

Analytical Solution of the Thermal Behavior of a Circulating Porous Heat Exchanger

R. Henda¹, W. Quesnel² and Z. Saghir³

Abstract: The transient thermal behavior of a two-dimensional circulating porous bed is analytically investigated. A one-energy equation model, representing both the gas and solid phases via a unified temperature, is employed to describe the thermal behavior of the circulating bed. The latter is essentially a tube and shell heat exchanger commonly used in technologically important applications. The model equation is transformed into a simpler set of partial differential equations using an analytical procedure. The analytical solution, based on the method of separation of variables and the principle of superposition, is formulated for the calculation of the temperature distribution in the radial and axial directions of the bed. The temperature distribution can be determined under different process parameters and conditions. Convergence criteria of the solution are derived for typical process conditions. The developed closed-form solution of the transient one-equation energy model provides a simple and convenient means for estimating the thermal behavior of the circulating bed.

Keyword: Circulating bed, Heat transfer, Analytical solution, Convergence criteria, One-equation model.

1 Introduction

Circulating gas-solid heat exchangers are finding use in many practical applications as a means for the storage and transport of heat. The main advantages of circulating heat exchangers are the rela-

tively simple and inexpensive shell-and-tube design and the low maintenance costs, see Rautenbach and Katz (1996). Heat transfer in two-phase systems is generally modeled using either the one-equation model or the two-equation model. More details are in Quintard and Whitaker (1993). The one-equation model assumes local thermal equilibrium (LTE) whereby the average temperatures of the two phases are sufficiently close to each other so that they can be represented by a single spatial average temperature. This model is more convenient to use provided that some conditions are met as many studies have suggested (Vafai and Sozen (1990), Whitaker (1991), Quintard and Whitaker (1995)). For instance, the one-equation model is no longer valid when the thermal properties of the two phases differ widely, or when the solid particles are not small, or when convective transport is important, see Duval, Fichot and Quintard (2004). Under these circumstances the two-equation model, based on local thermal non-equilibrium (LTNE), must be used to investigate heat transfer in two-phase systems.

Early analytical solutions have been obtained for simplified one- and two equation models (Azelius (1926), Schumann (1929), Arpaci and Clark (1962), Jang and Lee (1974), Burch, Allen and Peavy (1976), Riaz (1977)). These studies neglect the diffusive and/or convective terms in the energy equations along the lines advanced by Schumann (1929). Over the last two decades, there has been a growing interest in analytical investigations of various aspects of two-phase systems. For instance, Vafai and Kim (1989) and Lee and Vafai (1999) have analytically investigated channeling effects in packed beds. Recently, Kuznetsov (1994, 1997) developed an analytical solution for the simplified LTNE in a parallel plate channel subject to constant heat flux

¹ School of Engineering, Laurentian Univ., Sudbury, ON Canada

² Xstrata Plc, Technology Center, Falconbridge, ON Canada

³ Dept. Mechanical Engineering, Ryerson Univ., Toronto, ON Canada

boundary conditions. One of the major advantages of analytical solutions is the way in which they can provide insight into global and general aspects of problems. It is not surprising that there is today a renewal of interest in analytical methods, especially with the wide availability of symbolic manipulation and computer algebra through robust software packages, e.g., see Editorial Note (2003). Ideally, the powerful combination of advanced analytical methods with efficient computational methods should be applied more widely.

This work is a continuation of our previous study on heat transfer in a moving packed bed by Henda and Falcioni (2006). The present study has been undertaken to formulate a transient and two-dimensional (2-D) model of a circulating gas-solid heat exchanger under the condition of local thermal equilibrium. The objective is to obtain an analytical solution of the model using the method of separation of variables (MSV) and the principle of superposition.

2 Governing Equations

The physical system under consideration is illustrated in Figure 1. It consists of a cylindrical two-phase (solid phase σ and stagnant gas phase γ) circulating solid bed subjected to a hot gas outside the tube wall. Energy is transferred to the bed via both conduction and convection heat transfer. The governing energy balance equations for both phases are given by

σ -phase:

$$(1 - \varepsilon)\rho_s C_{ps} \frac{\partial T_s^*}{\partial t^*} + (1 - \varepsilon)\rho_s C_{ps} v \frac{\partial T_s^*}{\partial x^*} = \nabla \cdot (k_s \nabla T_s^*) - ha_v (T_s^* - T_f^*) \quad (1)$$

γ -phase:

$$\varepsilon \rho_f C_v \frac{\partial T_f^*}{\partial t^*} = \nabla \cdot (k_f \nabla T_f^*) + ha_v (T_s^* - T_f^*) \quad (2)$$

where T_s^* , T_f^* , are the solid- and fluid-phase temperatures, respectively, and t , x , and r are time, axial position, and radial position in the circulating bed, respectively. The properties h , a_v , ε , and v are the interstitial heat transfer coefficient, surface area per unit volume of solid, bed voidage,

and linear velocity, respectively. ρ_s and ρ_f , C_{ps} and C_v , and k_s and k_f are the solid- and fluid-phase densities, specific heat capacities and thermal conductivities, respectively.

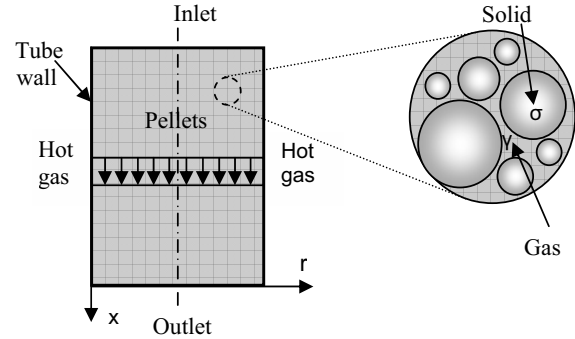


Figure 1: Schematics of the physical system

Under the assumption of LTE and constant physical properties, the one-equation energy model can be obtained by letting the interstitial heat transfer coefficient, h , tend to infinity in Equations 1 and 2. This assumption is valid under the current conditions of the circulating bed and in line with the findings of Henda and Falcioni (2006). The bed voidage is assumed to be of constant value, for the bed particles are randomly distributed as they travel down the tube. The one-equation energy balance model reduces to the following partial differential equation

$$\frac{\partial T^*}{\partial t^*} + \frac{(1 - \varepsilon)\rho_s C_{ps} v}{\langle \rho C_p \rangle} \frac{\partial T^*}{\partial x^*} = \frac{\langle k \rangle}{\langle \rho C_p \rangle} \left[\frac{\partial^2 T^*}{\partial x^{*2}} + \frac{1}{r^*} \frac{\partial}{\partial r^*} \left(r^* \frac{\partial T^*}{\partial r^*} \right) \right] \quad (3)$$

The terms $\langle \rho C_p \rangle$ and $\langle k \rangle$ are the volume-averaged heat capacity and conductivity of the medium, respectively.

In order to make the model more tractable and in a form that is compatible with the archival work of Carslaw and Jaeger (1959), the following dimensionless parameters and variables are defined

$$\alpha = \frac{(1 - \varepsilon)\rho_s C_{ps} v}{\langle \rho C_p \rangle} \quad (4)$$

$$\beta = \frac{\langle k \rangle}{\langle \rho C_p \rangle} \quad (5)$$

and

$$T = \frac{T^* - T_{in}^*}{T_o^* - T_{in}^*} \quad (6)$$

$$t = t^* \frac{\alpha^2}{\beta} \quad (7)$$

$$x = x^* \frac{\alpha}{\beta} \quad (8)$$

$$r = r^* \frac{\alpha}{\beta} \quad (9)$$

where T , t , x , and r are the dimensionless bed temperature, time, axial distance, and radial position, respectively.

Equation (3) can be expressed as

$$\frac{\partial T}{\partial t} + \frac{\partial T}{\partial x} = \frac{\partial^2 T}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) \quad (10)$$

The circulating bed is subjected to the following initial and boundary conditions

$$\text{IC: } T = 1, \quad t = 0 \quad (11)$$

BCs:

$$T = 0, \quad x = 0 \quad (\text{Dirichlet}) \quad (12)$$

$$\frac{\partial T}{\partial x} = 0, \quad x = x_e \quad (\text{Danckwerts}) \quad (13)$$

$$\frac{\partial T}{\partial r} = 0, \quad r = 0 \quad (\text{Symmetry}) \quad (14)$$

$$\frac{\partial T}{\partial r} = -\eta T, \quad r = r_w \quad (\text{Cauchy}) \quad (15)$$

with

$$\eta = \frac{h_w}{(1 - \varepsilon)\rho_s C_{ps} v} \quad (16)$$

Equation 10, along with the set of initial and boundary conditions, i.e., Eqs. 11-16, constitute the complete one-equation energy balance model describing the thermal behavior of the circulating bed under investigation.

3 Analytic Solution

To solve the governing equation the method of separation of variables has been used. The latter assumes the separation of solution function $T(x, r, t)$ into two partial solution functions: a dimensionless function in x and t , viz., $T(x, t)$, and a dimensionless function in r and t , viz., $T(r, t)$, in the product form

$$T(x, r, t) = T(x, t) \times T(r, t) \quad (17)$$

Using the principle of superposition it can be shown that $T(x, t)$ and $T(r, t)$ are the solutions of the following equations, respectively

$$\frac{\partial T}{\partial t} + \frac{\partial T}{\partial x} = \frac{\partial^2 T}{\partial x^2} \quad (18)$$

And

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \quad (19)$$

Only the analytic solution of Eq. 19 along with proper initial and boundary conditions, i.e., Eq. 11 and Eqs. 14-16, respectively, is documented in Carslaw and Jaeger (1959) using MSV as follows

$$T(r, t) = \sum_{m=1}^{\infty} \frac{2\eta}{r_w} e^{-\delta_m^2 t} \frac{J_o(r\delta_m)}{(\eta^2 + \delta_m^2)J_o(r_w\delta_m)} \quad (20)$$

where coefficients δ_m are the positive roots of the transcendental equation, see Carslaw and Jaeger (1959),

$$\delta_m \dot{J}_o(r_w\delta_m) + \eta J_o(r_w\delta_m) = 0 \quad (21)$$

and where J_o and \dot{J}_o are Bessel function of zero order and first kind and its first derivative, respectively.

The analytic solution of the remaining convection-diffusion type partial differential equation, i.e., Eq. 18, and the corresponding initial and boundary conditions, i.e., Eqs. 11-13, can be obtained using MSV by assuming that function $T(x, t)$ can be separated into two eigenfunctions: a time-dependent function $F(t)$ and a space-dependent function $G(x)$ in the form

$$T(x, t) = F(t) \times G(x) \quad (22)$$

Substituting Eq. 22 into Eq. 18 yields

$$\frac{\dot{G}(x) - \ddot{G}(x)}{G(x)} = -\frac{\dot{F}(t)}{F(t)} = \lambda \quad (23)$$

where λ is a real number such that $\lambda > 1/4$.

Eigenfunctions $G(x)$ and $F(t)$ must satisfy the initial and boundary conditions of Eq. 18. After some manipulations the resulting solution $T(x, t)$ can be shown to be

$$T(x, t) = \sum_{m=1}^{\infty} a_m e^{x/2} \sin(\gamma_m x) e^{-(\gamma_m^2 + 1/4)t} \quad (24)$$

where coefficients γ_m are the positive roots of the transcendental equation

$$2\gamma_m \cos(\gamma_m x_e) + \sin(\gamma_m x_e) = 0 \quad (25)$$

Coefficients a_m are obtained by considering the initial condition, Eq. 11, and utilizing the orthogonality property of the sine function in the expression of the eigenfunction $G(x)$. The resulting equation is expanded in terms of an infinite series of orthogonal functions leading to the following expression

$$a_m = \frac{\int_0^{x_e} \sin^2(\gamma_m x) dx}{\int_0^{x_e} e^{(-x/2)} \sin(\gamma_m x) dx} \quad (26)$$

The partial solutions $T(x, t)$ and $T(r, t)$ are combined as per Eq. 17 to yield

$$T(x, r, t) = \sum_{m=1}^{\infty} a_m e^{x/2} \sin(\gamma_m x) e^{-(\gamma_m^2 + 1/4)t} \times \sum_{n=1}^{\infty} \frac{2\eta}{r_w} e^{-\delta_m^2 t} \frac{J_o(r\delta_m)}{(\eta^2 + \delta_m^2)J_o(r_w\delta_m)} \quad (27)$$

Finally, Eq. 27 can be expressed in a simpler form as

$$T(x, r, t) = \frac{2\eta}{r_w} e^{x/2} \sum_{n=1, m=1}^{\infty} \left[a_n \sin(\gamma_n x) \frac{J_o(r\delta_m)}{(\eta^2 + \delta_m^2)J_o(r_w\delta_m)} \times e^{-(\gamma_m^2 + \delta_m^2 + 1/4)t} \right] \quad (28)$$

where the inner loop is the sum over index m .

Equation 28 solves for the temperature distribution in the circulating bed as a function of time, axial ordinate, and radial position. For convenience, the dimensionless temperature of the bed is defined by

$$\Theta(x, r, t) = 1 - T(x, r, t) \quad (29)$$

4 Results and Discussion

The distribution of the temperature along the radial and axial coordinates of the circulating heat exchanger has been obtained using the analytic series approximation given by Eq. 28 and Eq. 29. Unless otherwise stated, the calculation results correspond to a physical system with dimensions $x_e = 10$ and $r_w = 2/3$, and to parameter $\eta = 0.01$.

Table 1: Effect of the number of terms in Eqs. 20 and 24 on the dimensionless temperature at $x = 10$, $r = 0$, and $t = 1$

m'	m''	Θ
1	11	0.3592233830
2*	11	0.3592233828
3	11	0.3592233828
2	6	0.4333889448
2	9	0.3587187524
2	10	0.3592966753
2	11*	0.3592233828
2	12	0.3592233828
2	14	0.3592233828

4.1 Convergence and Accuracy

The analytic series solution given by Eq. 28 is not unconditionally convergent. Convergence is only attained after a number of terms in the series has been tallied, i.e., the solution must be independent of the number of terms, m and n . If the partial solutions (Eqs. 20 and 24) are convergent, then the analytic solution of the problem at hand is convergent and constitutes a Cauchy product. Under the prevailing conditions of this study the value of the fractional term in the Bessel function of Eq. 20 is smaller than unity. In

this case the term $2\eta \exp(-\delta_m^2 t)/r_w$ must be very small, say 10^{-6} , to secure accuracy of Eq. 20. The magnitudes of the term a_m and sine function in Eq. 24 are both less than one. The term $\exp(x/2)\exp(-[\gamma_m^2 + 0.25]t)$ must be very small, say 10^{-6} , to attain accuracy of Eq. 24. Two criteria can be developed to evaluate the number of sufficient terms m' and m'' to get accurate convergence of Eqs. 20 and 24, respectively. These criteria can be shown to be

$$m' = \frac{2}{3\pi} \sqrt{(\ln(2\eta/r_w) + 14)/t + 1/2} \quad (30)$$

and

$$m'' = \frac{10}{\pi} \sqrt{(0.5x + 14)/t - 1/4 + 1/2} \quad (31)$$

by noticing that the magnitude of coefficient δ_m is within $[(m-1)3\pi/2, 3m\pi/2]$ and the magnitude of coefficient γ_m is within $[(m-1)\pi/10, m\pi/10]$. Accuracy of the calculated solutions depends on many parameters as expressed in Eqs. 30 and 31. For instance, the sufficient number of terms to be considered in Eq. 28 decreases as time, t , increases. Table 1 illustrates the effect of the number of terms m' and m'' , in the series solution, on the estimated temperature, Θ , at position $(x = 10, r = 0)$, and at time $t = 1$. The reported data in Table 1 have been calculated from Eq. 28. The magnitudes of m' and m'' have also been calculated from criteria expressed by Eqs. 30 and 31, in order to attain sufficient accuracy of the solution, and have been found to be equal to 2 and 14, respectively. The latter are in good accordance with the data (in asterisk) reported in Tab. 1.

4.2 Temperature Distribution

Figure 2 depicts the space distribution of the temperature of the circulating bed, denoted Theta, and its evolution with time, t . As it can be noticed from Figs. 2 (a-c), the temperature of the bed increases with time from the inlet to the exit of the bed in a wave-like form. The temperature of the bed tends to steady-state for large values of time as shown in Fig. 2 (c).

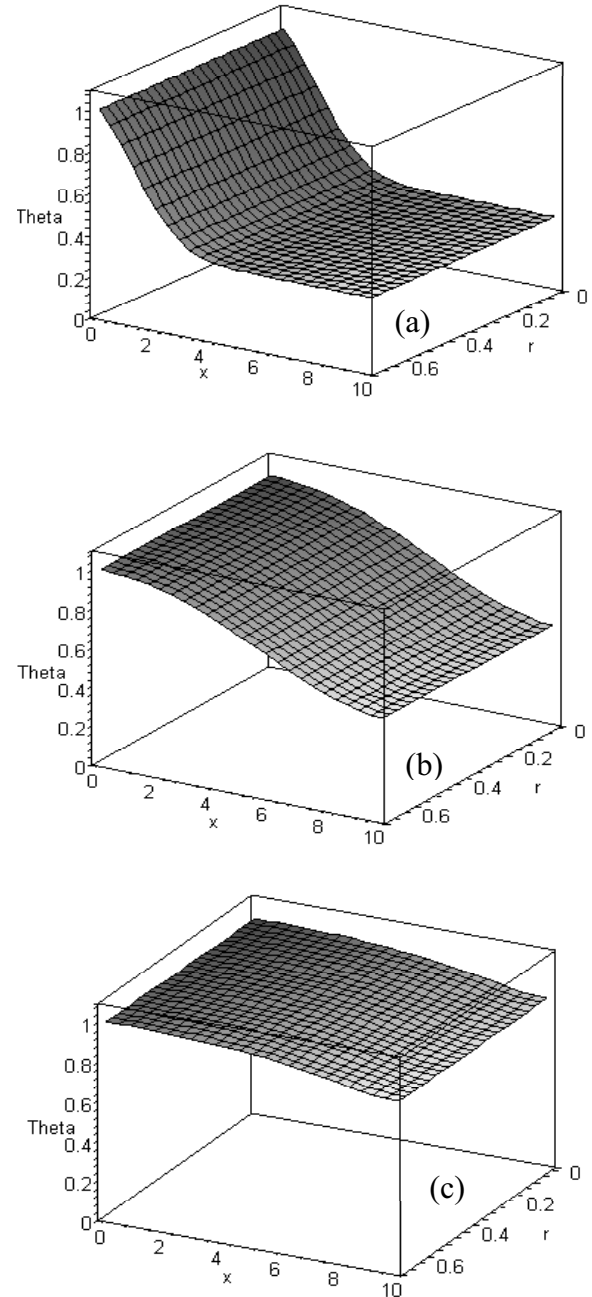


Figure 2: Space distribution of the circulating bed temperature at different points in time: $t = 1$ (a), $t = 5$ (b), and $t = 10$ (c)

5 Conclusions

The thermal behavior of a circulating heat exchanger has been considered. The analytic solution is obtained using the method of separation of variables and the principle of superposition. The

ranges of application of the analytic solution are also determined through the derivation of convergence criteria. The findings show that the temperature propagates throughout the bed in a wave-like form, and tends to steady state for large values of time. The closed form solution can serve as a valuable benchmark for verifying the accuracy of approximate algorithms and numerical methods for the solution of similar problems.

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Nomenclature

a_v	Interfacial area per unit volume m^{-1}
C_p	Specific heat capacity at constant pressure J/kg/K
C_v	Specific heat capacity at constant volume J/kg/K
h	Interstitial heat transfer coefficient $W/m^2/K$
h_w	Bed-to-wall heat transfer coefficient $W/m^2/K$
J_0	Zero order Bessel function of the first kind
k	Thermal conductivity $W/m/K$
r	Radial position m
r_w	Radius of bed m
T^*/T	Absolute/dimensionless temperature of bed K / -
t^*/t	Absolute/dimensionless time s / -
v	Linear velocity of bed m/s
x	Axial position m
x_e	Length of bed m

Greek Symbols

α	Convective coefficient m/s
β	Diffusive coefficient m^2/s
δ	Coeff. of transcendental equation
γ	Coeff. of transcendental equation; gas phase
η	Dimensionless parameter
ε	Bed voidage
Θ	Dimensionless modified temperature of bed

ρ	Density kg/m^3
σ	Solid phase
∇	Nabla operator

Subscripts

o	Reference value (initial value for T)
f	Fluid phase
in	Inlet
m, n	Indices
m', m''	Number of terms in series solution
s	Solid phase

Exponents

*	Absolute quantity
•/••	First / second derivative

Notations

F	Time-dependent eigenfunction
G	Space-dependent eigenfunction

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