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Article

Pythagorean Neutrosophic Planar Graphs with an Application in Decision-Making

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Abstract: Graph theory has a significant impact and is crucial in the structure of many real-life situations. To simulate uncertainty and ambiguity, many extensions of graph theoretical notions were created. Planar graphs play a vital role in modelling which has the property of non-crossing edges. Although crossing edges benefit, they have some drawbacks, which paved the way for the introduction of planar graphs. The overall purpose of the study is to contribute to the conceptual development of the Pythagorean Neutrosophic graph. The basic methodology of our research is the incorporation of the analogous concepts of planar graphs in the Pythagorean Neutrosophic graphs. The significant finding of our research is the introduction of Pythagorean Neutrosophic Planar graphs, a conceptual blending of Pythagorean Neutrosophic and Planar graphs. The idea of Pythagorean Neutrosophic multigraphs and dual graphs are also introduced to deal with the ambiguous situations. This paper investigates the Pythagorean Neutrosophic planar values, which form the edges of the Pythagorean neutrosophic graphs. The concept of Pythagorean Neutrosophic dual graphs, isomorphism, co-weak and weak isomorphism have also been explored for Pythagorean Neutrosophic planar graphs. A decision-making algorithm was proposed with a numerical illustration by using the Pythagorean Neutrosophic fuzzy graph.

Keywords: Pythagorean neutrosophic planar graph; planarity value; isomorphism; dual graphs; multigraph

1 Introduction

Graphs are illustrative representations that express the relation between objects and their data. When the relationships are ambiguous, a graph can be implemented as a fuzzy graph model, which has the same structure as a crisp graph but works with ambiguous data. The fuzzy set theory for dealing with incomplete and vague information originated from the work of Zadeh [1]. Following the fuzzy sets, Intuitionistic sets [2], in which the elements possess membership (μ) and non-membership (ρ) grades with the condition that $\mu + \rho \leq 1$. By considering the vagueness of information, adding some restrictions leads to the extension and development of the neutrosophic set by Smarandache [3], which assigns a truth, indeterminacy, and false membership grade to the elements, with the condition that the sum of the membership grades is within the range of 0 and 3. To expand this concept, Yager [4] proposed the concept of Pythagorean sets, which have an added relaxation in their condition as $\mu^2 + \rho^2 \leq 1$. The fusion of Pythagorean and Neutrosophic sets resulted in the development of the Pythagorean Neutrosophic set, which allows the element to have the membership (μ), indeterminacy (σ) and non-membership grade (γ) with the constraint that $\mu^2 + \rho^2 + \gamma^2 \leq 2$.

Kaufmann [5], based on fuzzy relation [6], developed the idea of Fuzzy Graphs (FGs). Later, Rosenfeld [7] defined the basic properties of fuzzy relations, which are generalized with a fuzzy set as a base set, and fuzzy analogs of graphic theoretical concepts like bridges and trees were established with their properties. Bhattacharya [8] introduced the notions of eccentricity, and center explored how a fuzzy group can be associated with a fuzzy graph. Fundamental operations on FGs and their properties were discussed by Mordeson et al. in [9].

Shannon et al. introduced intuitionistic fuzzy relations in [10], and intuitionistic fuzzy graphs (IFG) with their properties were investigated in [11]. The operations of IFGs were described by Parvathi et al. in [12]. In [13,14], Akram et al. established ideas such as strong IFGs and IF hypergraphs. Pythagorean Fuzzy Graphs (PFGs) and their applications were explored by Naz et al. in [15], and the energy of a Pythagorean Fuzzy Graph (PFG) was studied by Akram et al. in [16], followed by the assertion of some PFG operations by Akram et al. in [17]. Akram et al. [18] proposed certain graphs using the base Pythagorean and the abstraction of the fuzzy dual graph with the investigation of its properties in [19].

Yager proposed the concept of fuzzy multiset in [20], and fuzzy planar graphs were developed along with their properties in [21] and [22]. Alshehri et al. [23] established some exciting proofs of Intuitionistic fuzzy planar graphs, and the concept of bipolar fuzzy planar graphs was described in [24]. Strong neutrosophic graphs were introduced in [25], and single-valued neutrosophic graphs were instituted by Broumi et al. [26]. Akram et al. introduced neutrosophic graphs and neutrosophic soft graphs along with their applications [27]. Single-valued neutrosophic hypergraphs and intuitionistic neutrosophic soft graphs were studied in [28,29]. The recent development of planar graphs in this area can be seen in [30–34]. The concept of the Pythagorean Neutrosophic graphs [35] was developed using the Pythagorean Neutrosophic set [36] and further into other graph-theoretical concepts in [37–40].

In this study, the graph-theoretical results are applied in the Pythagorean Neutrosophic fuzzy environment. The concept of planar graphs is more captivating because of their complexity. In designing circuits, circuits or lines are arranged so they do not intersect to avoid circuit problems, and planar graphs can be used to tackle this problem. Though all developments in fuzzy graph theory have advantages, the Pythagorean Neutrosophic graphs have their advantage with more fuzzified inputs. This research article elaborates on the abstraction of Pythagorean Neutrosophic Multi Graphs (PNMGs), Pythagorean Neutrosophic Planar Graphs (PNPGs), and Pythagorean Neutrosophic Dual Graphs (PNDGs). The potential applications of these graphs can be used to assess and design a variety of real-world challenges. Planarity is a crucial feature that is investigated in this work. A significant addition to the literature is the development of the Pythagorean neutrosophic planar and multigraphs with their characterizations.

The following is the order in which this article is organized: Section 2 deals with introducing the Pythagorean Neutrosophic Multi Graphs (PNMGs) and investigating their properties. Section 3 proposes the concept of the Pythagorean Neutrosophic Planar Graphs (PNPGs) along with the results and the investigation of its characteristics. An algorithm for decision-making using Pythagorean Neutrosophic fuzzy graphs has been proposed with the examination of a practical numerical illustration in Section 4 and the work is concluded in Section 5.

2 Pythagorean Neutrosophic Multigraphs

Definition 2.1. A Pythagorean Neutrosophic Multi Set ($PNMS$) C of a non-void set H is grouped by functions, '*count M*', '*count NM*' and '*count I*' of C symbolized by CM_C, CNM_C, and CI_C and given as $CM_C, CNM_C, CI_C: H \rightarrow R$ with R, a collection of all multisets from interval $[0,1]$. A $PNMS$ C is represented by $C = \left\{< z, \left(\mu_C^1(z), \mu_C^2(z), \ldots, \mu_C^g(z)\right), \left(\sigma_C^1(z), \sigma_C^2(z), \ldots, \sigma_C^g(z)\right), \left(\gamma_C^1(z), \gamma_C^2(z), \ldots, \gamma_C^g(z)\right) > | z \in H\right\}$. where the M sequence $\left(\mu_C^1(z), \mu_C^2(z), \ldots, \mu_C^g(z)\right)$, the I sequence $\sigma_C^1(z), \sigma_C^2(z), \ldots, \sigma_C^g(z)$ and the NM sequence $\left(\gamma_C^1(z), \gamma_C^2(z), \ldots, \gamma_C^g(z)\right)$ may be increasing (or) decreasing order, and sum of $\mu_C^f(z), \sigma_C^f(z), \gamma_C^f(z) \in [0,1]$ satisfies the criteria $0 \leq sup\ \mu_C^f(z) + sup\ \sigma_C^f(z) + sup\ \gamma_C^f(z) \leq 2$ for $z \in H$ and $C = \{< z, \mu_C(z)_f, \sigma_C(z)_f, \gamma_C(z)_f > /z \in H, f = 1, 2, \ldots, g\}$.

Definition 2.2. $C = \{< s, \mu_C(s)_k, \sigma_C(s)_k, \gamma_C(s)_k > /w \in H, k = 1, 2, \ldots, q\}$ and $O = \{< s, \mu_O(s)_k, \sigma_O(s)_k, \gamma_O(s)_k > /s \in H, k = 1 - q\}$ be two PNMSs in H. Then,

1. $C \subseteq O$ iff $\mu_C(s)_k \leq \mu_O(s)_k, \sigma_C(s)_k \leq \sigma_O(s)_k, \gamma_C(s)_k \geq \gamma_O(s)_k$ for $k = 1 - q$ and $s \in H$;
2. $C = O$ iff $C \subseteq O$ and $O \subseteq C$.
3. $C^c = \{< s, \gamma_C(s)_k, 1 - \sigma_C(s)_k, \mu_C(s)_k > /s \in H, k = 1 - q\}$.
4. $C \cup O = \{s, \mu_C(s)_k \vee \mu_O(s)_k, \sigma_C(s)_k \vee \sigma_O(s)_k, \gamma_C(s)_k \wedge \gamma_O(s)_k, /s \in H, k = 1 - q\}$.
5. $C \cap O = \{s, \mu_C(s)_k \wedge \mu_O(s)_k, \sigma_C(s)_k \wedge \sigma_O(s)_k, \gamma_C(s)_k \vee \gamma_O(s)_k, /s \in H, k = 1 - q\}$.

Definition 2.3. Let $C = (\mu_C, \sigma_C, \gamma_C)$ be a Pythagorean Neutrosophic (PN) set on V and $O = \left\{ds, \mu_O(ds)_k, \sigma_O(ds)_k, \gamma_O(ds)_k\ i = 1 - \frac{m}{ds} \in V \times V\right\}$ a PNMS of $V \times V$ with $\mu_O(ds)_k \leq \min\{\mu_O(d), \mu_O(s)\}$, $\sigma_O(ds)_k \leq \min\{\sigma_O(d), \sigma_O(s)\}$, $\gamma_O(ds)_k \leq \max\{\gamma_O(d), \gamma_O(s)\}$, $\forall\ k = 1 - m$. $G = (C, O)$ is a PN Multi Graph (PNMG). $\mu_O(ds)_k, \sigma_O(ds)_k, \gamma_O(ds)_k$ Symbolize the M, I and NM value of ds in G, correspondingly. m represents the count of edges among the vertices. In $PNMG$ G, O represents a PN Multi Edge set ($PNME$).

Example 2.1. Let the multigraph be $G = (V, E)$ with $V = \{a, b, c, d\}$, $E = \{ab, bc, bc, bc, bd\}$.

Let $C = (\mu_C, \sigma_C, \gamma_C)$ be a PN set on V and $O = (\mu_O, \sigma_O, \gamma_O)$ be a $PNME$ set on $V \times V$ defined as

$$C = \{< a, .5, .3, .3 >, < b, .4, .2, .4 >, < c, .5, .4, .3 >, < d, .4, .3, .4 >\},$$

$$O = \{< ab, .3, .2, .3 >< bc, .3, .2, .3 >, < bc, .2, .1, .2 >, < bc, .4, .2, .4 >, < bd, .3, .2, .2 >\}.$$

Definition 2.4. Let $O = \{< ds, \mu_O(ds)_k, \sigma_O(ds)_k, \gamma_O(ds)_k >, k = 1 - m | ds \in V \times V\}$, a $PNME$ in $PNMG$ G. Degree of a vertex $d \in V$ is

$$deg(d) = \left(\sum_{k=1}^{m} \mu_O(ds)_k, \sum_{k=1}^{m} \sigma_O(ds)_k, \sum_{k=1}^{m} \gamma_O(ds)_k\right), \ \forall s \in V.$$

Example 2.2. The vertices a, b, c *and* d in *example* 2.1, hold the degrees

$$deg(a) = (.3, .2, .3),\ deg(b) = (1.2, .7, 1.1),\ deg(c) = (.9, .5, .9),\ deg(d) = (.3, .2, .2).$$

Definition 2.5. *Let* $O = \{(ds,\ \mu_O(ds)_k,\ \sigma_O(ds)_k,\ \gamma_O(ds)_k,\ k = 1\ to\ m/ds\ \in\ V \times V\}$ be a $PNMS$ of $PNMG\ G$. Multi edge ds of G is strong if

$$\frac{1}{2}\ min\ [\mu_C(d),\ \mu_C(s)] \leq \mu_O(ds)_k, \quad \frac{1}{2}\ min\ [\sigma_C(d),\ \sigma_C(s)] \leq \sigma_O(ds)_k,$$

$$\frac{1}{2}\ max\ [\gamma_C(d),\ \gamma_C(s)] \geq\ \gamma_O(ds)_k, \quad for\ all\ k = 1\ to\ m.$$

Definition 2.6. *Let* $O = \left\{(ds,\ \mu_O(ds)_k,\ \sigma_O(ds)_k,\ \gamma_O(ds)_k,\ k = 1\ to\ \frac{m}{ds} \in\ V \times V\right\}$ be a $PNMS$ in $PNMG\ G$.

1. G has order,

$$O(G) = \sum_{d \in V} \mu_C(d),\ \sum_{d \in V} \sigma_C(d),\ \sum_{d \in V} \gamma_C(d),$$

2. G has size,

$$S(G) = \left(\sum_{k=1}^{n} \mu_O(ds)_k, \sum_{k=1}^{n} \sigma_O(ds)_k, \sum_{k=1}^{n} \gamma_O(ds)_k\right), \forall \mathrm{ds} \in \mathrm{V} \times \mathrm{V}.$$

3. The total degree of $d\ \in\ V$ is,

$$td_G(d) = \left(\sum_{k=1}^{n} \mu_O(ds)_k,\ \sum_{k=1}^{n} \sigma_O(ds)_k,\ \sum_{k=1}^{n} \gamma_O(ds)_k\right),\ \ \forall d\ \in\ V.$$

Definition 2.7. Let $O = \{(ds,\ \mu_O(ds)_k,\ \sigma_O(ds)_k,\ \gamma_O(ds)_k,\ k = 1\ to\ m/ds\ \in\ V \times V\}$, a $PNME$ in $PNMG\ G$. G is complete if $\min[\mu_C(d),\ \mu_C(s)] = \mu_O(ds)_k$, $\min[\sigma_C(d),\ \sigma_C(s)] = \sigma_O(ds)_k$, $\max[\gamma_C(d),\ \gamma_C(s)] = \gamma_O(ds)_k$, $\forall\ k = 1 -\ m$ and $\forall\ d, s\ \in\ V$.

Example 2.3. Consider $PNMG\ G$ in Fig. 1. Using the calculation as defined, it is verified that G in Fig. 1 is a complete $PNMG$.

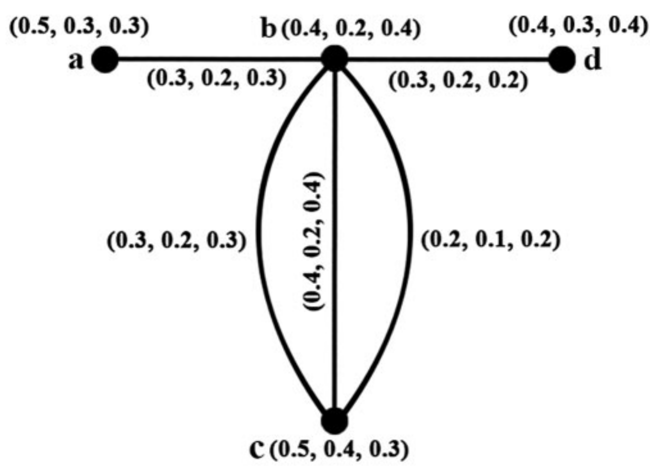

Figure 1: Pythagorean neutrosophic multigraph

Definition 2.8. If each node of G has the same degree of M, I and NM values, then G is a regular $PNMG$.

Definition 2.9. Let G be a $PNMG$ such that $O = \{ds,\ \mu_O(ds)_i,\ \sigma_O(ds)_i,\ \gamma_O(ds)_i, i = 1\ to\ m/ds \in V \times V\}$.

1. The edge ds has degree

$$D_G(ds) = \left((deg_\mu)_G(d) + (deg_\mu)_G(s) - 2\mu_O(ds)_i\right), ((deg_\sigma)_G(d) + (deg_\sigma)_G(s) - 2\sigma_O(ds)_i), ((deg_\sigma)_G(d) + (deg_\sigma)_G(s) - 2\sigma_O(ds)_i).$$

2. The edge ds has degree

$$tD_G(ds) = \left((deg_\mu)_G(d) + (deg_\mu)_G(s) - 2\mu_O(ds)_i\right), ((deg_\sigma)_G(d) + (deg_\sigma)_G(s) - 2\sigma_O(ds)_i), ((deg_\sigma)_G(d) + (deg_\sigma)_G(s) - 2\sigma_O(ds)_i),$$

where $((ds)_i)$ is the i^{th} edge between d & s.

Definition 2.10. If the degree of M, I and NM of every edge in $PNMG\ G$ are equal, then G is Edge Regular (ER).

Example 2.4. Degree of the edges in *Example* 1 are $D_G(ab) = (1.2, .7, 1.1)$, $D_G(bd) = (1.2, .7, 1.2)$, $D_G(bc) = (1.2, .7, 1.1)$, $D_G(bc) = (1.1, .7, 1)$, $D_G(bc) = (1.3, .8, 1.2)$ and total degree of edges are $tD_G(ab) = (1.5, 0.9, 1.4)$, $tD_G(bd) = (1.5, 0.9, 1.4)$, $tD_G(bc) = (1.5, 0.9, 1.4)$, $tD_G(bc) = (1.5, 0.9, 1.4)$, $tD_G(bc) = (1.5, 0.9, 1.4)$.

Theorem 2.1. Let $G = (C, O)$ be a $PNMG$. If G is regular and edge regular $PNMG$, then the $M\ \mu(ds)_k$, $I\ \sigma(ds)_k$, $NM\ \gamma(ds)_k$ for every line $(ds) \in V \times V$ are constants.

Proof. Consider, $G = (C, O)$, a $PNMG$; G is regular and edge regular $PNMG$. There exists constants p_1, p_2, p_3 and q_1, q_2, q_3 for regular and edge regular correspondingly so that for every node,

$deg\ _G(d) = \left((deg_\mu)_G(d), (deg_\sigma)_G(d), (deg_\sigma)_G(d)\right) = p_1, p_2, p_3$ for every $edge\ ds \in V \times V$,

$$\begin{aligned} DG(ds) &= ((D\mu)_G(ds), (D\sigma)_G(ds), (D\gamma)_G(ds)), \\ &= \left((deg_\mu)_G(d) + (deg_\mu)_G(s) - 2\mu_O(ds)_k\right), ((deg_\sigma)_G(d) + (deg_\sigma)_G(s) - 2\sigma_O(ds)_k), ((deg_\sigma)_G(d) \\ &\quad + (deg_\sigma)_G(s) - 2\gamma_O(ds)_k). \\ &= (q_1, q_2, q_3). \end{aligned}$$

Thus, for the M, I and NM values, $p_1 + p_1 - 2\mu_O(ds)_k = 2q_1$, $p_1 + p_1 - \mu_O(ds)_k = 2q_1$, $2p_1 - 2q_1 = 2\mu_O(ds)_k$, $p_1 - q_1 = \mu_O(ds)_k$, Similarly, $p_2 - q_2 = \sigma_O(ds)_k$ and $p_3 - q_3 = \gamma_O(ds)_k$.

Thus, the M, I and NM values of a regular $PNMG$ with edge regular are constant.

Theorem 2.2. Let $G = (C, O)$ be a $PNMG$ on $G^* = (V, E)$. If G^* is y–regular multigraph, $\mu_O(ds)_k$, $\sigma_O(ds)_k$ and $\gamma_O(ds)_k$ are constants for every edge $ds \in V \times V$, then G is a regular and edge regular $PNMG$.

Proof. Consider $G^* = (V, E)$ as a $y-$ regular multigraph. Consider $\mu_O(ds)_k = q_1, \sigma_O(ds)_k = q_2$ and $\gamma_O(ds)_k = q_3$. For every vertex $d \in V$,

$$\begin{aligned} deg_G(d) &= \left((deg_\mu)_G(d), (deg_\sigma)_G(d), (deg_\sigma)_G(d)\right), \\ &= \left(\sum_{d \neq s} \mu_O(ds)_k, \sum_{d \neq s} \sigma_O(ds)_k, \sum_{d \neq s} \gamma_O(ds)_k\right), \\ &= (y \times q_1, y \times q_2, y \times q_3), \end{aligned}$$

$$= \left(\sum_{d \neq s} \mu_O(yx)_k,\ \sum_{d \neq s} \sigma_O(yx)_k,\ \sum_{d \neq s} \gamma_O(yx)_k \right),$$
$$= \left((deg_\mu)_G(s),\ (deg_\sigma)_G(s), (deg_\sigma)_G(s) \right) = deg_G(s).$$

For every edge, $ds \in V \times V$,

$$DG(ds) = ((D\mu)_G(ds),\ (D\sigma)_G(ds), (D\gamma)_G(ds))$$
$$= \left((deg_\mu)_G(d) + (deg_\mu)_G(s) - 2\mu_O(ds)_k \right),\ ((deg_\sigma)_G(d) + (deg_\sigma)_G(s) - 2\sigma_O(ds)_k),$$
$$((deg_\sigma)_G(d) + (deg_\sigma)_G(s) - 2\gamma_O(ds)_k).$$
$$= ((y \times q_1) + (y \times q_1) - 2(q_1), (y \times q_2) + (y \times q_2)$$
$$- 2(q_2), (y \times q_3) + (y \times ; q_3)\ -\ 2(q_3))$$
$$= (2q_1\ (y\ -\ 1),\ 2q_2\ (y\ -\ 1),\ 2q_3\ (y\ -\ 1)).$$

Thus, G is regular and edge regular $PNMG$.

Definition 2.11. The strength of PN edge fj is determined by the value

$$S_{fi} = \left((S_\mu)_{fi},\ (S_\sigma)_{fi}, (S_\gamma)_{fi} \right) = \left(\frac{\mu_O(fj)_i}{\min(\mu_C(f), \mu_C(j))}, \frac{\sigma_O(fj)_i}{\min(\sigma_C(f), \sigma_C(j))}, \frac{\gamma_O(fj)_i}{\min(\gamma_C(f), \gamma_C(j))} \right).$$

An edge fj of a $PNMG$ is PN strong if $(S_\mu)_{fj} \geq 0.5, (S_\sigma)_{fj} \geq 0.5,\ (S_\gamma)_{fj} \geq 0.5$.

Definition 2.12. Let G be a $PNMG$ such that O has 2 edges $(ab, \mu_o(ab)_k, \sigma_o(ab)_k,\ \gamma_o(ab)_k)$ and $(cd,\ \mu_o(ab)_y,\ \sigma_o(ab)_y,\ \gamma_o(ab)_y)$, which intersect at P, k and y are fixed integers. At P, the intersecting value is defined by,

$$S_P = \left((S_\mu)_P,\ (S_\sigma)_P, (S_\gamma)_P \right) = \left(\frac{(S_\mu)_{ab} + (S_\mu)_{cd}}{2} + \frac{(S_\sigma)_{ab} + (S_\sigma)_{cd}}{2} + \frac{(S_\gamma)_{ab} + (S_\gamma)_{cd}}{2} \right).$$

The planarity decreases when the number of points of intersection in $PNMG$ increases.

S_P is inversely proportional to the planarity for $PNMG$.

3 Pythagorean Neutrosophic Planar Graphs

The concept of the Pythagorean Neutrosophic planar graphs has been discussed.

Definition 3.1. Let G be a $PNMG$ and the point of intersections among the edges be $P_1,\ P_2, \ldots,\ P_m$, G is a PN Planar Graph ($PNPG$) with PN Planarity Value ($PNPV$) $f = (f_\mu, f_\sigma, f_\gamma)$, where

$$f = (f_\mu, f_\sigma, f_\gamma) = \left(\frac{1}{1 + \{(S_\mu)_{P_1} + (S_\mu)_{P_2} + \ldots + (S_\mu)_{P_m}\}}, \frac{1}{1 + \{(S_\sigma)_{P_1} + (S_\sigma)_{P_2} + \ldots + (S_\sigma)_{P_m}\}}, \frac{1}{1 + \{(S_\gamma)_{P_1} + (S_\gamma)_{P_2} + \ldots + (S_\gamma)_{P_m}\}} \right).$$

$0 \leq f_\mu \leq 1,\ 0 \leq f_\sigma \leq 1,\ 0 \leq f_\gamma \leq 1$. The $PNPV$ is $(1, 1, 1)$ for a geometrical representation of $PNPG$ if it has no intersecting point.

Example 3.1. Take a multigraph $G^* = (V,\ E)$ such that $V = \{a,\ b,\ c,\ d,\ e\}$, $E = \{ab,\ ac,\ ad,\ ad,\ bc, bd,\ cd,\ ce,\ ae,\ de,\ be\}$. Let $C = (\mu_C,\ \sigma_C,\ \gamma_C)$ be a PN set on V and $O = (\mu_O,\ \sigma_O,\ \gamma_O)$ be a $PNME$

set on $V \times V$ as are described as,

$C = \{< a, .5, .5, .2 >, < b, .6, .7, .3 >, < c, .4, .6, .4 >, < d, .7, .5, .3 >, < e, .8, .6, .5 >\}$,

$O = \{< ab, .5, .4, .2 >, < ac, .4, .5, .3 >, < ad, .5, .5, .3 >, < ad, .4, .4, .2 >,$
$< bc, .4, .6, .4 >, < bd, .6, .5, .2 >, < cd, .3, .5, .3 >, < ae, .5, .5, .4 >,$
$< ce, .3, .6, .5 >, < de, .7, .5, .4 >, < be, .6, .6, .4 >\}$.

The $PNMG$ has two points of intersection in Fig. 2 (P_1 *and* P_2).P_1 is a point among the lines $(ad, .5, .5, .3)$ and $(bc, .4, .6, .4)$ and P_2 is a point among the edges $(ad, .4, .4, .2)$ and $(bc, .4, .6, .4)$.

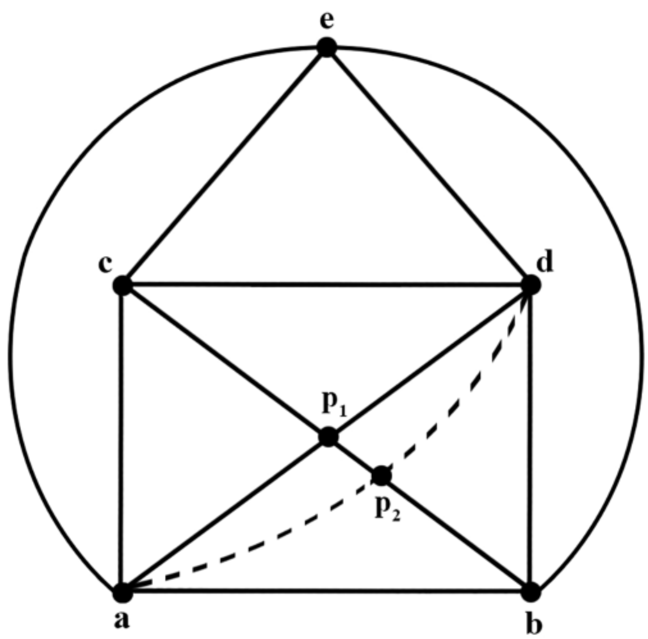

Figure 2: Pythagorean neutrosophic planar graph

The strength for the edges ab, ad and bc are $S_{ad} = \left(\frac{.5}{.5}, \frac{.5}{.5}, \frac{.3}{.3}\right) = (1, 1, 1)$, $S_{ad} = \left(\frac{.4}{.5}, \frac{.4}{.5}, \frac{.2}{.3}\right) = (.8, .8, .67)$, $S_{bc} = \left(\frac{.4}{.4}, \frac{.6}{.6}, \frac{.4}{.4}\right) = (1, 1, 1)$. For P_1, intersecting value S_{P_1} is $(1, 1, 1)$ and for P_2, S_{P_2} is $(.9, .9, .835)$.

Therefore $PNPV$ for the $PNMG$ given in Fig. 2 is $(.345, .345, .353)$.

Theorem 3.1. Let G be a complete $PNMG$. The $PNPV, f = (f_\mu, f_\sigma, f_\gamma)$ of G is given by $f_\mu = \frac{1}{1 + n_p}$, $f_\sigma = \frac{1}{1 + n_p}$ and $f_\gamma = \frac{1}{1 + n_p}$ such that $f_\mu + f_\sigma + f_\gamma \leq 3$, where n_p is the count of point of intersection among the lines in G.

Definition 3.2. A $PNPG$ G is called strong ($\mathcal{S}PNPG$) if the $PNPV f = (f_\mu, f_\sigma, f_\gamma)$ of the graph is $f_\mu \geq 0.5, f_\sigma \geq 0.5, f_\gamma \geq 0.5$.

Theorem 3.2. Let G be a $\mathcal{S}$ $PNPG$. The number of points of intersections among $\mathcal{S}$ lines in G is utmost one.

Proof. Let G be a $\mathcal{S}$ $PNPG$. Consider G has at least 2 points of intersections P_1 and P_2 between 2 S lines in G. For any $\mathcal{S}$ edge $(wq, \mu_O(wq)_i, \sigma_O(wq)_i, \gamma_O(wq)_i)$, $\mu_O(wq)_i \geq \frac{1}{2}\min\{\mu_C(w), \mu_C(q)\}$, $\sigma_O(wq)_i \geq \frac{1}{2}\min\{\sigma_C(w), \sigma_C(q)\}$, $\gamma_O(wq)_i \leq \frac{1}{2}\max\{\gamma_C(w), \gamma_C(q)\}$.

Thus, $(S_\mu)_{wq}, (S_{\mu\sigma})_{wq}, (S_\gamma)_{wq} \geq .5$. Thus, for two intersecting $\mathcal{S}$ edges $(wq, \mu_O(wq)_k, \sigma_O(wq)_k, \gamma_O(wq)_k)$ and $(cd, \mu_O(cd)_j, \sigma_O(cd)_j, \gamma_O(cd)_j)$,

$$\frac{(S_\mu)_{wq} + (S_\mu)_{cd}}{2} + \frac{(S_\sigma)_{wq} + (S_\sigma)_{cd}}{2} + \frac{(S_\gamma)_{wq} + (S_\gamma)_{cd}}{2} \geq .5,$$

(i.e.,) $(S_\mu)_{P_1}, (S_\sigma)_{P_1} \geq .5, (S_\gamma)_{P_1} \leq .5$, Likewise, $(S_\mu)_{P_2}, (S_\sigma)_{P_2} \geq .5, (S_\gamma)_{P_2} \leq .5$.

$$\Rightarrow 1 + (S_\mu)_{P_1} + (S_\mu)_{P_2} \geq 2, 1 + (S_\sigma)_{P_1} + (S_\sigma)_{P_2} \geq 2, 1 + (S_\gamma)_{P_1} + (S_\gamma)_{P_2} \geq 2.$$

$$f_\mu = \frac{1}{1 + (S_\mu)_{P_1} + (S_\mu)_{P_2}} \leq .5\, f_\sigma = \frac{1}{1 + (S_\sigma)_{P_1} + (S_\sigma)_{P_2}} \leq .5\, f_\gamma = \frac{1}{1 + (S_\gamma)_{P_1} + (S_\gamma)_{P_2}} \geq .5.$$

This becomes a contradiction to the fact PN graph is a $\mathcal{S}PNPG$. Thus the number of points of intersections between $\mathcal{S}$ edges cannot be two. If the count of point of intersections of PN edges increases, the $PNPV$ decreases. When the count of the point of intersection of $\mathcal{S}$ edges is 1, then the $PNPV$ $f_\mu \geq 0.5, f_\sigma \geq 0.5, f_\gamma \geq 0.5$. A $\mathcal{S}PNPG$ is a $PNPG$ without any crossing between edges. Thus, the largest number of points of intersections among the $\mathcal{S}$ edges in G is 1 parameter. The region bounded by PN edges is a face of a PN graph. Every PN Face (PNF) in its boundary is characterized by PN edges. If every edge in the boundary of a PNF have $\mu_O, \sigma_O, \gamma_O$ values $(1, 1, 1)$ and $(0, 0, 0)$, then it is a crisp face. When one among those edges is removed or has $\mu_O, \sigma_O, \gamma_O$ values $(0, 0, 0)$ and $(1, 1, 1)$ correspondingly, the PNF does not exist. The existence of a PNF depends on the minimal strength of PN edges in its boundary. A PNF and its $\mu_O, \sigma_O, \gamma_O$ Values of PNG are expressed below.

Definition 3.3. Let G be a $PNPG$ and $O = \{(ds, \mu_O(ds)_k, \sigma_O(ds)_k, \gamma_O(ds)_k, k = 1 \text{ to } m/ds \in V \times V\}$. A PNF of G is a region and is bounded by the set of PN lines $E' \subset E$, of a pictorial demonstration of G. The M, I *and* NM of PNF are:

$$\min\left\{\frac{\mu_O(ds)_k}{\min\{\mu_C(d), \mu_C(s)\}}, k = 1, 2, \ldots \frac{m}{ds} \in E'\right\},$$
$$\min\left\{\frac{\sigma_O(ds)_k}{\min\{\sigma_C(d), \sigma_C(s)\}}, k = 1, 2, \ldots \frac{m}{ds} \in E'\right\},$$
$$\max\left\{\frac{\gamma_O(ds)_k}{\max\{\gamma_C(d), \gamma_C(s)\}}, k = 1, 2, \ldots \frac{m}{ds} \in E'\right\}.$$

Definition 3.4. A PNF is $(\mathcal{S})$ PNF if the value of M, I is larger than 0.5, NM is below 0.5, and weak otherwise. The infinite region in every $PNPG$ is termed as an outer PNF, and other faces are called inner $PNFs$.

Example 3.2. The $PNPG$ as in Fig. 3, has the following faces: PNF F_1 is bounded by the edges $(ab, .4, .4, .1)$, $(bc, .5, .5, .1)$, $(ac, .4, .4, .1)$. Outer PNF F_2 surrounded by edges $(ac, .4, .4, .1)$, $(ad, .4, .4, .1)$, $(bd, .5, .5, .1)$, $(bc, .5, .5, .1)$. PNF F_3 is bounded by lines $(ab, .4, .4, .1)$, $(bd, .5, .5, .1)$, $(ad, .4, .4, .1)$.

Clearly, the M, I and NM value of a PNF F_1 is $(.8, .8, .5)$. Thus, F_1 is a $\mathcal{S}PNF$.

Definition 3.5. Let G be a PNPG and let $O = \{(ds, \mu_O(ds)_k, \sigma_O(ds)_k, \gamma_O(ds)_k, k = 1 - m/ds \in V \times V\}$. Let $F_1, F_2, \ldots, F_k$ be the $\mathcal{S}PNFs$ of G. The PN Dual Graph ($PNDG$) of G is a $PNPG$ $G' = (V', C', O')$ with $V = \{x_k, k = 1 - k\}$, and the vertex x_k of G' is for F_k of G. The M, I, NM values of vertices are $C' = (\mu'_C, \sigma'_C, \gamma'_C) : V' \rightarrow [0,1]^3$ such that

$$\mu'_C(x_k) = \max\left\{\mu'_O(ua)_k,\ k = 1\ to\ \frac{p}{ua}\ is\ in\ the\ boundary\ of\ \mathcal{S}PNF\ F_k\right\},$$

$$\sigma'_C(x_k) = \max\left\{\sigma'_O(ua)_k,\ k = 1\ to\ \frac{p}{ua}\ is\ in\ the\ boundary\ of\ \mathcal{S}PNF\ F_k\right\},$$

$$\gamma'_C(x_k) = \min\left\{\gamma'_O(ua)_k,\ k = 1\ to\ \frac{p}{ua}\ is\ in\ the\ boundary\ of\ \mathcal{S}PNF\ F_k\right\}.$$

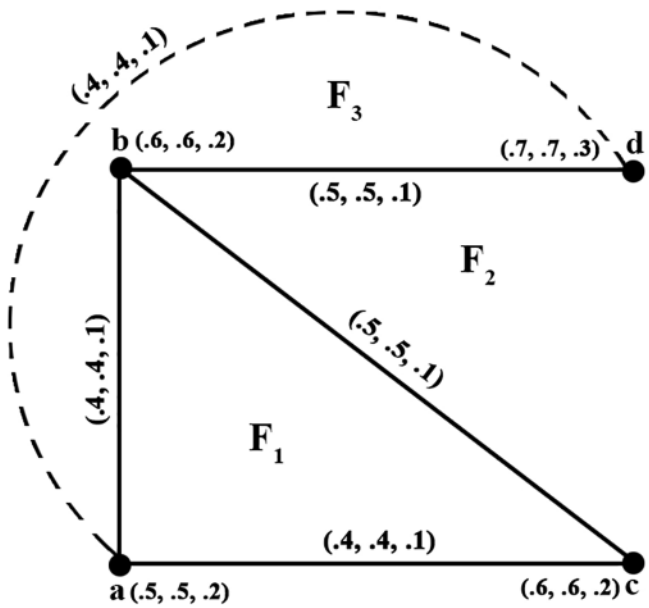

Figure 3: Faces in pythagorean neutrosophic planar graph

Two common faces F_k and F_b of G might exist between one common line. There may be more than 1 edge among 2 vertices x_k and x_b in $PNDG\ G'$. $\mu'_O\ (x_k x_b)$ represent the M value of the l^{th} edge among x_k and x_b and $\gamma'_O(x_k x_b)$ represent the NM value of the l^{th} edge amidst x_k and x_b. M, I and NM values of PN edges of the $PNDG$ are presented by $\mu'_O(x_k x_b)_l = \mu'_O(ua)_b$, $\sigma'_O(x_k x_b)_l = \sigma'_O(ua)_b$, $\gamma'_O(x_k x_b)_l = \gamma'_O(ua)_b$, with $(ua)_b$ is an edge in the boundary between 2 $\mathcal{S}PNF$ F_k and F_b and $l = 1$ to S, where $\mathcal{S}$ is the count of lines among x_k and x_b. $PNDG$ of $PNPG$ does not hold point of intersection of edges for a some representation, so it is $PNPG$ with $PNPV$ (1, 1, 1). The PNF of $PNDG$ can be similarly expressed as in $PNPG$.

Theorem 3.3. Let G be a $PNPG$ whose count of vertices, total of PN edges, count of $\mathcal{S}PNF$ are symbolized by m, p, n correspondingly. G' be the $PNDG$ of G, then count of vertices, edges, PNF of G' equals m, p, n correspondingly.

Theorem 3.4. Let $G = (V, C, O)$ be a $PNPG$ without weak lines and the $PNPG$ of G be $G' = (V', C', O')$. The M, I and NM values of PN lines of G' equals values of G.

Definition 3.6. Let $G = (C, O)$ be a $PNPG$ where $O = \left\{(ds, \mu_O(ds)_k, \sigma_O(ds)_k, \gamma_O(ds)_k, b) = 1\ to\ \frac{n}{ab} \in V \times V\right\}$. Let $F_1, F_2, \ldots, F_k$ be $\mathcal{S}PNFs$ of G. Then $PNDG$ of G is a $PNPG$ $G' = (C', O')$, where $V' = \{r_b, b = 1\ to\ K\}$ and the vertex r_b of G' is taken for F_b of G.

The M, I, NM by mapping $G' = (V', C', O'): V' \to [0,1]^3$ such that

$$\mu_C'(r_b) = max\ \left\{\mu_O'(ua)_b,\ b = 1\ to\ \frac{m}{ab}\ is\ in\ the\ neighbourhood\ of\ \mathcal{S}PNF\ F_b\right\},$$

$$\sigma_C'(r_b) = max\ \left\{\sigma_O'(ua)_b,\ b = 1\ to\ \frac{m}{ab}\ is\ in\ the\ neighbourhood\ of\ \mathcal{S}PNF\ F_b\right\},$$

$$\gamma_C'(r_b) = min\ \left\{\gamma_O'(ua)_b,\ b = 1\ to\ \frac{m}{ab}\ is\ in\ the\ neighbourhood\ of\ \mathcal{S}PNF\ F_b\right\}.$$

Between F_k and F_b of $G,$ at least one common edge may occur. Among two vertices, there may exist beyond a single edge $r_k r_b$ in PNDG G'. M, I and NM values of PN edges of $PNDG$ are $\mu_C'(r_k r_b)_S = \mu_O^S(ua)_k$, $\sigma_C'(r_k r_b)_S = \sigma_O^S(ua)_k$, $\gamma_C'(r_k r_b)_S = \gamma_O^S(ua)_k$ where $(ab)^S$ is in the surrounding among $\mathcal{S}PN$ faces F_k and F_b and $S = 1 - l$, is the count of common edges in the neighborhood of F_k and F_b. The $PNDG\ G'$of $PNDG\ G$has no crossing among lines for some definite geometric representation, $PNPG$ of $PNPV$ (1, 1, 1).

Example 3.3. Take a $PNG\ G = (V, C, O)$ as displayed in Fig. 4 with $V = \{a_1, a_2, a_3, a_4, a_5\}$. Let C and O be PN vertex set and PN edge set.

$C =< (a_1, .7, .5, .3), (a_1, .69, .55, .4), (a_1, .35, .45, .5), (a_4, .76, .8, .3) > .$

$O =< (a_1a_2, .6, .48, .2), (a_2a_3, .3, .4, .43), (a_3a_4, .3, .4, .45), (a_4a_1, .65, .4, .25), (a_1a_3, .3, .4, .4) > .$

$PNF\ F_1$ is enclosed by the edges $(a_1a_3, .3, .4, .4)$, $(a_3a_4, .3, .4, .45)$, $(a_4a_1, .65, .4, .25)$.

$PNF\ F_2$ is enclosed by the edges $(a_1a_2, .6, .48, .2)$, $(a_2a_3, .3, .4, .43)$, $(a_1a_3, .3, .4, .4)$.

$PNF\ F_3$ is enclosed by the edges $(a_1a_2, .6, .48, .2)$, $(a_2a_3, .3, .4, .43)$, $(a_3a_4, .3, .4, .45)$, $(a_4a_1, .65, .4, .25)$.

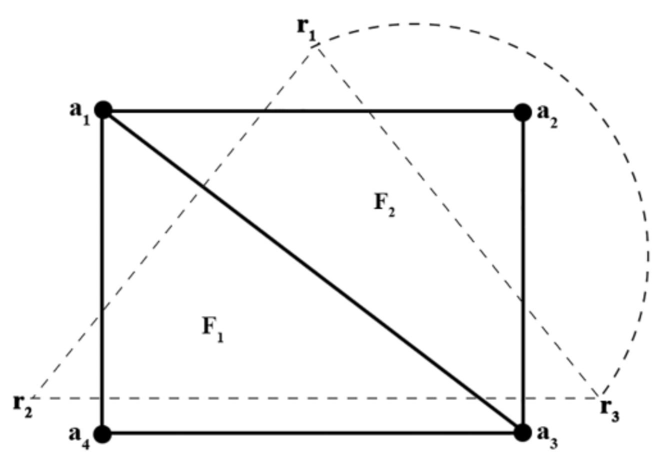

Figure 4: Pythagorean neutrosophic dual graph

We symbolize the vertices of PN Dual Graph $(PNDG)$ by a dot and edges by dashed lines. We take a vertex for each face of $PNDG$ with $V' = \{r_1, r_2, r_3, r_4\}$.

$\mu_{C'}(r_1) = max\ \{.3, .3, .65\} = .65, \sigma_{C'}(r_1) = max\ \{.4, .4, .4\} = .4, \gamma_{C'}(r_1) = min\ \{.4, .45, .25\} = .25,$

$\mu_{C'}(r_2) = max\ \{.6, .3, .3\} = .6, \sigma_{C'}(r_2) = max\ \{.48, .4, .4\} = .48, \gamma_{C'}(r_2) = min\ \{.2, .43, .4\} = .2,$

$$\mu_{C'}(r_3) = max\ \{.6, .3, .3, .65\} = .65,\ \sigma_{C'}(r_3) = max\ \{.48, .4, .4, .4\} = .48,\ \gamma_{C'}(r_3) = min\ \{.2, .43, .45, .25\} = .2$$

The vertex set V' has the vertices $< r_1, (.65, .4, .25) >, < r_2, (.6, .48, .2) >, < r_3, (.65, .48, .2) >$.

There is one common edge a_1a_3 amidst F_1, F_2 in G. Thus, there exists a single line among vertices r_1 and r_2 in $PNDG$ of G. The edges for the $PNDG$ are constructed as in Fig. 4.

Definition 3.7. An isomorphism of two $PNPGs$ G_1 and G_2, $y : G_1 \to G_2$ is a bijective mapping $y : V_1 \to V_2$ that holds the following

1. $\mu_{C_1}(r) = \mu_{C_2}(y(r)),\ \sigma_{C_1}(r) = \sigma_{C_2}(y(r)),\ \gamma_{C_1}(r) = \gamma_{C_2}(y(r))$.
2. $\mu_{O_1}(rs) = \mu_{O_2}(y(r)\ y(s)),\ \sigma_{O_1}(rs) = \sigma_{O_2}(y(r)\ y(s)),\ \gamma_{O_1}(rs) = \gamma_{O_2}(y(r)\ y(s)),\ \forall\ r \in V_1,\ \forall\ rs \in E_1$.

Example 3.4. Consider two $PNPGs$ $G_1 = (C_1, O_1)$ and $G_2 = (C_2, O_2)$ as in Fig. 5 such that

$C_1 = \{< \mathrm{a}_1, .8, .7, .2 >, < \mathrm{a}_2, .7, .6, .4 >, < \mathrm{a}_3, .6, .5, .5 >, < \mathrm{a}_4, .5, .4, .4 >\}$,

$O_1 = \{< \mathrm{a}_1\mathrm{a}_2, .6, .5, .3 >, < \mathrm{a}_2\mathrm{a}_3, .5, .4, .4 >, < \mathrm{a}_3\mathrm{a}_4, .4, .3, .3 >, < \mathrm{a}_4\mathrm{a}_1, .3, .2, .3 >\}$,

$C_2 = \{< \mathrm{r}_1, .5, .4, .4 >, < \mathrm{r}_3, .7, .6, .4 >, < \mathrm{r}_2, .6, .5, .5 >, < \mathrm{r}_4, .8, .7, .2 >\}$,

$O_2 = \{< \mathrm{r}_1\mathrm{r}_2, .4, .3, .3 >, < \mathrm{r}_2\mathrm{r}_3, .5, .4, .4 >, < \mathrm{r}_3\mathrm{r}_4, .6, .5, .3 >, < \mathrm{r}_4\mathrm{r}_1, .3, .2, .3 >\}$.

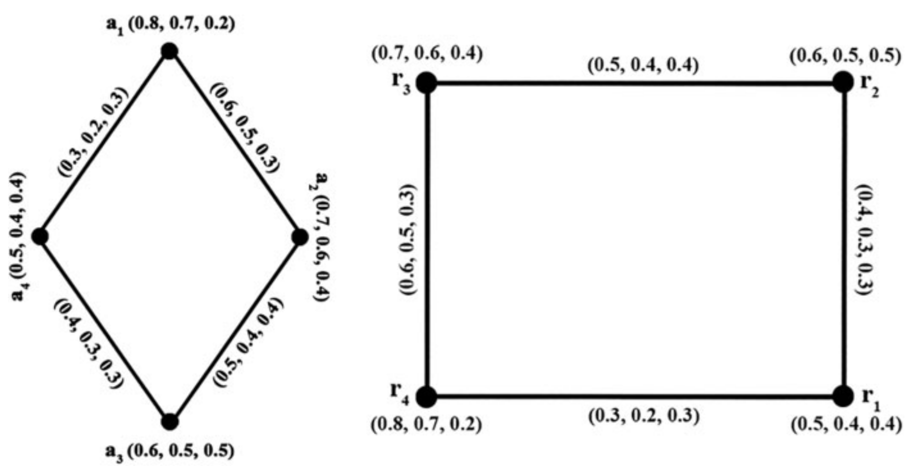

Figure 5: Pythagorean neutrosophic planar graph

$y : V_1 \to V_2$ given by $y(a_1) = r_4,\ y(a_2) = r_3,\ y(a_3) = r_2,\ y(a_4) = r_1$ satisfies

$\mu_{C_1}(r_k) = \mu_{C_2}(y(r_k)),\ \sigma_{C_1}(r_k) = \sigma_{C_2}(y(r_k)),\ \gamma_{C_1}(r_k) = \gamma_{C_2}(y(r_k)),\ \mu_{O_1}(r_kr_b) = \mu_{O_2}(y(r_k)y(r_b)), \sigma_{O_1}(r_kr_b) = \sigma_{O_2}(y(r_k)y(r_b)),\ \gamma_{O_1}(r_kr_b) = \gamma_{O_2}(y(r_k)y(r_b))$ for all $r_k \in V_1$, $r_kr_b \in E_1$, where $i, j = 1$ to 4. Thus G_1 is isomorphic to G_2.

The M, I and NM of the edges of $PNDG$ are $\mu_{O'}(r_1r_2) = \mu_O(a_1a_3) = .3$, $\sigma_{O'}(r_1r_2) = \sigma_O(a_1a_3) = .4$, $\gamma_{O'}(r_1r_2) = \gamma_O(a_1a_3) = .4$, $\mu_{O'}(r_2r_3) = \mu_O(a_1a_2) = .6$, $\sigma_{O'}(r_2r_3) = \sigma_O(a_1a_2) = .48$, $\gamma_{O'}(r_2r_3) = \gamma_O(a_1a_2) = .2$, $\mu_{O'}(r_3r_1) = \mu_O(a_3a_4) = .65$, $\sigma_{O'}(r_3r_1) = \sigma_O(a_3a_4) = .4$, $\gamma_{O'}(r_3r_1) = \gamma_O(a_3a_4) = .45$, $\mu_{O'}(r_1r_3) = \mu_O(a_4a_1) = .65$, $\sigma_{O'}(r_1r_3) = \sigma_O(a_4a_1) = .4$, $\gamma_{O'}(r_1r_3) = \gamma_O(a_4a_1) = .25$, $\mu_{O'}(r_2r_3) = \mu_O(a_2a_3) = .3$, $\sigma_{O'}(r_2r_3) = \sigma_O(a_2a_3) = .4$, $\gamma_{O'}(r_2r_3) = \gamma_O(a_2a_3) = .43$.

Thus, the $PNDG$ edge set is,

$O' = <(r_1r_2, .3, .4, .4), (r_2r_3, .6, .48, .2), (r_3r_1, .65, .4, .45), (r_1r_3, .65, .4, .25), (r_2r_3, .3, .4, .43)>$. Thus G_1 is a PNDG of G_2.

Definition 3.8. A weak isomorphism of two $PNPGs$ G_1 and G_2, $y: G_1 \to G_2$ is a bijective mapping $y: V_1 \to V_2$ that holds the following:

1. y is a homomorphism.
2. $\mu_{C_1}(r) = \mu_{C_2}(y(r)), \sigma_{C_1}(r) = \sigma_{C_2}(y(r)), \gamma_{C_1}(r) = \gamma_{C_2}(y(r)) \, \forall \, r \in V_1$.

Example 3.5. Consider two $PNPG$, $G_1 = (C_1, O_1)$ and $G_2 = (C_2, O_2)$ as in Fig. 6 such that

$C_1 = \{< \mathrm{a}_1, .9, .5, .3 >, < \mathrm{a}_2, .8, .6, .2 >, < \mathrm{a}_3, .7, .4, .4 >, < \mathrm{a}_4, .6, .3, .4 >, < \mathrm{a}_5, .5, .4, .3 >\}$,

$O_1 = \{< \mathrm{a}_1\mathrm{a}_2, .7, .4, .2 >, < \mathrm{a}_2\mathrm{a}_3, .6, .4, .3 >, < \mathrm{a}_3\mathrm{a}_4, .4, .3, .3 >, < \mathrm{a}_4\mathrm{a}_1, .5, .3, .2 >,$

$< \mathrm{a}_4\mathrm{a}_5, .5, .3, .2 >, < \mathrm{a}_5\mathrm{a}_1, .4, .3, .1 >, < \mathrm{a}_2\mathrm{a}_5, .4, .3, .15 >\}$,

$C_2 = \{< \mathrm{r}_1, .7, .4, .4 >, < \mathrm{r}_2, .6, .3, .4 >, < \mathrm{r}_3, .8, .6, .2 >, < \mathrm{r}_4, .5, .4, .3 >, < \mathrm{r}_5, .9, .5, .3 >\}$,

$O_2 = \{< \mathrm{r}_1\mathrm{r}_2, .4, .2, .1 >, < \mathrm{r}_2\mathrm{r}_3, .5, .3, .2 >, < \mathrm{r}_3\mathrm{r}_5, .5, .2, .1 >, < \mathrm{r}_3\mathrm{r}_4, .3, .2, .1 >, < \mathrm{r}_2\mathrm{r}_4, .4, .2, .1 >$

$< \mathrm{r}_4\mathrm{r}_5, .2, .2, .5 >\}$.

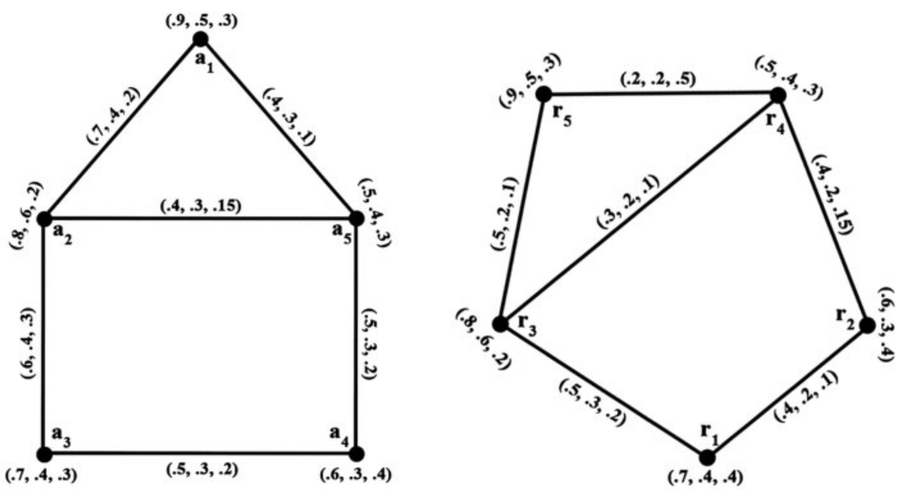

Figure 6: Pythagorean neutrosophic planar graphs

A mapping y: $V_1 \to V_2$ given by $y(a_1) = r_5$, $y(a_2) = r_3$, $y(a_3) = r_1$, $y(a_4) = r_2$, $y(a_5) = r_4$ satisfies $\mu_{C_1}(r_k) = \mu_2(y(r_k), \sigma_{C_1}(r_k) = \sigma_2(y(r_k))$, $\gamma_{C_1}(r_k) = \gamma_2(y(r_k))$ for all $r_k \in V_1$, where $k, b = 1, 2, 3, 4, 5$.

But $\mu_{C_1}(r_kr_b) \neq \mu_{C_2}(y(r_k)\, y(r_b)), \sigma_{C_1}(r_kr_b) \neq \sigma_{C_2}(y(r_k)\, y(r_b)), \gamma_{C_1}(r_kr_b) \neq \gamma_{C_2}(y(r_k)\, y(r_b))$.

Thus G_1 is a weak isomorphic to G_2.

Definition 3.9. A co-weak isomorphism of two $PNPGs$ G_1 and G_2, y: $G_1 \to G_2$ is a bijective mapping $y: V_1 \to V_2$ that holds

1. y is a homomorphism.
2. $\mu_{C_1}(rs) = \mu_{C_2}(y(r)\, y(s)), \sigma_{C_1}(rs) = \sigma_{C_2}(y(r)\, y(s)), \gamma_{C_1}(rs) = \gamma_{C_2}(y(r)\, y(s)), \forall \, rs \in E_1$.

Example 3.6. Take $PNPG$, $G_1 = (C_1, O_1)$ and $G_2 = (C_2, O_2)$ as in Fig. 7 such that

$C_1 = \{< a_1, .8, .7, .6 >, < a_2, .7, .6, .5 >, < a_3, .9, .5, .4 >, < a_4, .8, .6, .2 >, < a_5, .7, .5, .3 >\}$

$O_1 = \{< a_1a_2, .6, .5, .4 >, < a_2a_3, .5, .4, .35 >, < a_1a_4, .65, .5, .3 >, < a_2a_5, .55, .4, .25 >, < a_4a_5, .6, .4, .1 >, < a_2a_5, .6, .4, .35 >\}$

$C_2 = \{< p_1, .75, .65, .5 >, < p_2, .6, .5, .45 >, < p_3, .8, .4, .3 >, < p_4, .7, .5, .1 >, < p_5, .65, .45, .25 >\}$

$O_2 = \{< p_1p_2, .6, .5, .4 >, < p_2p_3, .5, .4, .35 >, < p_1 p_4, .65, .5, .3 >, < p_2 p_5, .55, .4, .25 >, < p_2 p_4, .5, .4, .2 >, < p_3 p_5, .6, .4, .2 >, < p_4 p_5, .6, .4, .1 >, < p_2 p_5, .6, .4, .35 >\}$.

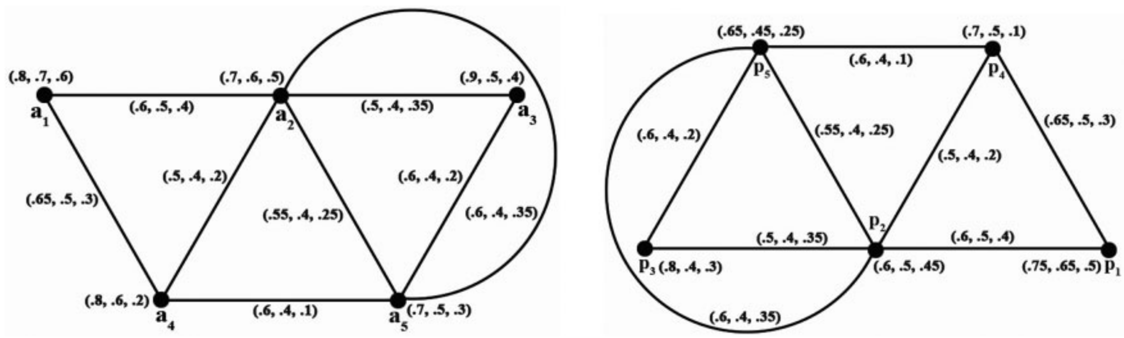

Figure 7: Pythagorean neutrosophic planar graphs

A mapping $y: V_1 \rightarrow V_2$ illustrated by $y(a_1) = p_1$, $y(a_2) = p_2$, $y(a_3) = p_3$, $y(a_4) = p_4$, $y(a_5) = p_5$ satisfies $\mu_{O_1}(r_kr_b) = \mu_{O_2}(y(r_k)y(r_b))$, $\sigma_{O_1}(r_kr_b) = \sigma_{O_2}(y(r_k)y(r_b))$, $\gamma_{O_1}(r_kr_b) = \gamma_{O_2}(y(r_k)y(r_b))$.

forall $r_kr_b \in E_1$, where $k, b = 1, 2, 3, 4, 5$ but $\mu_{C_1}(r_k) \neq \mu_{C_2}(y(r_k))$, $\sigma_{C_1}(r_k) \neq \sigma_{C_2}(y(r_k))$,$\gamma_{C_1}(r_k) \neq \gamma_{C_2}(y(r_k))$. Thus G_1 is a weak isomorphic to G_2.

4 Application in Decision Making Problem

4.1 Algorithm

The following algorithm is our proposed technique for multi-criteria decision making.

Step 1: Input the alternatives $B = (B_1, B_2, \ldots, B_n)$ and set of criteria's $C = (C_1, C_2, \ldots, C_m)$ and create the PNF relation $\left(M^{(k)} = m_{lp}^{(k)}\right)_{nxn}$ according to each criteria.

Step 2: Aggregate all $m_{lp}^{(k)} = \left(\mu_{lp}^{(k)}, \beta_{lp}^{(k)}, \sigma_{lp}^{(k)}\right)$ $(l, p = 1, 2, \ldots, n)$ regarding criteria C_j and derive $M^{(k)} = (m_{lp})_{nxn}$ where m_{lp} is the value assigned for alternate m_l over m_p according to criteria C_j by PNF averaging (PNFA) operator. $p_i^{(k)} = PNFA\ \left(m_{i1}^{(k)}, m_{i2}^{(k)}, \ldots, m_{in}^{(k)}\right)$, (k = 1, 2, ..., m)

$$= \left(\sqrt{1 - \left(\prod_{j=1}^{n}(1-\mu_{ij}^2)\right)^{1/n}}, \sqrt{1 - \left(\prod_{j=1}^{n}(1-\beta_{ij}^2)\right)^{1/n}}, \left(\prod_{j=1}^{n}(1-\sigma_{ij}^2)\right)^{1/n}\right), i = 1, 2, \ldots, m$$

Step 3: Calculate the aggregated value of each criteria C_m and compute the aggregated matrix.

Step 4: Use the score function,

$$S\left(B_i^{(j)}\right) = \frac{1 + \mu + \beta - \sigma}{3}, (i = 1\ to\ n, j = 1\ to\ m).$$

calculate the score matrix for the problem.

Step 5: Calculate the choice matrix by the function

$$S(B_i) = \sum_{j=1}^{n} \frac{B_{ij}}{n},\ (i = 1\ to\ m).$$

Step 6: Now arrange the alternatives in an order and choose the maximum as the optimal decision.

4.2 Numerical Approach

In this competitive world, time is the most precious asset for everyone. In the given 24 h a day, saving time and using it for multiple duties, and chores is an important quality. Even though time management is in our hands, travelling from one point destination for the people who doesn't drive is hard in recent times, cabs and travelling applications is one of the trending and useful facility in our cities. By the existing trends and techniques, the fastest and money-saving possibility is vital in day-to-day life. Consider the following scenario: a person wants to travel from a point to his destination and is given a set of mobile booking applications to choose a ride. Let there be these 5 cabs booking applications namely B_i $(i = 1\ to\ 5)$ that are effective nowadays. The decision-makers provide their priors by comparing these applications concerning criteria's C_j $(j = 1, 2, 3, 4, 5)$.

$C_1 = Availability,\ C_2 = Travelling\ speed,\ C_3 = Safety,\ C_4 = Cost,\ C_5 = User\ friendly$

Step 1: The alternatives are $B = (B_1, B_2, B_3, B_4, B_5)$ and the criteria's are $C = (C_1, C_2, C_3, C_4, C_5)$. The Pythagorean neutrosophic fuzzy relation according to criteria $C_i = (i = 1, 2, \ldots, 5)$ is given in structure as in Fig. 8 and values are detailed in Tables 1 to 5.

Step 2 and 3: Using PNFA operator, the aggregated values are calculated as follows:

$$C_1 : B_1^{(1)} = (.5269, .3984, .0423),\ B_2^{(1)} = (.4687, .3543, .08662),\ B_3^{(1)} = (.521, .325, .0321),$$
$$B_4^{(1)} = (.7275, .3791, .04975),\ B_5^{(1)} = (.69123, .61604, .0321).$$

$$C_2 : B_1^{(2)} = (.5967, .2851, .0234),\ B_2^{(2)} = (.7223, .26435, .0184),\ B_3^{(2)} = (.48703, .3332, .04975),$$
$$B_4^{(2)} = (.2121, .2851, .0321),\ B_5^{(2)} = (.6953, .5701, .0321).$$

$$C_3 : B_1^{(3)} = (.6088, .3248, .0321),\ B_2^{(3)} = (.51595, .4729, .0558),\ B_3^{(3)} = (.7763, .3248, .0243),$$
$$B_4^{(3)} = (.5847, .3922, .0443),\ B_5^{(3)} = (.6214, .3543, .0321).$$

$$C_4 : B_1^{(4)} = (.6247, .3407, .0321),\ B_2^{(4)} = (.5523, .3332, .0321),\ B_3^{(4)} = (.5376, .3734, .0377),$$
$$B_4^{(4)} = (.70114, .3017, .0321),\ B_5^{(4)} = (.6955, .3248, .0377).$$

$$C_5 : B_1^{(5)} = (.5316, .347, .0497),\ B_2^{(5)} = (.7029, .3161, .0243),\ B_3^{(5)} = (.5523, .3922, .0423),$$
$$B_4^{(5)} = (.5931, .3108, .0423),\ B_5^{(5)} = (.6807, .2766, .0243).$$

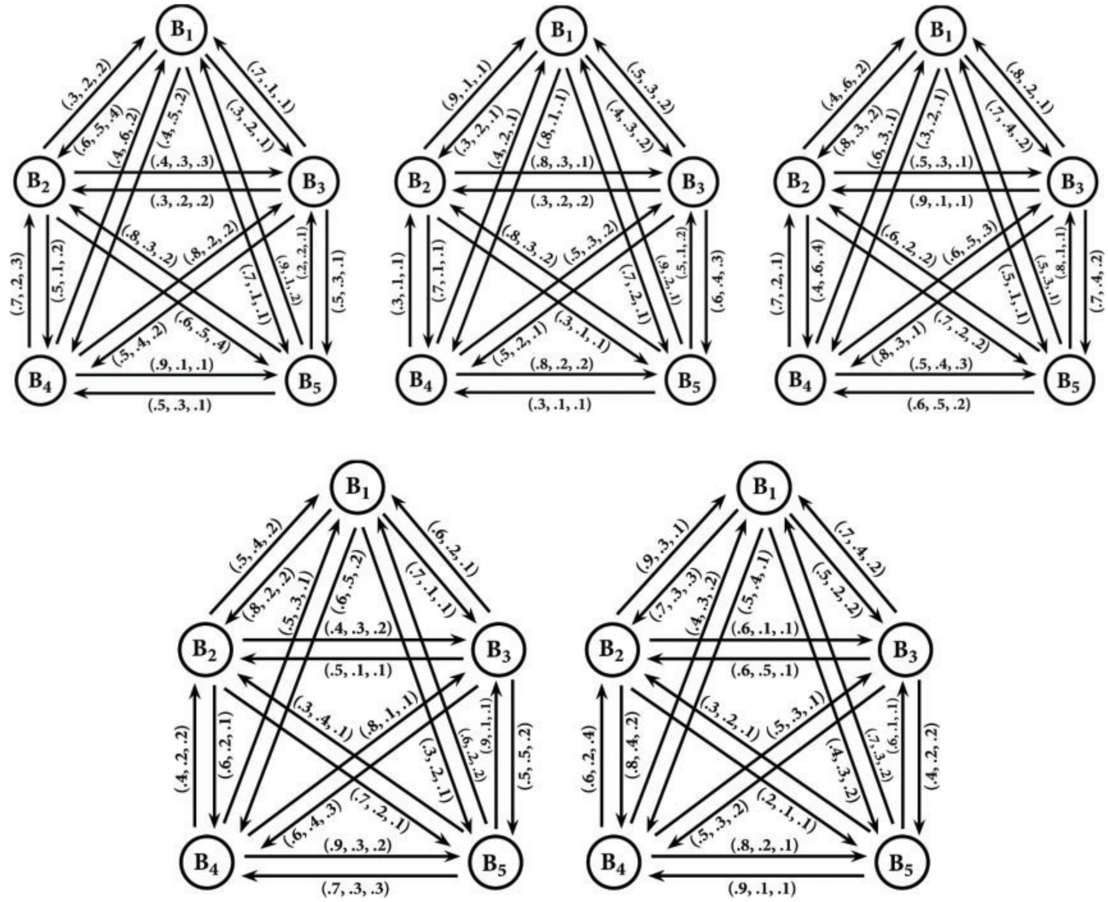

Figure 8: Pythagorean neutrosophic fuzzy directed graph for $M^{(k)}$ ($k = 1, 2, 3, 4, 5$)

Table 1: Pythagorean neutrosophic fuzzy relation of criteria C_1

M^1	B_1	B_2	B_3	B_4	B_5
B_1	(.46, .46, .46)	(.6, .5, .4)	(.3, .2, .1)	(.4, .5, .2)	(.7, .1, .1)
B_2	(.3, .2, .2)	(.46, .46, .46)	(.4, .3, .3)	(.5, .1, .2)	(.6, .5, .4)
B_3	(.7, .1, .1)	(.3, .2, .2)	(.46, .46, .46)	(.5, .4, .2)	(.5, .3, .1)
B_4	(.4, .6, .2)	(.7, .2, .3)	(.8, .2, .2)	(.46, .46, .46)	(.9, .1, .1)
B_5	(.9, .1, .2)	(.8, .3, .2)	(.2, .2, .1)	(.5, .3, .1)	(.46, .46, .46)

Table 2: Pythagorean neutrosophic fuzzy relation of criteria C_2

M^2	B_1	B_2	B_3	B_4	B_5
B_1	(.46, .46, .46)	(.3, .2, .1)	(.4, .3, .2)	(.8, .1, .1)	(.7, .2, .1)

(Continued)

Table 2: Continued

M^2	B_1	B_2	B_3	B_4	B_5
B_2	(.9, .1, .1)	(.46, .46, .46)	(.8, .3, .1)	(.7, .1, .1)	(.3, .1, .1)
B_3	(.5, .3, .2)	(.3, .2, .2)	(.46, .46, .46)	(.5, .2, .1)	(.6, .4, .3)
B_4	(.4, .2, .1)	(.3, .1, .1)	(.5, .3, .2)	(.46, .46, .46)	(.8, .2, .2)
B_5	(.9, .2, .1)	(.8, .3, .2)	(.5, .1, .2)	(.3, .1, .1)	(.46, .46, .46)

Table 3: Pythagorean neutrosophic fuzzy relation of criteria $\boldsymbol{C}_3$

M^3	B_1	B_2	B_3	B_4	B_5
B_1	(.46, .46, .46)	(.8, .3, .2)	(.7, .4, .2)	(.3, .2, .1)	(.5, .1, .1)
B_2	(.4, .6, .2)	(.46, .46, .46)	(.5, .3, .1)	(.4, .6, .4)	(.7, .2, .2)
B_3	(.8, .2, .1)	(.9, .1, .1)	(.46, .46, .46)	(.8, .3, .1)	(.7, .4, .2)
B_4	(.6, .3, .1)	(.7, .2, .1)	(.6, .5, .3)	(.46, .46, .46)	(.5, .4, .3)
B_5	(.5, .3, .1)	(.6, .2, .2)	(.8, .1, .1)	(.6, .5, .2)	(.46, .46, .46)

Table 4: Pythagorean neutrosophic fuzzy relation of criteria $\boldsymbol{C}_4$

M^4	B_1	B_2	B_3	B_4	B_5
B_1	(.46, .46, .46)	(.8, .2, .2)	(.7, .1, .1)	(.6, .5, .2)	(.3, .2, .1)
B_2	(.5, .4, .2)	(.46, .46, .46)	(.4, .3, .2)	(.6, .2, .1)	(.7, .2, .1)
B_3	(.6, .2, .1)	(.5, .1, .1)	(.46, .46, .46)	(.6, .4, .3)	(.5, .5, .2)
B_4	(.5, .3, .1)	(.4, .2, .2)	(.8, .1, .1)	(.46, .46, .46)	(.9, .3, .2)
B_5	(.6, .2, .2)	(.3, .4, .1)	(.9, .1, .1)	(.7, .3, .3)	(.46, .46, .46)

Table 5: Pythagorean neutrosophic fuzzy relation of criteria $\boldsymbol{C}_5$

M^5	B_1	B_2	B_3	B_4	B_5
B_1	(.46, .46, .46)	(.7, .3, .3)	(.5, .2, .2)	(.5, .4, .1)	(.4, .3, .2)
B_2	(.9, .3, .1)	(.46, .46, .46)	(.6, .1, .1)	(.8, .4, .2)	(.2, .1, .1)
B_3	(.7, .4, .2)	(.6, .5, .1)	(.46, .46, .46)	(.5, .3, .2)	(.4, .2, .2)
B_4	(.4, .3, .2)	(.6, .2, .4)	(.5, .3, .1)	(.46, .46, .46)	(.8, .2, .1)
B_5	(.7, .3, .2)	(.3, .2, .1)	(.6, .1, .1)	(.9, .1, .1)	(.46, .46, .46)

Step 4: By using the score function, the score matrix is calculated in Table 6.

Step 5: Deriving the choice values of alternatives using the function, we get

$S(B_1) = .62702, S(B_2) = .6324,\ S(B_3) = .629, S(B_4) = .6192, S(B_5) = .691.$

Table 6: Score matrix of alternatives to concerning criteria

Criteria\value	B_1	B_2	B_3	B_4	B_5
C_1	.628	.579	.605	.686	.758
C_2	.6195	.656	.488	.488	.744
C_3	.634	.644	.644	.644	.648
C_4	.644	.618	.657	.657	.661
C_5	.6096	.665	.621	.621	.644

Step 6: The order of ranking obtained for the problem is

$B_5 > B_2 > B_3 > B_1 > B_4$.

Thus, the alternate with maximum value B_5 is chosen to be the optimal decision.

4.3 Comparative Analysis

The proposed model is compared with the decision-making method in [39] and is verified that the same ranking is obtained. Table 7 provides a comparison of both algorithms, showing the optimal alternative and results. Both algorithms provide the same optimum decision, as can be seen in the comparison table.

Table 7: Comparison analysis

Method	Ranking	Optimal alternative
Decision-making method in [39]	$B_5 > B_2 > B_3 > B_1 > B_4$	B_5
Our proposed method	$B_5 > B_2 > B_3 > B_1 > B_4$	B_5

5 Discussion

Graph theory is known for its vast applications in various fields most vitally in designing networking problems. In particular, numerous graph theoretical concepts have been introduced to model vagueness in networking problems. PN graphs, an extension of FGs, and a fusion of Pythagorean and Neutrosophic graphs have better flexibility to be applied in real-world problems. The article has initiated the concept of PN Multi Graph and PN Planar Graph employing the concept of PNGs. The concept of PN Dual Graph, isomorphism, weak and co-weak isomorphism has been explored for PN Planar Graphs and their results have been examined. An algorithm has been proposed using Pythagorean Neutrosophic fuzzy graphs with a numerical example for a real-life problem. The limitation of the set and study is that it is limited when it is compared with the newly proposed sets, but the advancements pave for the new concept of planar graphs in Pythagorean neutrosophic environment. The advantage of this proposed study is this set is more fuzzifying than the previous studies because of the set and their properties. This research can be extended further to investigate Interval- valued PNGs, bipolar PNGs, and their implementation in real-life situations.

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