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Partitioned Formulation for Solving 3D Frictional Contact Problems with BEM using Localized Lagrange Multipliers

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Summary

This work presents an interface treatment method based on localized Lagrange Multipliers (LLM) to solve frictional contact problems between two 3D elastic bodies. The connection between the solids is done using a displacement frame intercalated between the interfaces meshes, and the LLM are collocated at the interface nodes. The Boundary Elements Method (BEM) is used to compute the influence coefficients of the surface points involved, and contact conditions are imposed using projection functions. The LLM provides a partitioned formulation which preserves software modularity, facilitates non-matching meshes treatment and passes the contact patch test [4].

Introduction

In the present work, a new methodology for solving 3D frictional contact interfacing based on localize Lagrange Multipliers [3-5] is presented. The formulation is based on the methodology proposed by Rebel et al. [5] and extended by J. A. González et al. [7] for solving the contact problem of two surfaces, introducing an intermediate contact frame with independents degrees of freedom and treated with a BEM discretization. The LLM connect the frame with the contacting bodies.

As we use the BEM for solid modelling, we will work under the small displacements assumption. This fact simplify the jacobian matrix expressions presented in [5] and [7]. Another new feature presented is the way of finding the contact state; in the present work contact conditions are imposed mathematically using the augmented Lagrangian formulation and projection functions developed by the authors in [6].

The contact frame

Let consider two sub-structures, Ω_1 and Ω_2, in contact. The formulation of the contact problem considers a contact frame between the two bodies and reformulates the problem in terms of the frame using LLM.

Contact tractions coming from each sub-domain and acting on the frame are represented by λ and $\bar{\lambda}$. These tractions are expressed using two locally orthonormal base system connected to the frame: $\mathscr{B}_{\mathbf{p}} = [\mathbf{a_1}|\mathbf{a_2}|\mathbf{n}]$ which is used to describe

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λ, and $\bar{\mathscr{B}}_{\mathbf{p}} = [\bar{\mathbf{a_1}}|\bar{\mathbf{a_2}}|\bar{\mathbf{n}}]$ which is used for $\bar{\lambda}$, where $\mathbf{a_1}$ and $\mathbf{a_2}$ are the orthogonal vectors contained in the frame tangent plane at the considered point, and the vector $\mathbf{n}$ points outside the solid Ω_1. The base system $\mathscr{B}_p$ in the same position, opposite to $\bar{\mathscr{B}}_p$.

The motion of the two solids is described by the small displacements field $\mathbf{u}$ and $\bar{\mathbf{u}}$, respectively. The displacements are added to the referent configuration $\mathbf{X}$ and $\bar{\mathbf{X}}$, so the current configuration of each sub-structure is

$$\mathbf{x} = \mathbf{X} + \mathbf{u} \quad ; \quad \bar{\mathbf{x}} = \bar{\mathbf{X}} + \bar{\mathbf{u}} \tag{1}$$

The motion of the contact frame is described using its small displacements $\mathbf{v}$ from its initial configuration $\mathbf{Y}$, providing the current position:

$$\mathbf{y} = \mathbf{Y} + \mathbf{v} \tag{2}$$

This motion will be restricted to maintain the frame just in the middle between the two contact interfaces. To do that we define the relative slip of each body regarding to the frame, expressed in a frame system of reference.

$$\mathbf{k} = \mathscr{B}_{\mathrm{p}}^{\mathrm{T}}(\mathbf{x} - \mathbf{y}) \quad ; \quad \bar{\mathbf{k}} = \bar{\mathscr{B}}_{\mathrm{p}}^{\mathrm{T}}(\bar{\mathbf{x}} - \mathbf{y}) \quad ; \quad \mathbf{k} = \bar{\mathbf{k}} \tag{3}$$

The above definition is based on [3-5] and [7-8], and says when a point is in contact $k_n = 0$, and how is the tangential slip $\mathbf{k}_t$.

Contact interface restrictions

The contact conditions can be summarised in the no penetration condition and the Coulomb law. To formulate these restriction the *augmented Lagrange multiplier* variable $\lambda(r) = \lambda + r\mathbf{k}$ with a penalty parameter $r > 0$, is introduced. The restrictions are applied by the following projection functions:

$$\mathbb{P}_{\Re_-}(x) = min(x, 0) \quad ; \quad \mathbb{P}_{\mathscr{C}_R}(x, y) = \left\{ \begin{array}{cc} [x,y]^T & if \quad x^2 + y^2 \leq R^2 \\ R\frac{[x,y]^T}{\sqrt{x^2+y^2}} & if \quad x^2 + y^2 > R^2 \end{array} \right\} \tag{4}$$

In the equation (4a) we are projecting on the negative real set. The equation (4b) projects one point in 2-D inside a disk of radius g. So the no penetration condition and the Coulomb law are expressed respectively as

$$\lambda_n = \mathbb{P}_{\Re_-}(\lambda_n + r_n k_n) \quad ; \quad \lambda_t = \mathbb{P}_{\mu\lambda_n}(\lambda_t + r_t \mathbf{k}_t) \tag{5}$$

Weak formulation

Let consider each sub-structure independent. Using the variational formulation proposed by Park and Felippa [3], we can derive the equilibrium equations of our

constrained system adding the contributions of each sub-structure (treated as if they were entirely free), Ω_1 and Ω_2, to the interface constraint functional associated with the contact phenomena

$$\delta\Pi^{total} = \delta\Pi^1 + \delta\Pi^2 + \delta\Pi_c \tag{6}$$

The contact interface potential $\delta\Pi_c$ can be expressed as the sum of each body contribution to the contact frame,

$$\delta\Pi_c = \delta\Pi_i + \delta\bar{\Pi}_i \tag{7}$$

each one containing the following two terms:

$$\delta\Pi_i = \int_{\Gamma_c} \delta\left\{\lambda^T\left[\mathscr{B}_p^T(\mathbf{x}-\mathbf{y})-\mathbf{k}\right]\right\}d\Gamma \quad + \quad \int_{\Gamma_c}\delta\mathbf{k}^T\mathbb{P}(\lambda(r))d\Gamma \tag{8}$$

The first term in (8) is related with the kinematic positioning of the frame, equation (3) that is enforced in a weak sense using the variational form. The second one represents the virtual work of the contact forces. Assuming small displacements, the derivation of the expression (8) takes the form

$$\begin{array}{l}\delta\Pi_i = \int_{\Gamma_c}\delta\lambda^T\left\{\mathscr{B}_p^T(\mathbf{X}-\mathbf{Y})-\mathbf{k}\right\}d\Gamma \quad + \\ \qquad\qquad \int_{\Gamma_c}(\delta\mathbf{u}^T-\delta\mathbf{v}^T)\left\{\mathscr{B}_p\lambda\right\}d\Gamma \quad + \int_{\Gamma_c}\delta\mathbf{k}^T\left\{-\lambda+\mathbb{P}(\lambda(r))\right\}d\Gamma\end{array} \tag{9}$$

Frame functional discretization

The discretization of the contact problem will be defined in terms of couples formed by a set of interface nodes and its associated frame element. Those couples are calculated for every interface node belongs to contact potential zone, and consist of the node and its projection over the frame. The fields involved in the problem are interpolated in the following way:

$$u_i = \mathbf{N}(\underline{\xi})\mathbf{u}_i \quad ; \quad \bar{u}_i = \mathbf{N}(\underline{\xi})\bar{\mathbf{u}}_i \quad ; \quad v_i = \mathbf{N}_v(\underline{\xi})\mathbf{v}_i \quad ; \quad k_i = \mathbf{N}_k(\underline{\xi})\mathbf{k}_i \tag{10}$$

where the variables: $\mathbf{u}_i$, $\bar{\mathbf{u}}_i$, $\mathbf{v}_i$ and $\mathbf{k}_i$, on the right side of equations (10), express the nodal values in the component i. The interpolation of the slip vector can use a different frame discretization, that is the reason why $\mathbf{N}_v$ and $\mathbf{N}_k$ are diferent.

The localized Lagrange multipliers are collocated in the contacting interfaces nodes using Dirac's delta functions:

$$\lambda_i = \delta(\underline{\xi} - \underline{\xi}_p)\lambda_p \tag{11}$$

with $\underline{\xi} = (\xi_1, \xi_2)$ and $\underline{\xi}_p$ frame coordinates of the sub-structure over the frame. Using this discretization, the integral in where are involved λ become an evaluation of the functions on ξ_p. So, the equation (9) becomes in the matrix expression:

$$\delta\Pi_i = \delta\lambda^T\{\mathbf{B}^T(\mathbf{X}-\mathbf{Y}) + \mathbf{B}^T(\mathbf{u}-\mathbf{C}_f\mathbf{v}) - \mathbf{C}_s\mathbf{k}\} + \delta\mathbf{u}^T\{\mathbf{B}\lambda\} - \delta\mathbf{v}^T\{\mathbf{C}_f^T\lambda\} - \delta\mathbf{k}^T\{\mathbf{C}_s^T\lambda\} \tag{12}$$

where:

$$\begin{array}{l} \mathbf{C}_f = \int_{\Gamma_c} \delta(\underline{\xi} - \underline{\xi}_p)\mathbf{N}_v d\Gamma = \mathbf{N}_v(\underline{\xi}_p) \\ \mathbf{C}_s = \int_{\Gamma_c} \delta(\underline{\xi} - \underline{\xi}_p)\mathbf{N}_k d\Gamma = \mathbf{N}_k(\underline{\xi}_p) \end{array} \quad ; \quad \mathbf{B} = \sum_{p=1}^{np} \mathscr{L}_{up}^T \mathscr{B}_p \mathscr{L}_{\lambda p} \tag{13}$$

being $\mathscr{L}_{\square p}$ the Boolean finite element operator who extract the variable associated with the contact interface node p from the global $\square$ unknowns vector.

Boundary elements discrete equations

The discrete Boundary Element equations for a continua are well known and can be found in many classical texts like [2].

$$\mathbf{Hu} - \mathbf{Gp} = \mathbf{b} \tag{14}$$

where the vector $\mathbf{b}$ contain the applied boundary conditions. Using BEM, our variables will be displacements and tractions instead of displacement and forces, so the application of the LLM requires the "lumping" of the tractions over each interface in the contact region. This can be done satisfying the energy equivalence between the tractions acting on the boundary elements belong to the contact zone, $\mathbf{p}_c$, and the localize multipliers, λ, acting on its nodes as [8]. So the expressions are

$$\delta\Pi^{lump1} = \delta\mathbf{u}^T \{\mathbf{Mp} - \mathbf{E}\lambda\} \quad ; \quad \delta\Pi^{lump2} = \delta\bar{\mathbf{u}}^T \left\{\bar{\mathbf{M}}\bar{\mathbf{p}} - \bar{\mathbf{E}}\bar{\lambda}\right\} \tag{15}$$

in which the matrix $\mathbf{M}$, $\bar{\mathbf{M}}$, $\mathbf{E}$ and $\bar{\mathbf{E}}$, are

$$\mathbf{E} = \sum_{i=1}^{np} \mathscr{L}_{ui}^T B_i \mathscr{L}_{\lambda i} \quad ; \quad \mathbf{M} = \sum_{i=1}^{np}\sum_{j=1}^{np} \mathscr{L}_{ui}^T M_{ij} \mathscr{L}_{pj} \quad ; \quad M_{ij} = \int_{\Gamma_c} N_i N_j d\Gamma \tag{16}$$

Non-linear equations system

Substituting all the virtual work expressions on the equation (6) and carrying out the variations, the nonlinear equation set is achieved imposing the stationary point of the total virtual work and adding the boundary elements terms outside the potential contact zone, obtaining the following equilibrium equation set:

$$\Theta(\mathbf{w}) = \begin{bmatrix} \mathbf{H} & \mathbf{0} & -\mathbf{G} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{H}} & \mathbf{0} & -\bar{\mathbf{G}} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{B}^T\mathbf{C}_f & -\mathbf{C}_s \\ \mathbf{0} & \bar{\mathbf{B}}^T & \mathbf{0} & \mathbf{0} & -\bar{\mathbf{B}}^T\bar{\mathbf{C}}_f & -\bar{\mathbf{C}}_s \\ \mathbf{0} & \mathbf{0} & \mathbf{C}_f^T\mathbf{M} & \bar{\mathbf{C}}_f^T\bar{\mathbf{M}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{P}_\lambda & \bar{\mathbf{P}}_\lambda & \mathbf{0} & \mathbf{P}_k \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \bar{\mathbf{u}} \\ \mathbf{p} \\ \bar{\mathbf{p}} \\ \mathbf{v} \\ \mathbf{k} \end{Bmatrix} - \begin{bmatrix} \mathbf{b} \\ \bar{\mathbf{b}} \\ -\mathbf{B}^T(\mathbf{X}-\mathbf{Y}) \\ -\bar{\mathbf{B}}^T(\bar{\mathbf{X}}-\mathbf{Y}) \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} = \mathbf{0} \tag{17}$$

The first two matrix rows, in the expression above, are the BE elastic equations of each free solid. The next two are the kinematics positioning of the frame reference to each solid. The following is the equilibrium over the frame, and the last one is

the contact restrictions. The energy equivalence between the tractions acting on the boundary elements belong to the contact zone ($\mathbf{p}_c$) and the localize multipliers (λ) acting on its nodes, have been condensed statically.

The expressions for the contact restrictions are: $\mathbf{P}_\lambda = \sum_{p=1}^{np} \mathscr{L}_{kp}^T \mathbf{P}_{\lambda p} \mathscr{L}_{\lambda p}$

$$\bar{\mathbf{P}}_\lambda = \sum_{p=1}^{np} \mathscr{L}_{kp}^T \bar{\mathbf{P}}_{\lambda p} \mathscr{L}_{\lambda p} \quad ; \quad \mathbf{P}_k = \sum_{p=1}^{np} \mathscr{L}_{kp}^T \mathbf{P}_{kp} \mathscr{L}_{kp} + \sum_{p=1}^{np} \mathscr{L}_{kp}^T \bar{\mathbf{P}}_{kp} \mathscr{L}_{kp} \tag{18}$$

where the matrix: $\mathbf{P}_{\lambda p}$ and $\mathbf{P}_{kp}$ associated with the point p, take the values presented in [7] according to its contact situation, and considering the simplification in the projection function presented in [6].

Solving the non-linear system

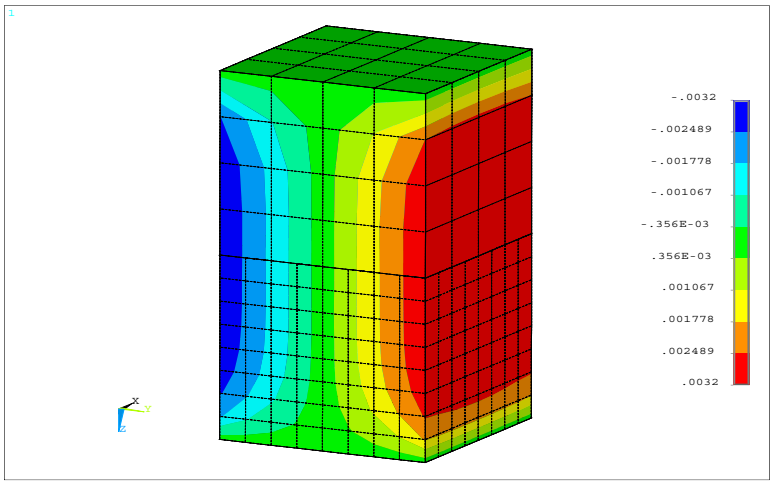
The system (17) can be expressed in a general way as $\Theta(\mathbf{w}) = \mathbf{A}\mathbf{w} - \mathbf{f}$. To solve this nonlinear system the Generalized Newton's Method with line search (GNMLS) formulated by Pang [1] has been used. The jacobian matrix $\mathbf{A}^{(k)}$ takes different values according to the contact state in the iteration (k). The next contact state $\mathbf{w}^{(k+1)}$ is found by the *line search process* which stars from the current position $\mathbf{w}^{(k)}$ and uses the vector $\Delta\mathbf{w}^{(k)}$ weighted by the parameter $\alpha^{(k)}$ ($\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} + \alpha^{(k)}\Delta\mathbf{w}^{(k)}$).

If we define the variable: *tentative solution* $\tilde{\mathbf{w}}^{(k+1)}$, as the next contact state in the case $\alpha = 1$, the system of equation we have to solve during each iteration, can be written as $\mathbf{A}^{(k)}\tilde{\mathbf{w}}^{(k+1)} - \mathbf{f} = 0$. Known $\tilde{\mathbf{w}}^{(k+1)}$, the solution for the next step is computed using the following expression: $\mathbf{w}^{(k+1)} = (1-\alpha^{(k)})\mathbf{w}^{(k)} + \alpha^{(k)}\tilde{\mathbf{w}}^{(k+1)}$ where the parameter $\alpha^{(k)}$ is modified until we satisfy the inequality: $\Psi(\mathbf{w}^{(k+1)}) \leq (1-2\sigma\alpha^{(k)})\Psi(\mathbf{w}^{(k)})$. $\Psi(\mathbf{w})$ is an error function defined as: $\Psi(\mathbf{w}) = \frac{1}{2}\Theta(\mathbf{w})^T\Theta(\mathbf{w})$. The solution is achieved when $\Psi(\mathbf{w}^{(k+1)}) \leq TOL$, so the Newton resolution finish.

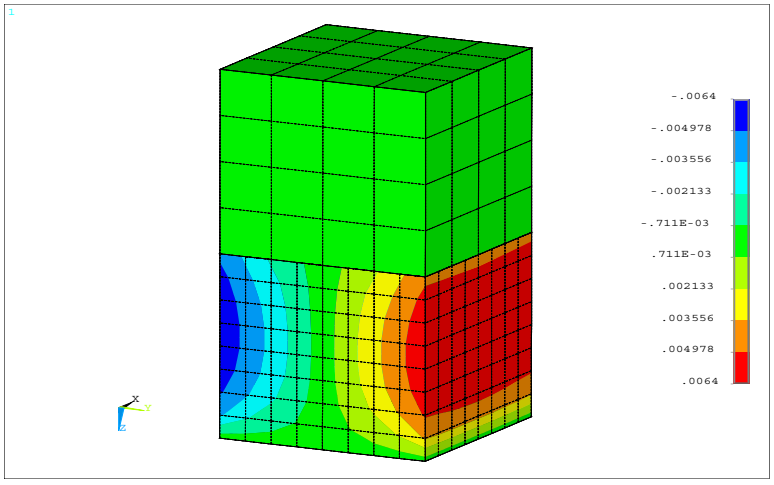
Applications

The problems considered are two. The first one is the contact between two 2x2x2mm cubes. Their material properties are $E_1 = E_2 = 10^4$ MPa (Young modulus) and $\nu_1 = \nu_2 = 0.3$ (Poisson ratio), and subjected to the boundary conditions: null displacements in the lower face of the lower cube, null displacements in x-y direction and 0.04mm in z direction in the upper face of the other cube. In the second problem the boundary conditions are the same, but the upper cube has a different Young modulus $E_2 = 10^8$ MPa. In both cases the Coulomb friction coefficient is $\mu = 0.1$, and the common interfaces are non-matching meshes.

Each case is an example of similar and dissimilar contact problem, respectively. In the figure (1a) we can see the displacements u_y of each solid and how there's no slip between the two solids. But the different tangential displacements u_y of each body, showed in figure (1b), reveals the slip produced in the second problem.



(a)



(b)

Figure 1: Solids displacements u_y in the cases of similar (a) and dissimilar (b) contact.

Conclusions

This paper presents an interface treatment method based on LLM to solve frictional contact problems between two 3D elastic bodies, using the BEM for solid modeling. The formulation proposed is an extension of the methodology developed by the authors in previous works, and provides a partitioned formulation which preserves software modularity, facilitates non-matching mesh treatment, passes the contact patch test, and facilitate the connection between different numerical techniques like the BEM and the FEM.

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