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On Three-dimensional Effects in Propagation of Surface-breaking Cracks

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Summary

In fracture mechanics, we have to discuss corner and edge singularities for two- and three-dimensional problems in isotropic and layered anisotropic continua. To say something about the behavior of crack propagation starting from corners and edges, we need the information about stress asymptotics in the vicinity of three-dimensional corner points. Thus, in this paper we can study two aspects: the interface crack in layered unisotropic materials with re-entrant corners and surface cracks for the homogeneous isotropic continua. To study the effect of geometrical singularities on the stress intensity factors, we have to define generalized stress intensity factors. We are starting with KONDRATIEV´s Lemma and starting from that, an elliptic boundary value problem with homogeneous DIRICHLET/NEUMANN boundary data which produce a singular field in the vicinity of corner points. In the next step, the weak form for the previous problem is discretized by using PETROV-GALERKIN finite element method and, as a result, we are getting a quadratic eigenvalue problem. The quadratic eigenvalue problem is solved iteratively by the ARNOLDI method, and finite element approximations of corner singularity exponents λ_I are computed. These eigenvalues λ_I are the basis for the definition of generalized stress intensity factors in the neighborhood of the corner points. For the *a* posteriori control on λ_I, an error estimator is developed on the basis of ZIEKIEWICZ-ZHU algorithm. The method is tested for some typical problems in fracture mechanics.

keywords: Fracture mechanics, stress singularities, eigenvalue problems, corner and edge effects.

Introduction

This paper deals with the computation of three-dimensional singularities in elasticity. Those singularities we have for non-smooth domains with corners, edges and cracks and if we have jumps for the material constants from one layer to each other. To say something about the strength and the life-cycle of materials, we need parameters like generalized stress intensity factors as parameters from the singular functions. The numerics behind computing singularities will be influenced by the regularity of the solution. On the basis of CÉA´S Lemma, the convergence behavior

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of those methods depends on the type of trial and test functions. Classical polynomials can produce problems with the convergence behavior to avoid or reduce those problems, we can work with adaptive mesh generation or considering the type of singularity in the trial and test spaces. The type of singularities can be described for the displacements in the following form:

$$|\mathbf{x}|^{\lambda} \sum_{n=0}^{N} \log^{n} |\mathbf{x}| \, \mathbf{U}_n(\mathbf{x}/|\mathbf{x}|). \tag{1}$$

We understand asymptotic solutions from type (1) for that $\lambda < 1$.

The Singularity Problem in Elasticity for $\mathbb{R}^3$

Given is Ω of $\mathbb{R}^3$ as a bounded domain which goes conform with the cone (see Fig. 1).

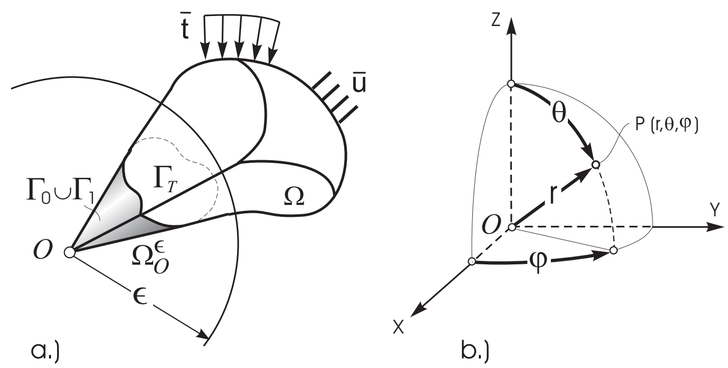

Figure 1: Solid Body with Singular, Conical Point O (a). Cartesian and Spherical Coordinates at O (b)

We are defining $\mathscr{K}$:

$$\mathscr{K} := \{\mathbf{x} \in \mathbb{R}^3 : 0 < |\mathbf{x}| < \infty, \mathbf{x}/|\mathbf{x}| \in \mathscr{S}\} \tag{2}$$

in an ε-region $\mathscr{U}_O^{\varepsilon} := \{\mathbf{x} \in \mathbb{R}^3 : 0 < |\mathbf{x}| < \varepsilon\}$ which has the same coordinate point O, so that we have

$$\Omega_O^{\varepsilon} := \mathscr{K} \cap \mathscr{U}_O^{\varepsilon} = \Omega \cap \mathscr{U}_O^{\varepsilon}. \tag{3}$$

On Ω we have the mixed boundary value problem in elasticity

$$\begin{aligned} \mathfrak{L}\mathbf{u} := \mathfrak{D}^T \mathbf{C} \mathfrak{D}\, \mathbf{u} = \mathbf{f} \quad & \text{on } \Omega, \\ \mathbf{u} = \bar{\mathbf{u}} \quad & \text{on } \partial\Omega_0, \\ \mathfrak{T}\mathbf{u} := \mathbf{t}(\mathbf{u}) = \bar{\mathbf{t}} \quad & \text{on } \partial\Omega_1, \end{aligned} \tag{4}$$

We are looking for a solution of an equivalent mixed boundary value problem of Ω_O^ε for which the transmission boundary $|\mathbf{x}| = \varepsilon$ has such DIRICHLET boundary conditions $\hat{\mathbf{u}}$ that the solution of (4) is identical with the local solution around the tip. Additionally, we define for $|\mathbf{x}| < \varepsilon$ only homogeneous boundary conditions. We are looking for $\mathbf{u}$ of the LAMÉ-System:

$$\begin{aligned} \mathfrak{L}\mathbf{u} &= \mathbf{f} \quad \text{on } \Omega_O^\varepsilon, \\ \mathbf{u} &= \hat{\mathbf{u}} \quad \text{on } \Gamma_T, \\ \mathbf{u} &= \mathbf{0} \quad \text{on } \Gamma_0, \\ \mathfrak{T}\mathbf{u} &= \mathbf{0} \quad \text{on } \Gamma_1. \end{aligned} \tag{5}$$

For the boundaries for DIRICHLET-, NEUMANN and transmission parts we have the following:

$$\begin{aligned} &\Gamma_0 := \{\mathbf{x} : 0 < |\mathbf{x}| < \varepsilon, \mathbf{x}/|\mathbf{x}| \in \gamma_0\}, \\ &\Gamma_1 := \{\mathbf{x} : 0 < |\mathbf{x}| < \varepsilon, \mathbf{x}/|\mathbf{x}| \in \gamma_1\}, \\ &\Gamma_T := \partial\Omega_0^\varepsilon \backslash \{\Gamma_0 \cup \Gamma_1\}, \end{aligned} \tag{6}$$

where $\gamma_0 \cup \gamma_1 = \partial\mathscr{S}$, $\gamma_0 \cap \gamma_1 = \emptyset$ defining $\partial\mathscr{S}$. We are using the disturbance theory [1], so that we are introducing the scaled coordinates $\mathbf{y} = \mathbf{x}/\varepsilon$ and after $\varepsilon \to 0$, the domain Ω_0^ε goes to the unbounded domain $\mathscr{K}$. We can transform the LAMÉ system to the following:

$$\begin{aligned} \mathfrak{L}\mathbf{u} &= \mathbf{0} \quad \text{on } \mathscr{K}, \\ \mathbf{u} &= \mathbf{0} \quad \text{on } \partial\mathscr{K}_0, \\ \mathfrak{T}\mathbf{u} &= \mathbf{0} \quad \text{on } \partial\mathscr{K}_1, \end{aligned} \tag{7}$$

where $\partial\mathscr{K}_0 := \{\mathbf{x} \in \partial\mathscr{K} : \mathbf{x}/|\mathbf{x}| \in \gamma_0\}$, $\partial\mathscr{K}_1 := \{\mathbf{x} \in \partial\mathscr{K} : \mathbf{x}/|\mathbf{x}| \in \gamma_1\}$ defines the DIRICHLET- and NEUMANN-part of the boundary $\partial\mathscr{K}$. For the spectral problem we come now to the following formulation:

$$\mathbf{u}(r,\theta,\varphi) = r^\lambda\, \mathbf{U}(\theta,\varphi), \tag{8}$$

for which we have to consider the following equation set:

$$\begin{aligned} \hat{\mathfrak{L}}(\lambda)\mathbf{U} &= \mathbf{0} \quad \text{on } \mathscr{S}, \\ \mathbf{U} &= \mathbf{0} \quad \text{on } \gamma_0, \\ \hat{\mathfrak{T}}(\lambda)\mathbf{U} &= \mathbf{0} \quad \text{on } \gamma_1, \end{aligned} \tag{9}$$

where γ_0 and γ_1 are DIRICHLET- NEUMANN-part of $\partial\mathscr{S}$. The operator in (9) is a so-called *'operator pencil'* $\mathfrak{A}(\lambda)$ for which we have the following properties (see proof in [4]):

1. $\mathfrak{A}(\lambda)$ is a FREDHOLM-operator for all $\lambda \in \mathbb{C}$.
2. The spectrum of $\mathfrak{A}(\lambda)$ consists of isolated eigenvalues with finite algebraic multiplicity.
3. If λ_0 is an eigenvalue of $\mathfrak{A}(\lambda)$, then this is also the case with $\bar{\lambda}_0, -1-\lambda_0, -1-\bar{\lambda}_0$, where the geometrical and algebraic multiplicity of λ_0 and $-1-\bar{\lambda}_0$ are identical.

We are coming to the kernel theorem formulated by KONDRATIEV [3]. If $\mathbf{u} \in [H^1(\Omega)]^3$, we have the following asymptotic series:

$$\mathbf{u} = \sum_{i=0}^{I} \sum_{k=0}^{k_j} K_{ik}\, r^{\lambda_i}\, \ln^k(r)\, \mathbf{U}_{ik}(\theta,\varphi), \tag{10}$$

where λ_i are eigenvalues of the operator pencil and are called *'singular exponents'*, $\mathbf{U}_{ik}$ are the generalized eigenvectors and K_{ik} are the amplitudes and are called *'generalized stress intensity factors'*. We have to consider that strain energy must be finite. In the following we are interested only in the *singular* part of the solution, thus we restrict ourselves to

$$-0.5 < \mathfrak{Re}(\lambda) < 1, \tag{11}$$

where we have the pontency logarithmic shape of the singularities. We understand asymptotic solutions from type $|\mathbf{x}|^{\lambda} \sum_{n=0}^{N} \log^n |\mathbf{x}|\, \mathbf{U}_n(\mathbf{x}/|\mathbf{x}|)$, for that $\lambda < 1$ and has infinite gradients of the displacements.

Weak Formulation of the Problem

We search for a solution for $\mathbf{u} \in [H^1(\Omega_0^{\varepsilon})]^3$ so that

$$\mathfrak{B}(\mathbf{u},\mathbf{v}) = 0, \quad \forall \mathbf{v} \in [H_0^1(\Omega_O^{\varepsilon})]^3. \tag{12}$$

We introduce now different trial and test functions which are associated with the operator pencil $\mathfrak{A}(\lambda)$:

$$\mathbf{u} = r^{\lambda}\, \mathbf{U}(\theta,\varphi) \in [H^1(\Omega_O^{\varepsilon})]^3, \tag{13}$$

$$\mathbf{v} = \Phi(r)\, \mathbf{V}(\theta,\varphi) \in [H_0^1(\Omega_O^{\varepsilon})]^3, \tag{14}$$

where $\Phi(r)$ is a scalar function with a compact support. Thus, we can formulate the following:

For $\mathscr{U} \in [H^1(\mathscr{S})]^3$ we get:

$$\hat{\mathfrak{B}}(\mathbf{U},\mathbf{V};\lambda) = 0, \quad \forall \mathbf{V} \in [H_0^1(\mathscr{S})]^3, \tag{15}$$

where $\hat{\mathfrak{B}}(\mathbf{U},\mathbf{V};\lambda)$ depends polynomially of the operator λ and represens the weak form of the operator pencil $\mathfrak{A}(\lambda)$. $(\lambda_i, \mathbf{U}_{ik})$ are only eigenpairs of $\mathfrak{A}(\lambda)$, if they are at the same time weak solutions of (15).

The approximation with finite elements leads to: $\mathbf{u}^h \in \mathbf{U}_h \subset [H^1(\Omega_0^\varepsilon)]^3$ so that

$$\mathfrak{B}(\mathbf{u}^h,\mathbf{v}^h) = 0, \quad \forall \mathbf{v}^h \in V_h \subset [H_0^1(\Omega_O^\varepsilon)]^3, \tag{16}$$

where we have the situation that $U_h \neq V_h$. Because we are working with different spaces for the trial and test functions, we are in the scheme GALERKIN-PETROV method. For the displacements we are formulating the functions in the sector $(r,\theta,\varphi) \in [0,\varepsilon] \times \Delta_i$, where we have the following finite elements initial formulations:

$$\begin{aligned} \mathbf{u}_i^h(r,\theta,\varphi) &= \quad r^\lambda\, \mathbf{N}(\theta,\varphi)\, \mathbf{T}_d^{-1}\, \mathbf{d}_i, \\ \mathbf{v}_i^h(r,\theta,\varphi) &= \Phi(r)\, \mathbf{N}(\theta,\varphi)\, \mathbf{T}_d^{-1}\, \mathbf{b}_i, \end{aligned} \tag{17}$$

We are coming to a non-symmetric stiffness-matrix, from which we are getting our fundamental equations for solving the eigenvalue problem:

$$\left[(\mathbf{K}-\mathbf{D}) + \lambda(\mathbf{D}^T - \mathbf{D} - \mathbf{M}) - \lambda^2(\mathbf{M})\right]^T \mathbf{d} = \mathbf{0}. \tag{18}$$

The Solution of the Eigenvalue Problem

We have to solve now the eigenvalue problem:

$$\left[\mathbf{P} + \bar{\lambda}\,\mathbf{Q} + \bar{\lambda}^2 \mathbf{R}\right] \mathbf{d} = \mathbf{0}, \tag{19}$$

with the definitions

$$\begin{aligned} \mathbf{P} &= \mathbf{K} + \frac{1}{4}\mathbf{M} - \frac{1}{2}(\mathbf{D}+\mathbf{D}^T), \\ \mathbf{Q} &= \left[\mathbf{D}^T - \mathbf{D}\right]^T, \\ \mathbf{R} &= -\mathbf{M}, \end{aligned} \tag{20}$$

The matrices $\mathbf{P},\mathbf{R}$ are now symmetric and $\mathbf{Q}$ is skew-symmetric. For applying the ARNOLDI-method [6, 5] we are doing the following transformation: $\bar{\lambda}\,\mathbf{x} = \bar{\lambda}^2 \mathbf{R}\,\mathbf{d}$ so that we get

$$\begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{d} \\ \mathbf{x} \end{bmatrix} = \bar{\lambda} \begin{bmatrix} -\mathbf{Q} & -\mathbf{I} \\ \mathbf{R} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{d} \\ \mathbf{x} \end{bmatrix}, \tag{21}$$

with $\mathbf{I}$ as the identity matrix.

Since for our fracture mechanics problems we need eigenvalues in the interval $0 < \mathfrak{Re}\bar{\lambda} < 1.5$, we do an additional transformation $\bar{\lambda} = 1/\theta$ so that we get

$$\underbrace{\begin{bmatrix} -\mathbf{Q} & -\mathbf{I} \\ \mathbf{R} & \mathbf{0} \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} \mathbf{d} \\ \mathbf{x} \end{bmatrix} = \theta \underbrace{\begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}}_{\mathbf{B}} \begin{bmatrix} \mathbf{d} \\ \mathbf{x} \end{bmatrix}. \tag{22}$$

So that we have the standard eigenvalue problem

$$\mathbf{X}\hat{\mathbf{d}} - \theta\hat{\mathbf{d}} = \mathbf{0}, \tag{23}$$

with $\mathbf{X} = \mathbf{B}^{-1}\mathbf{A}$ and $\hat{\mathbf{d}} = [\mathbf{d}, \mathbf{x}]^T$

Numerical Tests

We will discuss an elasticity problem with edge and corner singularities and this for homogeneous and inhomogeneous material properties. The singular exponents will be discussed in dependence of material data. We are working with an adaptive fine mesh series so that we can have the result for the first nine eigenvalues for an residumm of 10^{-4} with one ARNOLDI-step (see Fig. 2).

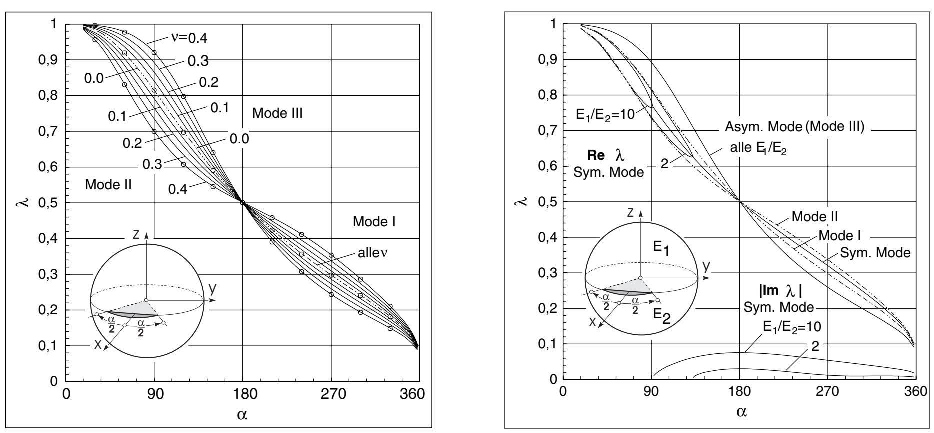

Figure 2: Wedge-shaped Crack with Homogeneous (left) and Inhomogeneous Material (right). Reference Solution is Given in [2].

We were able to show that we have derived a fast and accurate algorithm for computing eigenvalues of new questions in fracture mechanics.

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