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Symmetry Transformations and Exact Solutions of a Generalized Hyperelastic Rod Equation

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Abstract: In this paper, a nonlinear wave equation with variable coefficients is studied, interestingly, this equation can be used to describe the travelling waves propagating along the circular rod composed of a general compressible hyperelastic material with variable cross-sections and variable material densities. With the aid of Lou's direct method[1], the nonlinear wave equation with variable coefficients is reduced and two sets of symmetry transformations and exact solutions of the nonlinear wave equation are obtained. The corresponding numerical examples of exact solutions are presented by using different coefficients. Particularly, while the variable coefficients are taken as some special constants, the nonlinear wave equation with variable coefficients reduces to the one with constant coefficients, which can be used to describe the propagation of the travelling waves in general cylindrical rods composed of generally hyperelastic materials. Using the same method to solve the nonlinear wave equation, the validity and rationality of this method are verified.

Keywords: Generalized hyperelastic rod equation, symmetry transformation, Lou's direct method, exact solution.

1 Introduction

Hyperelastic materials such as rubber and rubber-like materials have many excellent properties, they all possess the typical characteristics of nonlinearity, high elasticity and large deformation, and their products are widely used in aerospace, transportation, petrochemical, and other fields. As these materials and their products are usually employed in certain environments, and all encounter the problems of deformation, instability, destruction and so on. Therefore, the dynamic stability related to these materials and structures has always been the focus of attention [Beatty (1987); Fu and Ogden (2001); Ou, Yao, Zhang et al. (2014)]. Thereafter, many significant works have been carried out. Some

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Remark 1: Based on CK's direct method, Lou and Ma (2005) skillfully constructed a modified direct method, known as Lou's direct method, this method without using any group theory and easy to use.

aspects of nonlinear elastodynamics for hyperelastic materials may be found in Chou-Wang et al. [Chou-Wang and Horgan (1989); Yuan and Zhang (2006, 2009); Yuan, Zhang and Zhang (2008); Akbarov and Ipek (2012); Negin, Akbarov and Erguven (2014)], and so on. There is a long history of research on the propagation of nonlinear travelling waves in the cylindrical rods composed of hyperelastic materials. Due to the nonlinear waves resistance to dissipation very well, so the nonlinear waves can be detected even after a very long distance. For example, long oil or gas pipelines, if we use linear waves to detect cracks, the result is not ideal since the linear waves are easy to dissipate. The applications of nonlinear waves can obtain more accurate results. The propagation of nonlinear waves can be used to determine material properties and defect detection [Dai and Fan (2004)]. Therefore, it is very important to study the propagation of nonlinear waves in cylindrical rods.

Wright [Wright (1985)] demonstrated the existence of some exact travelling waves of the cylindrical rods composed of incompressible materials under the effect of radial shear and inertia. Coleman et al. [Coleman and Newman (1990)] studied the axisymmetric motion of the cylindrical rod composed of an isotropic incompressible neo-Hookean material, and the analytical expressions of solitary waves and periodic waves are given. Cohen et al. [Cohen and Dai (1993)] investigated the propagation of nonlinear travelling waves in the cylindrical rod composed of a class of compressible Mooney-Rivlin materials, and obtained a system of nonlinear evolution equations for describing the symmetric motion of the rod. Using the reduced perturbation method, the authors reduced the governing equation to the KdV equation and proved that there exist finite amplitude waves. Moreover, Dai [Dai (1998)] considered a critical case that the coefficient of the dispersive term in the KdV equation vanishes, and then the authors focused on finite-length and finite-amplitude waves in the rod. The governing equation is further reduced to the Benjamin-Bona-Mahony (BBM) equation. The Camassa-Holm (CH) equation has also been found to model the propagation of nonlinear waves in cylindrical hyperelastic rods, see Dai et al. [Dai and Huo (2000)]. Xia et al. [Xia, Zhou and Qiao (2016)] proposed a new approach to calculate the multi-soliton solutions of the CH equation with the aid of Darboux transformation. Lv et al. [Lv, Niu, Yuan et al. (2016)] obtained an explicit solution of the breaking soliton system based on the reduced lax pairs by using the same method in Xia et al. [Xia, Zhou and Qiao (2016)]. Coclite et al. [Coclite, Holden and Karlsen (2005)] considered a generalized CH equation describing no nlinear dispersive waves in compressible hyperelastic rods. The authors established the existence of a strongly continuous semigroup of global weak solutions and presented a “weak equals strong” uniqueness result.

In this paper, we study a generalized hyperelastic rod equation, which can be also called a class of generalized KdV equations, in the following form Zhang et al. [Zhang (2007); Das and Ghosh (2016); Kumar, Bansal and Gupta (2016)]

$$u_t + \alpha(t)uu_x + \beta(t)u_{xxx} = 0, \tag{1}$$

where the first term is the evolution term, the second term is the convection term, and the third term is the dispersion term, u is a function with respect to x, t, the coefficients $\alpha(t)$ and $\beta(t)$ of the convection and dispersion terms are differentiable functions with respect to t.

If $\alpha(t)=6\rho^{3/4}a^{-1}, \beta(t)=\rho^{3/2}a^{2}$, interestingly, Eq. (1) reduces to [Dai and Huo (2002)]

$$u_t+6\rho^{\frac{3}{4}}a^{-1}uu_x+\rho^{\frac{3}{2}}a^2u_{xxx}=0. \tag{2}$$

It can be used to describe the propagation of solitary waves in a cylindrical rod composed of incompressible hyperelastic materials with variable cross-sections and variable densities. The density ρ and the radius a of the rod are functions of the variable t, and u represents the radial stretch function.

If $\alpha(t)=\dfrac{\pi_3-2\pi_6-4m\pi_7}{2\tau}, \beta(t)=\dfrac{m^2\delta}{4\varepsilon}$, Eq. (1) degenerates to the following form [Dai and Fan (2004)],

$$u_t+\frac{\pi_3-2\pi_6-4m\pi_7}{2\tau}uu_x+\frac{m^2\delta}{4\varepsilon}u_{xxx}=0, \tag{3}$$

significantly, Eq. (3) can be used to describe the propagation of weakly nonlinear waves of a hyperelastic cylindrical rod composed of a compressible Murnaghan material. In Eq. (3), u represents the ratio of axial stretch, and $\pi_i\,(i=3,6,7), m, \tau, \varepsilon, \delta$ are the material and geometry constants, respectively.

If $\alpha(t)=\dfrac{\eta}{2(3\eta+1)}\sigma_1, \beta(t)=\dfrac{\nu}{\varepsilon}\dfrac{\eta(4\eta+1)s}{8(2\eta+1)(3\eta+1)}$, Eq. (1) reduces to [Dai (1998)],

$$u_t+\frac{\eta}{2(3\eta+1)}\sigma_1uu_x+\frac{\nu}{\varepsilon}\frac{\eta(4\eta+1)s}{8(2\eta+1)(3\eta+1)}u_{xxx}=0. \tag{4}$$

And Eq. (4) can be used to describe the propagation of finite amplitude waves in an infinite cylindrical rod composed of a compressible Mooney-Rivlin material, in which u represents the ratio of axial stretch, the physical parameters $\eta, \sigma_1, s, \nu, \varepsilon$ are constants. It is proved that the solitary waves always propagate in the peak form in the hyperelastic rod.

If $\alpha(t)=\dfrac{\tau}{4\mu}-1, \beta(t)=\dfrac{1}{16}\dfrac{\nu}{\varepsilon}$, Eq. (1) becomes

$$u_t+\left(\frac{\tau}{4\mu}-1\right)uu_x+\frac{1}{16}\frac{\upsilon}{\varepsilon}u_{xxx}=0, \tag{5}$$

which is corresponding to (4.18) in Dai et al. [Dai and Huo (2002)]. Eq. (5) can be used to describe the propagation of longitudinal strain waves in an infinitely long cylindrical rod of generally incompressible materials. In Eq. (5) u represents the ratio of axial stretch, and $\tau, \mu, \varepsilon, \upsilon$ represent the material and structural constants, respectively. The authors Dai et al. [Dai and Huo (2002)] proved the existence of solitary waves expressed by the sech function, and as $\alpha(t)>0$, solitary waves propagate in the peak form; as $\alpha(t)<0$, solitary waves propagate in the valley form, however, the special case $\alpha(t)=0$ is not considered.

This paper is organized as follows. In Section 2, Eq. (1) is reduced with the aid of Lou's direct method, two sets of symmetry transformations are obtained and the corresponding exact solutions are given. In Section 3, the method is further used for the rod Eq. (5) with constant coefficients. The last section is a short summary and discussion.

2 Symmetry transformations and exact solutions of Eq. (1)

In this section, we consider the symmetry transformations and exact solutions of Eq. (1). Replacing the functions $\alpha(t)$ and $\beta(t)$ in Eq. (1) by the constants k_1, k_2 yields

$$u_t + k_1 u u_x + k_2 u_{xxx} = 0. \tag{6}$$

In order to obtain the symmetry transformations of Eq. (1), assume that

$$u = A + BU(X,T), \tag{7}$$

where A, B, U, X, T are functions with respect to x, t. Further, assume that U satisfies the following equation, which has the same form as Eq. (6) but with the new variables X, T, i.e.

$$U_T + k_1 U U_X + k_2 U_{XXX} = 0. \tag{8}$$

Substituting Eq. (7) into Eq. (1), then eliminating U_{XXX} by using Eq. (8), we have

$$\beta(t) B T_x^3 U_{TTT} + V(x,t,U) = 0, \tag{9}$$

where V is a complicated function which is independent of U_{TTT} and the specific representation is as follows,

$$\begin{aligned} V(x,t,U) &= A_t + B_t U + \alpha(t)(A+BU)\left[A_x + B_x U + B\left(U_X X_x + U_T T_x\right)\right] \\ &+ B\left(U_X X_t + U_T T_t\right) + \beta(t)\Big\{A_{xxx} + B_{xxx}U + 3B_{xx}(U_X X_x + U_T T_x) + 3B_x\big[X_{xx}U_X \\ &+ U_T T_{xx} + X_x\left(U_{XX} X_x + U_{XT} T_x\right) + T_x\left(U_{TX} X_x + U_{TT} T_x\right)\big] + B\big(X_{xxx}U_X + U_T T_{xxx} \\ &+ X_x\left[X_{xx}U_{XX} + X_x\left(\frac{-U_T - k_1 U U_X}{k_2} X_x + U_{XXT} T_x\right) + T_x\left(U_{XTX} X_x + U_{XTT} T_x\right)\right] \\ &+ 2X_{xx}U_{XT}T_x + 2T_{xx}\left(U_{TX} X_x + U_{TT} T_x\right) + T_x\left(X_{xx}U_{TX} + U_{TT}T_{xx} + U_{TTX}T_x X_x\right) \\ &+ X_x U_{XT} T_{xx} + 2X_{xx}U_{XX}X_x + T_x X_x\left(U_{TXX} X_x + U_{TXT} T_x\right)\big)\Big\}. \end{aligned} \tag{10}$$

Interestingly, Eq. (9) holds for any arbitrary solution U if and only if all coefficients of the derivatives of U are zero. Obviously, $\beta(t) B T_x^3 = 0$ is valid. Without loss of generality, set

$$T = T(t). \tag{11}$$

Using Eq. (11) to reduce Eq. (9), and collecting the constant terms and the coefficients of the derivatives of U, we obtain the determining equations for A, B, X, T, as follows,

$$\begin{aligned}
&A_t + \alpha(t)AA_x + \beta(t)A_{xxx} = 0,\\
&B_t + \alpha(t)AB_x + \alpha(t)A_xB + \beta(t)B_{xxx} = 0,\\
&BX_t + \alpha(t)ABX_x + 3\beta(t)B_{xx}X_x + 3\beta(t)B_xX_{xx} + \beta(t)BX_{xxx} = 0,\\
&3\beta(t)(B_xX_x^2 + BX_xX_{xx}) = 0, \quad \alpha(t)BB_x = 0,\\
&\alpha(t)B^2X_x - \beta(t)\frac{Bk_1X_x^3}{k_2} = 0, \quad BT_t - \beta(t)\frac{BX_x^3}{k_2} = 0.
\end{aligned} \tag{12}$$

Using Maple to solve Eq. (12), we consider the following two cases.

Case 1:

Let

$$\begin{aligned}
&A(x,t) = 0,\ B(t) = d_3,\ T(t) = \int \frac{\beta(t)d_1^3}{k_2}dt + d_4,\\
&X(x,t) = d_1x + d_2,\ \alpha(t) = \frac{\beta(t)k_1d_1^2}{d_3k_2},\ \beta(t) = \beta(t),
\end{aligned} \tag{13}$$

where $d_i (i = 1,2,3,4)$ are arbitrary constants. Then Eq. (1) can be reduced to

$$u_t + \frac{\beta(t)k_1d_1^2}{d_3k_2}uu_x + \beta(t)u_{xxx} = 0. \tag{14}$$

The symmetry transformation of Eq. (14) is given by

$$u = d_3U\left(d_1x + d_2, \int \frac{\beta(t)d_1^3}{k_2}dt + d_4\right). \tag{15}$$

It is easy to give the exact solution of Eq. (8), whose coefficients are constants, i.e.

$$U(X,T) = \frac{-C_3 + 8k_2 - C_2^3}{k_1C_2} - \frac{12k_2C_2^2 \tanh(C_1 + C_2X + C_3T)^2}{k_1}, \tag{16}$$

where $C_i (i = 1,2,3)$ are arbitrary constants.

Combining Eqs. (15) and (16), an exact solution of Eq. (14) can be obtained, namely,

$$u(x,t) = d_3\left(\frac{-C_3+8k_2C_2^3}{k_1C_2} - \frac{12k_2C_2^2\tanh^2\left(C_1+C_2\left(d_1x+d_2\right)+C_3\left(\int\frac{\beta(t)d_1^3}{k_2}dt+d_4\right)\right)}{k_1}\right). \tag{17}$$

Numerical examples of the exact solution (17) are given in Figs. 1 and 2.

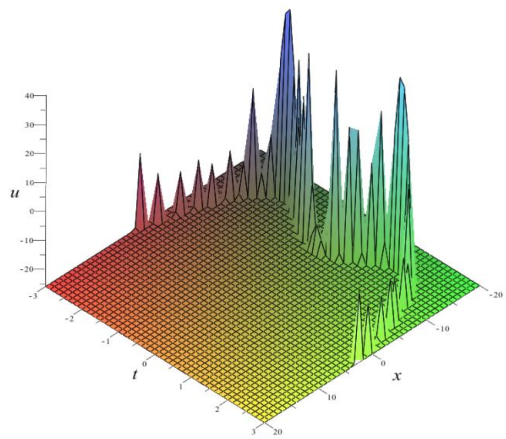

Figure 1: The exact solution (17) for $\beta(t)=t\cos t, k_1=1, k_2=3,$
$C_1=5, C_2=1, C_3=2, d_1=8, d_2=1, d_3=1, d_4=3$

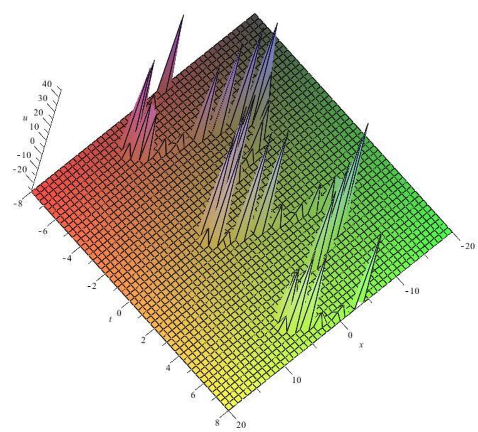

Figure 2: The exact solution (17) for $\beta(t)=\sin t, k_1=1, k_2=3,$
$C_1=5, C_2=1, C_3=2, d_1=8, d_2=1, d_3=1, d_4=2$

Case 2:

Let

$$A(x,t)=d_1,\ B(t)=-\frac{k_1d_1d_4^3}{d_3k_2},\ \alpha(t)=-\frac{\beta(t)d_3}{d_1d_4},\ \beta(t)=\beta(t),$$
$$T(t)=\int\frac{\beta(t)d_4^3}{k_2}dt+d_5,\ X(x,t)=d_4x+d_2+d_3\int\beta(t)dt. \tag{18}$$

Then Eq. (1) is reduced to

$$u-\frac{\beta(t)d_3}{d_1d_4}uu_x+\beta(t)u_{xxx}=0. \tag{19}$$

The symmetry transformation of Eq. (19) is

$$u=d_1-\frac{k_1d_1d_4^3}{d_3k_2}U\left(d_4x+d_2+d_3\int\beta(t)dt,\int\frac{\beta(t)d_4^3}{k_2}dt+d_5\right). \tag{20}$$

Combining Eqs. (16) and (20) leads to

$$u(x,t)=-\frac{k_1d_1d_4^3\left(\frac{-C_3+8k_2C_2^3}{k_1C_2}-\frac{12k_2C_2^2\tanh\left(C_1+C_2\left(d_4x+d_2+d_3\int\beta(t)dt\right)+C_3\left(\int\frac{\beta(t)d_4^3}{k_2}dt+d_5\right)\right)^2}{k_1}\right)}{d_3k_2}+d_1. \tag{21}$$

Numerical examples of the exact solution (21) may be found in Figs. 3 and 4.

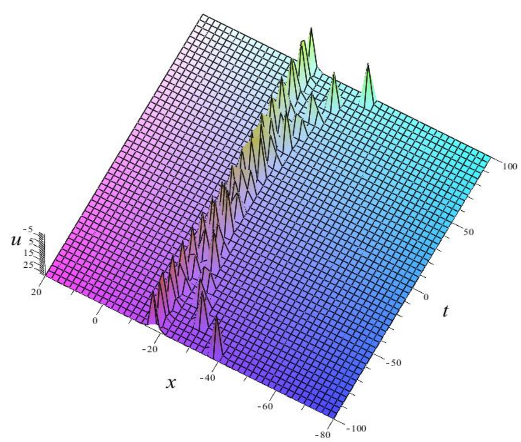

Figure 3: The exact solution (21) for $\beta(t)=\tan t, k_1=3, k_2=2,$
$C_1=6, C_2=1, C_3=4, d_1=9, d_2=3, d_3=3, d_4=1, d_5=2$

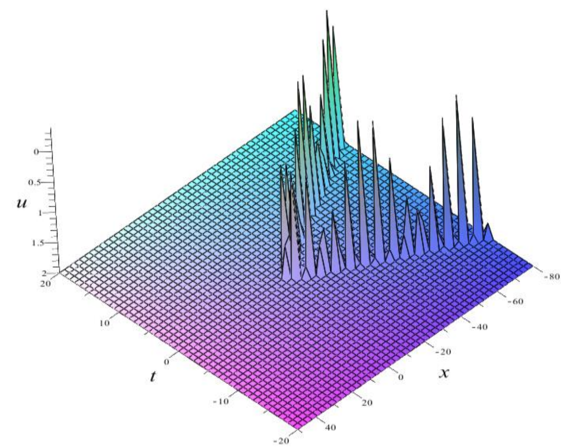

Figure 4: The exact solution (21) for $\beta(t)=\tanh t, k_1=3, k_2=1,$
$C_1=1, C_2=3, C_3=5, d_1=6, d_2=1, d_3=1, d_4=2, d_5=4$

3 Symmetry transformations and exact solutions of Eq. (5)

In this section, we solve Eq. (5) by using Lou's direct method. The coefficients are denoted by $\alpha(t)=M, \beta(t)=N$, where $M=\frac{\tau}{4\mu}-1, N=\frac{1}{16}\frac{\nu}{\varepsilon}$ are constants. We also, assume that U satisfies the following equation, which has the same form as Eq. (5) but with the variables X, T, i.e.

$$U_T + MUU_X + NU_{XXX} = 0. \tag{21}$$

Solving Eq. (21) gives the exact solution, as follows,

$$U(X,T)=\frac{-C_3+8NC_2^3}{MC_2}-\frac{12NC_2^2\tanh(C_1+C_2X+C_3T)^2}{M}, \tag{22}$$

where $C_i (i=1,2,3)$ are arbitrary constants.

In order to obtain the symmetry transformations of Eq. (5), we use the same assumption as Eq. (7). Substituting Eq. (7) into Eq. (5) and eliminating U_{XXX} by Eq. (21), we get

$$NBT_x^3 U_{TTT} + F(x,t,U) = 0, \tag{23}$$

where F is a complicated function which is independent of U_{TTT}. Eq. (23) is valid for an arbitrary solution U only when all the coefficients of the derivatives of U are zero. From Eq. (23), we know that $NBT_x^3 = 0$. Similar to the previous section, without loss of generality, we also assume that

$$T = T(t). \tag{24}$$

Using Eq. (24) to reduce Eq. (23), we obtain the following system of the restricted equations with respect to A, B, X, T, namely,

$$\begin{aligned}
&A_t + MAA_x + NA_{xxx} = 0,\\
&B_t + MAB_x + MA_x B + NB_{xxx} = 0,\\
&BX_t + MABX_x + 3NB_{xx}X_x + 3NB_x X_{xx} + NBX_{xxx} = 0,\\
&MBB_x = 0, \qquad 3N(B_x X_x^2 + BX_x X_{xx}) = 0,\\
&MB^2 X_x - MBX_x^3 = 0, \qquad BT_t - BX_x^3 = 0.
\end{aligned} \tag{25}$$

By means of Maple to solve System (25), we consider the following two cases.

Case 1:

Let

$$A(x,t) = 0, B(x,t) = D_1, T(t) = D_1^{(3/2)} t + D_3, X(x,t) = \sqrt{D_1}x + D_2, \tag{26}$$

where $D_i (i = 1,2,3)$ are arbitrary constants. The symmetry transformation of Eq. (5) is given by

$$u = D_1 U\left(\sqrt{D_1}x + D_2, D_1^{(3/2)} t + D_3\right). \tag{27}$$

From Eqs. (22) and (27), an exact solution of Eq. (5) is as follows

$$u(x,t) = \frac{-C_3 + 8NC_2^3}{MC_2} - \frac{12NC_2^2 \tanh\left(C_1 + C_2\left(\sqrt{D_1}x + D_2\right) + C_3\left(D_1^{(3/2)}t + D_3\right)\right)^2}{M}. \tag{28}$$

Figs. 5 and 6 display the exact solution (28) for different constants and physical parameters.

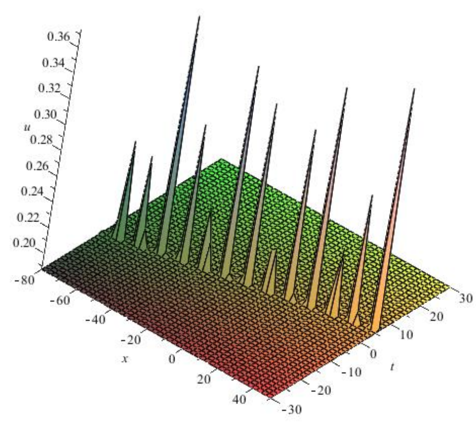

Figure 5: The exact solution (28) for $C_1 = 3, C_2 = 0.2, C_3 = -0.05,$ $D_1 = 32, D_2 = 1, D_3 = 1, M = 1, N = 0.039$

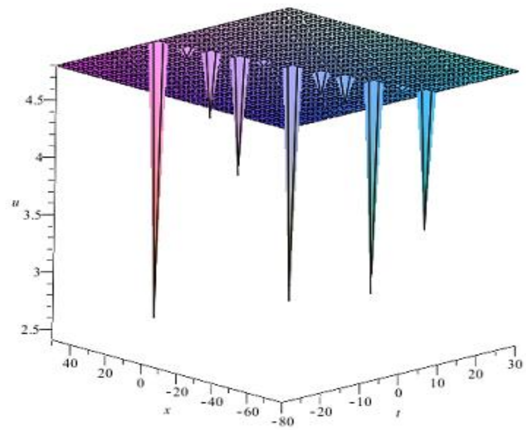

Figure 6: The exact solution (28) for $C_1 = 3, C_2 = 1, C_3 = 2,$
$D_1 = 5, D_2 = 1, D_3 = 1, M = -1/2, N = 0.039$

Case 2:

Let

$$A(x,t) = D_1, B(x,t) = D_2, T(t) = D_2^{(3/2)} t + D_4, X(x,t) = \sqrt{D_2}\left(D_1 M t + x\right) + D_3, \qquad (29)$$

where $D_i (i = 1,2,3,4)$ are arbitrary constants. The symmetry transformation of Eq. (5) has the form

$$u = D_1 + D_2 U\left(\sqrt{D_2}\left(D_1 M t + x\right) + D_3, D_2^{(3/2)} t + D_4\right). \qquad (30)$$

Combining Eqs. (22) and (30), we have

$$u(x,t) = D_1 + D_2 \frac{-C_3 + 8NC_2^3}{MC_2} - \frac{12NC_2^2 \tanh\left(C_1 + C_2\left(\sqrt{D_2}\left(D_1 M t + x\right) + D_3\right) + C_3\left(D_2^{(3/2)} t + D_4\right)\right)^2}{M}. \qquad (31)$$

Figs. 7 and 8 show the exact solution (31) for different constants and physical parameters.

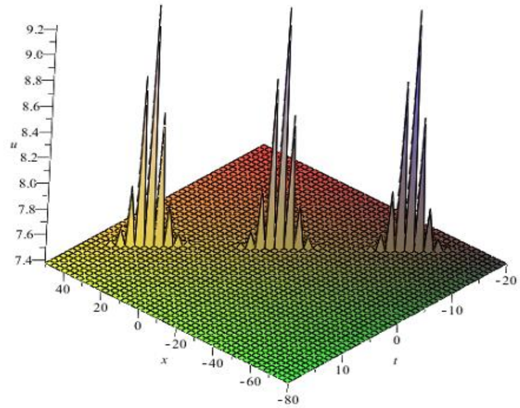

Figure 7: The exact solution (31) for $C_1 = 0.5, C_2 = 2, C_3 = -3,$
$D_1 = 1, D_2 = 3, D_3 = -1, D_4 = 5, M = 1, N = 0.039$

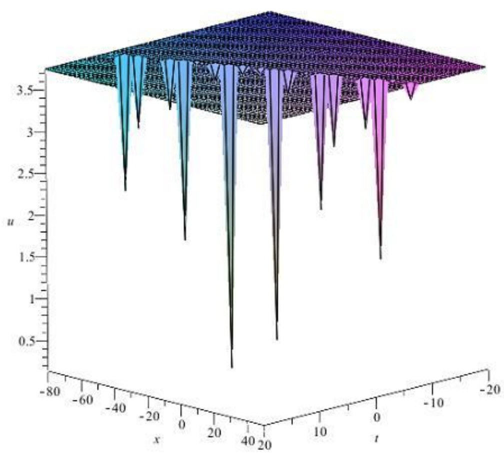

Figure 8: The exact solution (31) for $C_1 = 0.5, C_2 = 2, C_3 = 5,$
$D_1 = -5, D_2 = 2, D_3 = 1, D_4 = 5, M = -1/2, N = 0.039$

4 Conclusions

This paper examines a generalized hyperelastic rod equation with variable coefficients by using Lou's direct method, interestingly, this equation can be used to describe the propagation of travelling waves in the rod composed of compressible hyperelastic materials with variable cross-sections and variable densities. Two sets of symmetry transformations and exact solutions of the equation are obtained. As an example, Lou's direct method is also applied to the equation with constant coefficients which may be found in Dai et al. [Dai and Huo (2002)], and the forms of exact solutions are enriched.

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