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Topology optimization of structures with local and global stress constraints

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Summary

Topology structural optimization problems have been usually stated in terms of a maximum stiffness (minimum compliance) approach. In this kind of formulations, the aim is to distribute a given amount of material in a certain domain, so that the stiffness of the resulting structure is maximized for a given load case. In addition, no stress or displacement constraints are taken into account. This paper presents a different strategy: a minimum weight Finite Element formulation for optimization of continuum structures subjected to stress constraints. We propose two different approaches to take into account the stress constraints in the optimization formulation. The local constraints approach imposes a stress constraint in some distributed points of the domain. However, the global approach aggregates the effect of all the local constraints in a global function. The feasibility of these two approaches is demonstrated by solving some application examples.

Introduction

Structural optimization problems have been mainly written thereafter in terms of minimum weight formulations with non-linear constraints. These constraints usually limit the maximum allowable stresses and displacements. However, most of the topology optimization problems have been routinely stated in terms of minimum compliance (maximum stiffness) approaches. Essentially, in this kind of formulations a given amount of material must be distributed within a given domain while the stiffness of the resulting structure is maximized (the compliance is minimized) for a given load case [1].

The traditional minimum compliance formulations offer some obvious advantages. However, they also present several important drawbacks (mesh dependency, checkerboard layouts). In addition, the final design could be unfeasible in practice because stress and displacement constraints are not imposed.

On the other hand, since the most of structural design problems include stress and displacement constraints, it seems that these criteria should be mandatorily considered in structural topology optimization formulations. In this paper, we follow this strategy and propose a different approach that considers stress constraints.

The most intuitive formulation to impose stress constraints consists in considering a constraint at each point of the structure. Thus, the usual option is to consider stress constraints at one given point within each finite element of the mesh (the

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"local constraints approach"). This formulation is very robust and the solutions obtained are very realistic. In addition, no artificial techniques are required to obtain adequate solutions. However, this formulation presents some unexpected numerical effects when the relative density tends to zero (*e.g.* singularity phenomena).

Due to the large computing effort necessary to obtain optimal solutions when the number of elements increases, a number of different techniques have been proposed to reduce the number of constraints of the problem. One of these approaches consists in stating one or several global functions. This function would include all the constraints of the local approach. Thus, the number of constraints is drastically reduced. This technique is usually referred to as the "global stress constraints approach".

In this paper, we present a FEM based minimum weight with stress constraints (MWSC) approach for structural topology optimization problems. We have developed two different approaches: a local constraint one and a global constraint formulation based on the Kreisselmeier-Steinhauser function [2]. Finally, we present one application example that compares the results obtained with the global and the local approach.

Minimum weight with stress constraints formulation

The optimization problem can be formulated from a generic point of view as

$$\begin{array}{lll} \text{Minimize} & F(\rho) = \mathbf{Cost}(\rho) & \\ \text{subject to:} & G_\ell(\sigma_\mathbf{i}) \leq \mathbf{0} & \ell = 1,\ldots,N_{const} \\ & 0 < \rho_{min} \leq \rho_e \leq 1, & e = 1,\ldots,N_{elem} \\ & \rho_{min} = 0.001 & (usually) \end{array} \tag{1}$$

where $F(\bar{\rho})$ is the objective function and $G_\ell(\bar{\sigma}_i)$ are the stress constraints.

The objective function can be defined as

$$F(\rho) = \sum_{\mathbf{i=1}}^{\mathbf{N_{elem}}} \int_{\Omega_\mathbf{e}} (\rho_\mathbf{e})^{\mathbf{1/p}} \, \mathbf{d}\Omega \tag{2}$$

where the parameter p is a penalty parameter to avoid intermediate densities in the optimized solution [3] (If $p > 1$ intermediate densities are penalized). If $p = 1$ the objective function is the total weight of the structure. Ω is the domain occupied by each element.

Local Constraints

The value of the local constraints approach is computed by limiting the maximum value of the local stress obtained by means of a conventional Finite Element formulation. Moreover, the material failure is usually checked according to stress

failure criteria (e.g. Von Mises criterion). These criteria usually consider a "reference stress", based on the real stress tensor, to test the failure of the material. Thus, the maximum allowable values of this "reference stress" $\widehat{\sigma}(\bar{\sigma})$ at each point $\bar{r}_j^o$ can be considered by introducing the following inequalities:

$$g_j(\bar{\rho}) = \widehat{\sigma}\left(\bar{\sigma}^h(\bar{r}_j^o, \bar{\rho})\right) - \widehat{\sigma}_{max} \leq 0, \qquad g_j(\bar{\rho}) = \widehat{\sigma}_{min} - \widehat{\sigma}\left(\bar{\sigma}^h(\bar{r}_j^o, \bar{\rho})\right) \leq 0, \tag{3}$$

where $\widehat{\sigma}_{max}$ and $\widehat{\sigma}_{min}$ are the corresponding upper and lower limits of the stress failure criterion used.

Consequently, this approach requires to manage a huge number of highly non-linear constraints. So, the optimization problem is very complicated. We have solved this problem by using a Sequential Linear Programming algorithm based on the Simplex Method to obtain the search direction and a Quadratic Line Search algorithm to obtain the best advance factor based on the algorithms proposed by Navarrina and Casteleiro [4].

The Sensitivity Analysis required is developed following the general formulation proposed in [5]. The linear programming optimization algorithm requires full first order derivatives to obtain the right search direction and second order directional derivatives to obtain the advance factor. The calculations of the right search direction and the full first order derivatives require, in practice, almost the total computing time and data storage amount.

Moreover, the stress constraints approach usually presents the so called "singularity phenomena". The optimum solution of the problem is a singular point of the feasible solutions from a theoretical point of view, as it can be seen in [6]. Cheng and Jiang [7] explained the nature of this phenomenon, which is due to the discontinuos nature of the stresses when the density tends to zero. Thus, when the relative density is null the stress constraint disappears. However, when the relative density tends to zero the stress constraint can become violated.

This fact can be observed in some theoretical truss optimization problems like the proposed by Cheng and Guo [6]. Furthermore, it has been demonstrated that in other fields of structures optimization the singularity phenomena must be also considered [8].

Thus, due to this singularity phenomenon, the formulation needs to be relaxed to avoid this undesiderable situation. In this paper, we propose a different formulation based on the contributions of Duysinx and Cheng ([6],[8]) and Navarrina [3]. Following these ideas, we propose the following statement of the local stress constraints using the Von Mises failure criterion:

$$g(\rho) = \left[\widehat{\sigma}\left(\sigma^{\mathbf{h}}(\mathbf{r_j^o}, \rho)\right) - \widehat{\sigma}_{\mathbf{max}}\ \varphi_{\mathbf{j}}\right]\ \rho(\mathbf{r_j^o})^{\mathbf{q}} \leq \mathbf{0}, \tag{4}$$

where

$$\varphi_j = 1 - \varepsilon + \frac{\varepsilon}{\rho(\mathbf{r_j^o})} \tag{5}$$

The "relaxation parameter" ε usually varies from 0.001 to 0.1, and its value is reduced when the solution is near the optimum during the optimization process.

The exponent q is a parameter that allows us to impose real stress constraints ($q = 0$) or effective stress constraints ($q = 1$). We have observed that the effective stress constraints avoid some singularities [3].

Global stress constraint

The global stress constraint approach is a relatively new field of the topology optimization of continuum structures. This approach implies the substitution of all the local constraints by only one constraint which would include the local ones. This technique presents obvious advantages since the optimization problem is much easily solved than with the local approach because only one constraint has to be considered. In addition, the data storage amount and the computing time are decreased. The keystone of this approach is the global function. It is very important to find the best function to aggregate the local constraints in a global one. In this paper we propose a global formulation based on the Kreisselmeier-Steinhauser function as it was used by Martins and Poon [2]. However, we have introduced some simple modifications to solve a number of numerical effects observed with the original one. The proposed global function of stress constraints aggregation is

$$G_{KS}(g_j(\rho)) = \frac{1}{\mu}\, ln\left(\sum_{j=1}^{N_{const}} e^{\ \mu\ \left(\frac{\sigma_j}{\sigma_{j,max}\ \varphi_j}\ -1\right)}\right) \tag{6}$$

where σ_j is the local stress considered in the previous formulation and $\sigma_{j,max}$ is the maximum stress allowed according to the considered failure criterion. φ_j is the "relaxed stress coefficient" proposed in the local constraints approach (5). In addition, this modification also avoids the numerical overflow because the exponent would not take high values when the local constraints become violated.

The parameter μ allows to penalize higher stress values. When the value of μ increases, the solution is forced to become more feasible according to the local constraints approach. However, if we use high values the problem also becomes

more unstable due to the numerical inaccuracy in the calculation of the derivatives. In addition, some undesiderable numerical effects (overflow) may occur.

The maximum value of the aggregated constraint should be the upper limit of all the local stress constraints, that is

$$G_{KS_{max}}(g_j(\rho)) = \frac{1}{\mu}\, ln(N_{const}), \tag{7}$$

where N_{const} is the total number of local stress constraints considered.

Now, the problem (1) is stated as a constrained optimization problem with only one highly non-linear constraint. For this reason we have turned our attention to "barrier function" type optimization algorithm. We have tested different kinds of "barrier functions" (logarithmic and inverse) obtaining similar results with them.

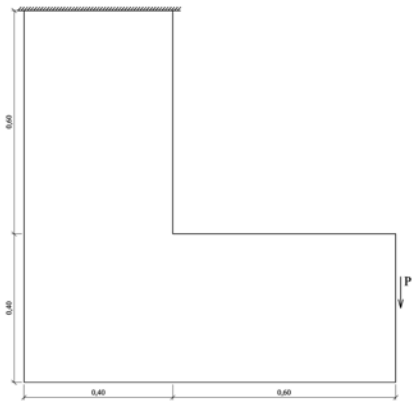

Figure 1: L-shape beam (units in meters).

Numerical example

We present a numerical example solved with the two formulations proposed to check them. The example solved is a two-dimensional structure. However, we show a three-dimensional solution, assuming the relative density to be the thickness of each element, to favor the comprehension of the solution obtained.

The example is a L-shape beam 1 m long and 1 m high (Figure 1). This beam is supported in the upper edge. Furthermore, a vertical concentrated force of $4\ 10^3$ kN is applied in the middle of the right vertical edge. In addition, self weight is also considered.

The beam is made of steel with an elastic limit of $\sigma_e = 230$ MPa and a Young Module of $E_e = 2.1\ 10^5$ MPa. The Poisson value is $\nu = 0.3$ and the mass density is $\gamma_{mat} = 76.5$ kN/m^3.

These solutions (figure 2) are very similar to the solutions obtained by Duysinx

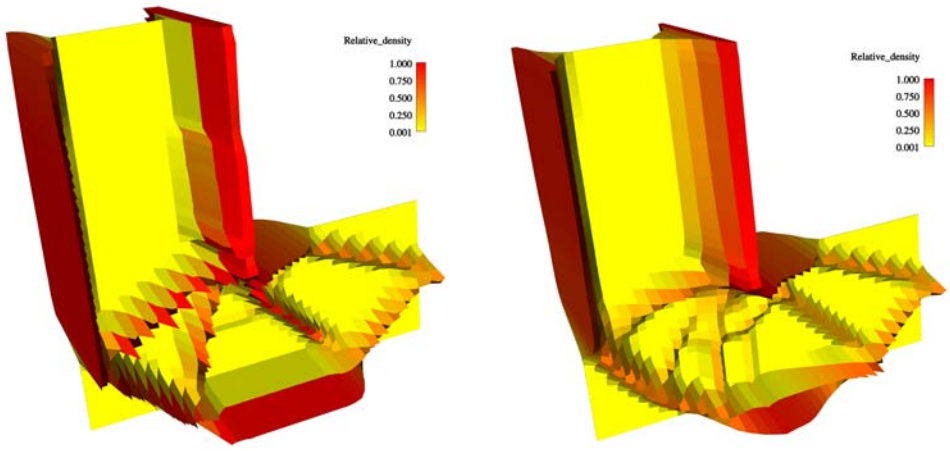

Figure 2: Local approach with q=1 (F=18.2% F_o) (left) and global approach $\mu = 20$, (F=17.6% F_o) (right), $\varepsilon = 0.01$ and $p = 4$, (Thickness multiplied by a factor of 0.5)

and Bendsøe [8].

Acknowledgement

This work has been partially supported by Grant Numbers DPI2002–00297 and DPI2004–05156 of the *"Ministerio de Ciencia y Tecnología"* of the Spanish Government, by Grant Numbers PGIDIT03–PXIC118001PN and PGIDIT03–PXIC118002PN of the *"Dirección Xeral de I+D"* of the *"Consellería de Innovación, Industria e Comercio"* of the *"Xunta de Galicia"*, and by research fellowships of the *"Universidade da Coruña"* and the *"Fundación de la Ingeniería Civil de Galicia"*.

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