon 1)d_2 Y_n(\alpha_2\bar{a})+\overline{M}\left(\left(\frac{n^2}{\alpha_2^2\bar{a}^2}-\frac{n^2}{\bar{a}^2}-1\right)Y_n(\alpha_2\bar{a})+\left(\frac{1}{\alpha_2\bar{a}}-\frac{1}{\bar{a}}\right)Y_{n+1}\right.$$

$$\left.(\alpha_2\bar{a})\right)\Bigg],$$

$$a_{17}=\Bigg[(\bar{c}_{11}+\mu_e H_0^2)\left(\left(\frac{n^2}{\alpha_3^2\bar{a}^2}-\frac{n}{\alpha_3\bar{a}}-1\right)Y_n(\alpha_3\bar{a})+\left(\frac{1}{\alpha_3\bar{a}}\right)Y_{n+1}(\alpha_3\bar{a})\right)+$$

$$(\bar{c}_{12}+\mu_e H_0^2)\left(\left(\frac{n}{\alpha_3\bar{a}}-\frac{n^2}{a^2}\right)Y_n(\alpha_3\bar{a})-\frac{1}{\bar{a}}Y_{n+1}(\alpha_3\bar{a})\right)-(\bar{c}_{13}+\mu_e H_0^2)$$

$$(\varepsilon 1)d_3 Y_n(\alpha_3\bar{a})+\overline{M}\left(\left(\frac{n^2}{\alpha_3^2\bar{a}^2}-\frac{n^2}{\bar{a}^2}-1\right)Y_n(\alpha_3\bar{a})+\left(\frac{1}{\alpha_3\bar{a}}-\frac{1}{\bar{a}}\right)Y_{n+1}(\alpha_3\bar{a})\right)\Bigg],$$

$$a_{18}=\left(\frac{n}{\bar{a}}\right)(\bar{c}_{11}-\bar{c}_{12})\left[-\alpha_4 Y_{n+1}(\alpha_4\bar{a})+\frac{(n-1)}{\bar{a}}Y_n(\alpha_4\bar{a})\right],$$

$a_{21}=\left[2\left(\frac{n}{\bar{a}}\right)\left\{-\alpha_1 J_{n+1}(\alpha_1\bar{a})+\frac{(n-1)}{\bar{a}}J_n(\alpha_1\bar{a})\right\}\right]$, $a_{22}=\left[2\left(\frac{n}{\bar{a}}\right)\left\{-\alpha_2 J_{n+1}(\alpha_2\bar{a})+\frac{(n-1)}{\bar{a}}J_n(\alpha_2\bar{a})\right\}\right]$,
$a_{23}=\left[2\left(\frac{n}{\bar{a}}\right)\left\{-\alpha_3 J_{n+1}(\alpha_3\bar{a})+\frac{(n-1)}{\bar{a}}J_n(\alpha_3\bar{a})\right\}\right]$, $a_{24}=\left(\frac{1}{\bar{a}}\right)^2[2\bar{a}\alpha_4 J_{n+1}(\bar{a}\alpha_4)+(\alpha_4^2\bar{a}^2+2n^2-2n)J_n(\bar{a}\alpha_4)]$, $a_{25}=\left[2\left(\frac{n}{\bar{a}}\right)\left\{-\alpha_1 Y_{n+1}\left(\alpha_1\bar{a}\right)+\frac{(n-1)}{\bar{a}}Y_n(\alpha_1\bar{a})\right\}\right]$,
$a_{26}=\left[2\left(\frac{n}{\bar{a}}\right)\left\{-\alpha_2 Y_{n+1}(\alpha_2\bar{a})+\frac{(n-1)}{\bar{a}}Y_n(\alpha_2\bar{a})\right\}\right]$, $a_{27}=\left[2\left(\frac{n}{\bar{a}}\right)\left\{-\alpha_3 Y_{n+1}(\alpha_3\bar{a})+\frac{(n-1)}{\bar{a}}Y_n(\alpha_3\bar{a})\right\}\right]$, $a_{28}=\left(\frac{1}{\bar{a}}\right)^2[2\bar{a}\alpha_4 Y_{n+1}(\bar{a}\alpha_4)+(\alpha_4^2\bar{a}^2+2n^2-2n)Y_n(\bar{a}\alpha_4)]$,
$a_{31}=[\alpha_1(d_1+\varepsilon 1)]\left[\left(\frac{n}{\bar{a}\alpha_1}\right)J_n(\alpha_1\bar{a})-J_{n+1}(\alpha_1\bar{a})\right]$,
$a_{32}=[\alpha_2(d_2+\varepsilon 1)]\left[\left(\frac{n}{\bar{a}\alpha_2}\right)J_n(\alpha_2\bar{a})-J_{n+1}(\alpha_2\bar{a})\right]$, $a_{33}=[\alpha_3(d_3+\varepsilon 1)]\left[\left(\frac{n}{\bar{a}\alpha_3}\right)J_n(\alpha_3\bar{a})-J_{n+1}(\alpha_3\bar{a})\right]$, $a_{34}=\left(\frac{n}{\bar{a}}\right)J_n(\alpha_4\bar{a})$, $a_{35}=[\alpha_1(d_1+\varepsilon 1)]\left[\left(\frac{n}{\bar{a}\alpha_1}\right)Y_n(\alpha_1\bar{a})-Y_{n+1}(\alpha_1\bar{a})\right]$,
$a_{36}=[\alpha_2(d_2+\varepsilon 1)]\left[\left(\frac{n}{\bar{a}\alpha_2}\right)Y_n(\alpha_2\bar{a})-Y_{n+1}(\alpha_2\bar{a})\right]$, $a_{37}==[\alpha_3(d_3+\varepsilon 1)]\left[\left(\frac{n}{\bar{a}\alpha_3}\right)Y_n(\alpha_3\bar{a})-Y_{n+1}(\alpha_3\bar{a})\right]$, $a_{38}=\left(\frac{n}{\bar{a}}\right)Y_n(\alpha_4\bar{a})$, $a_{41}=\left(\overline{M}-\frac{Q'_{e_1}}{k^2}+\frac{R'_{e_1}}{k^2}\right)\left[\left(\frac{n^2}{\alpha_1^2\bar{a}^2}-\frac{n^2}{\bar{a}^2}-1\right)J_n(\alpha_1\bar{a})+\left(\frac{1}{\alpha_1\bar{a}}-\frac{1}{\bar{a}}\right)J_{n+1}(\alpha_1\bar{a})\right]$, $a_{42}=(\overline{M}-\frac{Q'_{e_2}}{k^2}+\frac{R'_{e_2}}{k^2})\left[\left(\frac{n^2}{\alpha_2^2\bar{a}^2}-\frac{n^2}{\bar{a}^2}-1\right)J_n(\alpha_2\bar{a})+(\frac{1}{\alpha_2\bar{a}}-\frac{1}{\bar{a}})J_{n+1}(\alpha_2\bar{a})\right]$,

$$a_{43}=\left(\overline{M}-\frac{Q'e_3}{k^2}+\frac{R'e_3}{k^2}\right)\left[\left(\frac{n^2}{\alpha_3^2\bar{a}^2}-\frac{n^2}{\bar{a}^2}-1\right)J_n(\alpha_3\bar{a})+\left(\frac{1}{\alpha_3\bar{a}}-\frac{1}{\bar{a}}\right)J_{n+1}(\alpha_3\bar{a})\right],$$

$a_{44}=0$, $a_{45}=(\overline{M}-\frac{Q'_{e_1}}{k^2}+\frac{R'_{e_1}}{k^2})\left[\left(\frac{n^2}{\alpha_1^2\bar{a}^2}-\frac{n^2}{\bar{a}^2}-1\right)Y_n(\alpha_1\bar{a})\right.$

$$+\left(\frac{1}{\alpha_1\bar{a}}-\frac{1}{\bar{a}}\right)Y_{n+1}(\alpha_1\bar{a})\Bigg],$$

$$a_{46}=\left(\overline{M}-\frac{Q'e_2}{k^2}+\frac{R'e_2}{k^2}\right)\left[\left(\frac{n^2}{\alpha_2^2\bar{a}^2}-\frac{n^2}{\bar{a}^2}-1\right)Y_n(\alpha_2\bar{a})+\left(\frac{1}{\alpha_2\bar{a}}-\frac{1}{\bar{a}}\right)Y_{n+1}(\alpha_2\bar{a})\right],$$

$$a_{47}=\left(\overline{M}-\frac{Q'e_3}{k^2}+\frac{R'e_3}{k^2}\right)\left[\left(\frac{n^2}{\alpha_3^2\bar{a}^2}-\frac{n^2}{\bar{a}^2}-1\right)Y_n(\alpha_3\bar{a})+\left(\frac{1}{\alpha_3\bar{a}}-\frac{1}{\bar{a}}\right)Y_{n+1}(\alpha_3\bar{a})\right],$$

$$a_{48}=0\ ,$$

$$a_{51}=\Bigg[(\bar{c}_{11}+\mu_e H_0^2)\left(\left(\frac{n^2}{\alpha_1^2\bar{b}^2}-1-\frac{n}{\alpha_1\bar{b}}\right)J_n(\alpha_1\bar{b})+\left(\frac{1}{\alpha_1\bar{b}}\right)J_{n+1}\right)+$$

$$(\bar{c}_{12}+\mu_e H_0^2)\left(\left(\frac{n}{\alpha_1\bar{b}}-\frac{n^2}{\bar{b}^2}\right)J_n(\alpha_1\bar{b})-\frac{1}{\bar{b}}J_{n+1}(\alpha_1\bar{b})\right)-(\bar{c}_{13}+\mu_e H_0^2)$$

$$(\varepsilon 1)d_1J_n(\alpha_1\bar{b})+\overline{M}\left(\left(\frac{n^2}{\alpha_1^2\bar{b}^2}-\frac{n^2}{\bar{b}^2}-1\right)J_n(\alpha_1\bar{b})+\left(\frac{1}{\alpha_1\bar{b}}-\frac{1}{\bar{b}}\right)J_{n+1}(\alpha_1\bar{b})\right)\Bigg],$$

$$a_{52}=\Bigg[(\bar{c}_{11}+\mu_e H_0^2)\left(\left(\frac{n^2}{\alpha_2^2\bar{b}^2}-\frac{n}{\alpha_2\bar{b}}-1\right)J_n(\alpha_2\bar{b})+\left(\frac{1}{\alpha_2\bar{b}}\right)J_{n+1}(\alpha_2\bar{b})\right)+$$

$$(\bar{c}_{12}+\mu_e H_0^2)\left(\left(\frac{n}{\alpha_2\bar{b}}-\frac{n^2}{\bar{b}^2}\right)J_n(\alpha_2\bar{b})-\frac{1}{\bar{b}}J_{n+1}\right)-(\bar{c}_{13}+\mu_e H_0^2)(\varepsilon 1)d_2 J_n$$

$$(\alpha_2\bar{b})+\bar{M}\left(\left(\frac{n^2}{\alpha_2^2\bar{b}^2}-\frac{n^2}{b^2}-1\right)J_n(\alpha_2\bar{b})+\left(\frac{1}{\alpha_2\bar{b}}-\frac{1}{\bar{b}}\right)J_{n+1}(\alpha_2\bar{b})\right)\Bigg],$$

$$a_{53}=\Bigg[(\bar{c}_{11}+\mu_e H_0^2)\left(\left(\frac{n^2}{\alpha_3^2\bar{b}^2}-1-\frac{n}{\alpha_3\bar{b}}\right)J_n(\alpha_3\bar{b})+\left(\frac{1}{\alpha_3\bar{b}}\right)J_{n+1}(\alpha_3\bar{b})\right)+$$

$$(\bar{c}_{12}+\mu_e H_0^2)\left(\left(\frac{n}{\alpha_3\bar{b}}-\frac{n^2}{b^2}\right)J_n(\alpha_3\bar{b})+\left(-\frac{1}{b}\right)J_{n+1}(\alpha_3\bar{b})\right)-(\bar{c}_{13}+\mu_e H_0^2)$$

$$(\varepsilon 1)d_3 J_n(\alpha_3\bar{b})+\bar{M}\left(\left(\frac{n^2}{\alpha_3^2\bar{b}^2}-\frac{n^2}{\bar{b}^2}-1\right)J_n(\alpha_3\bar{b})+\left(\frac{1}{\alpha_3\bar{b}}-\frac{1}{\bar{b}}\right)J_{n+1}(\alpha_3\bar{b})\right)\Bigg],$$

$$a_{54}=\left(\frac{n}{\bar{b}}\right)(\bar{c}_{11}-\bar{c}_{12})\left[-\alpha_4 J_{n+1}(\alpha_4\bar{b})+\frac{(n-1)}{\bar{b}}J_n(\alpha_4\bar{b})\right],$$

$$a_{55}=\Bigg[(\bar{c}_{11}+\mu_e H_0^2)\left(\left(\frac{n^2}{\alpha_1^2\bar{b}^2}-\frac{n}{\alpha_1\bar{b}}-1\right)Y_n(\alpha_1\bar{b})+\left(\frac{1}{\alpha_1\bar{b}}\right)Y_{n+1}(\alpha_1\bar{b})\right)+$$

$$(\bar{c}_{12}+\mu_e H_0^2)\left(\left(\frac{n}{\alpha_1\bar{b}}-\frac{n^2}{\bar{b}^2}\right)Y_n(\alpha_1\bar{b})-\frac{1}{\bar{b}}Y_{n+1}(\alpha_1\bar{b})\right)-(\bar{c}_{13}+\mu_e H_0^2)$$

$$(\varepsilon 1)d_1 Y_n(\alpha_1\bar{b})+\bar{M}\left(\left(\frac{n^2}{\alpha_1^2\bar{b}^2}-\frac{n^2}{\bar{b}^2}-1\right)Y_n(\alpha_1\bar{b})+\left(\frac{1}{\alpha_1\bar{b}}-\frac{1}{\bar{b}}\right)Y_{n+1}(\alpha_1\bar{b})\right)\Bigg],$$

$$a_{56}=\Bigg[(\bar{c}_{11}+\mu_e H_0^2)\left(\left(\frac{n^2}{\alpha_2^2\bar{b}^2}-\frac{n}{\alpha_2\bar{b}}-1\right)Y_n(\alpha_2\bar{b})+\left(\frac{1}{\alpha_2\bar{b}}\right)Y_{n+1}(\alpha_2\bar{b})\right)+$$

$$(\bar{c}_{12}+\mu_e H_0^2)\left(\left(\frac{n}{\alpha_2\bar{b}}-\frac{n^2}{\bar{b}^2}\right)Y_n(\alpha_2\bar{b})-\frac{1}{\bar{b}}Y_{n+1}(\alpha_2\bar{b})\right)-(\bar{c}_{13}+\mu_e H_0^2)$$

$$(\varepsilon 1)d_2 Y_n(\alpha_2\bar{b})+\overline{M}\left(\left(\frac{n^2}{\alpha_2^2\bar{b}^2}-\frac{n^2}{\bar{b}^2}-1\right)Y_n(\alpha_2\bar{b})+(\frac{1}{\alpha_2\bar{b}}-\frac{1}{\bar{b}})Y_{n+1}$$

$$(\alpha_2\bar{b})\Bigg)\Bigg],$$

$$a_{57}=\Bigg[(\bar{c}_{11}+\mu_e H_0^2)\left(\left(\frac{n^2}{\alpha_3^2\bar{b}^2}-\frac{n}{\alpha_3\bar{b}}-1\right)Y_n(\alpha_3\bar{b})+(\frac{1}{\alpha_3\bar{b}})Y_{n+1}(\alpha_3\bar{b})\right)+$$

$$(\bar{c}_{12}+\mu_e H_0^2)\left(\left(\frac{n}{\alpha_3\bar{b}}-\frac{n^2}{\bar{b}^2}\right)Y_n(\alpha_3\bar{b})-\frac{1}{\bar{b}}Y_{n+1}(\alpha_3\bar{b})\right)-(\bar{c}_{13}+\mu_e H_0^2)$$

$$(\varepsilon 1)d_3 Y_n(\alpha_3\bar{b})+\overline{M}\left(\left(\frac{n^2}{\alpha_3^2\bar{b}^2}-\frac{n^2}{\bar{b}^2}-1\right)Y_n(\alpha_3\bar{b})+(\frac{1}{\alpha_3\bar{b}}-\frac{1}{\bar{b}})Y_{n+1}$$

$$(\alpha_3\bar{b})\Bigg)\Bigg],$$

$a_{58}=\left(\frac{n}{\bar{b}}\right)(\bar{c}_{11}-\bar{c}_{12})\left[-\alpha_4 Y_{n+1}(\alpha_4\bar{b})+\frac{(n-1)}{\bar{b}}Y_n(\alpha_4\bar{b})\right]$,
$a_{61}=\left[2\left(\frac{n}{\bar{b}}\right)\left\{-\alpha_1 J_{n+1}(\alpha_1\bar{b})+\frac{(n-1)}{\bar{b}}J_n(\alpha_1\bar{b})\right\}\right]$, $a_{62}=\left[2\left(\frac{n}{\bar{b}}\right)\left\{-\alpha_2 J_{n+1}(\alpha_2\bar{b})+\frac{(n-1)}{\bar{b}}J_n(\alpha_2\bar{b})\right\}\right]$,
$a_{63}=\left[2\left(\frac{n}{\bar{b}}\right)\left\{-\alpha_3 J_{n+1}(\alpha_3\bar{b})+\frac{(n-1)}{\bar{b}}J_n(\alpha_3\bar{b})\right\}\right]$, $a_{64}=\left(\frac{1}{\bar{b}}\right)^2\left[2\bar{b}\alpha_4 J_{n+1}(\bar{b}\alpha_4)+(\alpha_4^2\bar{b}^2+2n^2-2n)J_n(\bar{b}\alpha_4)\right]$, $a_{65}=\left[2\left(\frac{n}{\bar{b}}\right)\left\{-\alpha_1 Y_{n+1}(\alpha_1\bar{b})+\frac{(n-1)}{\bar{b}}Y_n(\alpha_1\bar{b})\right\}\right]$,
$a_{66}=\left[2\left(\frac{n}{\bar{b}}\right)\left\{-\alpha_2 Y_{n+1}(\alpha_2\bar{b})+\frac{(n-1)}{\bar{b}}Y_n(\alpha_2\bar{b})\right\}\right]$, $a_{67}=\left[2\left(\frac{n}{\bar{b}}\right)\left\{-\alpha_3 Y_{n+1}(\alpha_3\bar{b})+\right.\right.$

$\frac{(n-1)}{\bar{b}} Y_n(\alpha_3 \bar{b})\}\Big]$,

$a_{68} = \left(\frac{1}{\bar{b}}\right)^2 \left[2\bar{b}\alpha_4 Y_{n+1}(\bar{b}\alpha_4) + (\alpha_4^2\bar{b}^2 + 2n^2 - 2n) Y_n(\bar{b}\alpha_4)\right]$, $a_{71} = [\alpha_1(d_1 + \varepsilon 1)] \left[\left(\frac{n}{\bar{b}\alpha_1}\right) J_n(\alpha_1\bar{b}) - J_{n+1}(\alpha_1\bar{b})\right]$, $a_{72} = [\alpha_2(d_2 + \varepsilon 1)] \left[\left(\frac{n}{\bar{b}\alpha_2}\right) J_n(\alpha_2\bar{b}) - J_{n+1}(\alpha_2\bar{b})\right]$,

$a_{73} = [\alpha_3(d_3 + \varepsilon 1)] \left[\left(\frac{n}{\bar{b}\alpha_3}\right) J_n(\alpha_3\bar{b}) - J_{n+1}(\alpha_3\bar{b})\right]$, $a_{74} = \left(\frac{n}{\bar{b}}\right) J_n(\alpha_4\bar{b})$,

$a_{75} = [\alpha_1(d_1 + \varepsilon 1)] \left[\left(\frac{n}{\bar{b}\alpha_1}\right) Y_n(\alpha_1\bar{b}) - Y_{n+1}(\alpha_1\bar{b})\right]$, $a_{76} = [\alpha_2(d_2 + \varepsilon)] \left[\left(\frac{n}{\bar{b}\alpha_2}\right) Y_n(\alpha_2\bar{b}) - Y_{n+1}(\alpha_2\bar{b})\right]$, $a_{77} = [\alpha_3(d_3 + \varepsilon 1)] \left[\left(\frac{n}{\bar{b}\alpha_3}\right) Y_n(\alpha_3\bar{b}) - Y_{n+1}(\alpha_3\bar{b})\right]$

$a_{78} = \left(\frac{n}{\bar{b}}\right) Y_n(\alpha_4\bar{b})$, $a_{81} = (\overline{M} - \frac{Q' e_1}{k^2} + \frac{R' e_1}{k^2}) \left[\left(\frac{n^2}{\alpha_1^2\bar{b}^2} - \frac{n^2}{\bar{b}^2} - 1\right) J_n (\alpha_1\bar{b}) + (\alpha_1\bar{b} + (\frac{1}{\alpha_1\bar{b}} - \frac{1}{\bar{b}}) J_{n+1}(\alpha_1\bar{b})\right]$, $a_{82} = \left(\overline{M} - \frac{Q' e_2}{k^2} + \frac{R' e_2}{k^2}\right)\left[\left(\frac{n^2}{\alpha_2^2\bar{b}^2} - \frac{n^2}{\bar{b}^2} - 1\right) J_n\left(\alpha_2\bar{b}\right) + \left(\frac{1}{\alpha_2\bar{b}} - \frac{1}{\bar{b}}\right) J_{n+1}\left(\alpha_2\bar{b}\right)\right]$,

$a_{83} = (\overline{M} - \frac{Q' e_3}{k^2} + \frac{R' e_3}{k^2}) \left[\left(\frac{n^2}{\alpha_3^2\bar{b}^2} - \frac{n^2}{\bar{b}^2} - 1\right) J_n(\alpha_3\bar{b}) + (\frac{1}{\alpha_3\bar{b}} - \frac{1}{b}) J_{n+1}(\alpha_3\bar{b})\right]$,

$a_{84} = 0$, $a_{85} = (\overline{M} - \frac{Q' e_1}{k^2} + \frac{R' e_1}{k^2}) \left[\left(\frac{n^2}{\alpha_1^2\bar{b}^2} - \frac{n^2}{\bar{b}^2} - 1\right) Y_n(\alpha_1\bar{b}) + (\frac{1}{\alpha_1\bar{b}} - \frac{1}{\bar{b}}) Y_{n+1}(\alpha_1\bar{b})\right]$,

$$a_{86} = \left(\overline{M} - \frac{Q' e_2}{k^2} + \frac{R' e_2}{k^2}\right)\left[\left(\frac{n^2}{\alpha_2^2\bar{b}^2} - \frac{n^2}{\bar{b}^2} - 1\right) Y_n\left(\alpha_2\bar{b}\right) + \left(\frac{1}{\alpha_2\bar{b}} - \frac{1}{\bar{b}}\right) Y_{n+1}\left(\alpha_2\bar{b}\right)\right],$$

$$a_{87} = \left(\overline{M} - \frac{Q' e_3}{k^2} + \frac{R' e_3}{k^2} \right) \left[\left(\frac{n^2}{\alpha_3^2 \bar{b}^2} - \frac{n^2}{\bar{b}^2} - 1 \right) Y_n\left(\alpha_3 \bar{b}\right) + \left(\frac{1}{\alpha_3 \bar{b}} - \frac{1}{\bar{b}} \right) Y_{n+1}\left(\alpha_3 \bar{b}\right) \right],$$

$$a_{88} = 0.$$

References

Abd-Alla, A. M.; Othman, M. I. A.; Abo-Dahab, S. M. (2016): Reflection of Plane Waves from Electro-magneto-thermoelastic Half-space with a Dual-Phase-Lag Model. *CMC: Computers, Materials & Continua*, vol. 51, no. 2, pp. 63-79.

Ahmed, S. M.; Abd-Alla, A. M. (2002): Electromechanical wave propagation in a cylindrical poroelastic bone with cavity. *Applied Mathematics and Computation*, vol. 133, no. 2, pp. 257-286.

Abd-Alla, A. M.; Abo-Dahab, S. M.; Mahmoud, S. R. (2011): Wave propagation modeling in cylindrical human long wet bones with cavity. *Meccanica*, vol. 46, no. 6, pp.1413-1428.

Abd-Alla, A. M.; Abo-Dahab, S. M. (2013): Effect of magnetic field on poroelastic bone model for internal remodeling. *Applied Mathematics and Mechanics*, vol. 34, no. 7, pp. 889-906.

Abo-Dahab, S. M.; Abd-Alla, A. M.; Alqosami, S. (2014): Effect of Rotation on Wave Propagation in Hollow Poroelastic Circular Cylinder. *Mathematical Problems in Engineering*, vol. 2014, pp. 16.

Abd-Alla, A. M.; Yahya, G. A. (2013): Wave Propagation in a Cylindrical Human Long Wet Bone. *J. Comput. Theor. Nanosci*., vol. 10, no. 3, 750-755.

Biot, M. A. (1955): Theory of elasticity and consolidation for a porous anisotropic solid. *Journal of Applied Physics*, vol. 26, no. 2, pp. 182-185.

Biot, M. A. (1956): Theory of Propagation of elastic waves in a fluid-saturated porous solid. I: low-frequency range. *Acoustical Society of America*, vol. 28, no. 2, pp. 168-178.

Brynk, T.; Hellmich, C.; Fritsch, A.; Zysset, P.; Eberhardsteiner, J. (2011): Experimental poromechanics of trabecular bone strength: Role of Terzaghi's effective stress and of tissue level stress fluctuations. *Journal of Biomechanics*, vol. 44, no. 3, pp. 501-508.

Cardoso, L.; Cowin, S. C. (2012): Role of structural anisotropy of biological tissues in poroelastic wave propagation. *Mechanics of Materials*, vol. 44, pp. 174-188.

Cui, L. A.; Cheng, H. D.; Abousleiman, Y. (1997): Poroelastic solutions of an inclined borehole. Transactions of ASME. *Journal of Applied Mechanics*, vol. 64, pp.32-38.

Cowin, S. C. (1999): Bone poroelasticity. *Journal of Biomechanics*, vol. 32, pp. 217-238.

El-Naggar, A. M.; Abd-Alla, A. M.; Mahmoud, S. R. (2001): Analytical solution of electro-mechanical wave propagation in long bones. *Applied Mathematics and Computation*, vol. 119, pp. 77-98.

Gilbert, R. P.; Guyenne, P.; Ou, M. Y. (2012): A quantitative ultrasound model of the bone with blood as the interstitial fluid. *Mathematical and Computer Modelling*, vol. 55, no. 9-10, pp. 2029-2039.

Love, A. E. H. (1944): A theoretical on the mathematical theorey of elasticity. Dover, New York.

Matuszyk, P. J.; Demkowicz, L. F. (2014): Solution of coupled poroelastic/aco ustic/elastic wave propagation problems using automatic hp-adaptivity. *Computer Methods in Applied Mechanics and Engineering*, vol. 281, pp. 54-80.

Misra, J. C.; Samanta, S. C. (1984): Wave propagation in tubular bones, *International Journal of Solids and Structures*. vol. 20, no. 1, pp. 55-62.

Mathieu, V.; Vayron, R.; Soffer, E.; Anagnostou, F. (2012): Influence of healing time on the ultrasonic response of the bone-implant interface. *Ultrasound in Medicine & Biology*, vol. 38, no. 4, pp. 611-618.

Marin, M.; Abd-Alla, A. M.; Raducanu, D.; Abo-Dahab, S.M. (2015): Structural continuous dependence in micropolar porous bodies. *CMC: Computers, Materials & Continua*, vol. 45, no. 2, pp. 107-125.

Marin, M. (2010): Harmonic vibrations in thermoelasticity of microstretch materials. *J. Vibr. Acoust., ASME*, vol. 132, no. 4, pp. 1-6.

Marin, M. (1997): On weak solutions in elasticity of dipolar bodies with voids. *J. Comp.Appl. Math.*, vol. 82, no. 1-2, pp. 291-297.

Morin, C.; Hellmich, C. (2014): A multiscale poromicromechanical approach to wave propagation and attenuation in bone. *Ultrasonics*, vol. 54, no. 5, pp. 1251-1269.

Nguyen, V. H.; Lemaire, T.; Naili, S. (2010): Poroelastic behaviour of cortical bone under harmonic axial loading: A finite element study at the osteonal scale. *Medical Engineering & Physics*, vol. 32, no. 4, pp.384-390.

Papathanasopoulou, V. A.; Fotiadis, D.; Foutsitzi, I. G.; Massalas, C. V. (2002): A poroelastic bone model for internal remodeling. *International Journal of Engineering Science*, vol. 40, no. 5, pp. 511-530.

Potsika, V. T.; Grivas, K. N. ; Protopappas, V. C. ; Vavva, M. G. ; Raum, K.; Rohrbach, D. ; Polyzos, D.; Fotiadis, D. I. (2014): Application of an effective medium theory for modeling ultrasound wave propagation in healing long bones. *Ultrasonics*, vol. 54, no. 5, pp.1219-1230.

Parnell, W. J.; Vu, M. B.; Grimal, Q.; Naili, S. (2012): Analytical methods to determine the effective mesoscopic and macroscopic elastic properties of cortical bone. *Biomechanics and Modeling in Mechanobiology*, vol. 11, no. 6, pp. 883-901.

Qin, Q. H.; Qu, C.; Ye, J. (2005): Thermoelectroelastic solutions for surface bone remodeling under axial and transverse loads. *Biomaterials*, vol. 26, no. 33, pp. 6798-6810.

Sharma, K.; Marin, M. (2013): Effect of distinct conductive and thermodynamic temperatures on the reflection of plane waves in micropolar elastic half-space. *U.P.B. Sci. Bull., Series A-Appl. Math. Phys.*, vol. 75, no. 2, pp. 121-132.

Sharma, A.; Marin. M. (2013): Effect of distinct conductive and thermodynamic temperatures on the reflection of plane waves in micropolar elastic half-space. U.P.B. Sci. *Bull., Series A-Appl. Math. Phys.*, vol. 75, no. 2, pp. 121-132.

Shah, A. (2011): Flexural wave propagation in coated poroelastic cylinders with reference to fretting fatigue. *Journal of Vibration and Control*, vol. 17, no. 7, pp. 1049–1064.

Yoon, H. S.; Katz, J. L. (1976): Ultrasonic wave propagation in human cortic al bone—II. Measurements of elastic properties and microhardness. *Journal of Biomechanics*, vol. 9, pp. 459-462,9, pp. 463-464.

Wen, P. H. (2010): Meshless local Petrov–Galerkin (MLPG) method for wave propagation in 3D poroelastic solids, *Engineering Analysis with Boundary Elements*, vol. 34, no. 4, pp. 315-323.