 ICCES, vol.2, no.3, pp.67-73, 2007

Coupling of Natural Boundary Element Method and Finite Element Method for Three-dimensional Nonlinear Interface Problem

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Summary

In this paper, we apply the coupling of natural boundary element method and finite element method to solve a three-dimensional nonlinear interface problem. Two equations are coupled by interface conditions on the interface boundary. A spherical surface as the artificial boundary is introduced. The equivalent coupled variational problem is described. The existence and uniqueness of the solution of concerned problem as well as the estimates of its approximate solution are obtained. Some numerical examples are presented to demonstrate the effectiveness of this method.

keywords: Natural boundary reduction, Finite element method, Nonlinear interface problem, Coupled method.

Introduction

We consider a nonlinear elliptic differential equation in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ and a linear elliptic differential equation in $\Omega^c := \mathbb{R}^3 \backslash \overline{\Omega}$ and their solutions are connected by conditions on the interface boundary $\Gamma_0 = \partial\Omega$. For given $f \in L^2(\Omega), u_0 \in H^{1/2}(\Gamma_0), t_0 \in H^{-1/2}(\Gamma_0)$, the interface problem reads: find $u_1 \in H^1(\Omega),\ u_2 \in H^1_{loc}(\Omega^c)$ such that

$$-div(p(|\nabla u_1|)\cdot\nabla u_1) + u_1 = f \quad in\ \Omega, \tag{1}$$

$$-\Delta u_2 = 0 \quad in\ \Omega^c, \tag{2}$$

with

$$u_1 = u_2 + u_0, \quad p(|\nabla u_1|)\frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} + t_0 \quad on\ \Gamma_0, \tag{3}$$

and the radiation condition at infinity

$$u_2(x) = O(\frac{1}{|x|}) \quad for\ |x| \to \infty, \tag{4}$$

where $p(t) \in C^1(\{0\} \cup \mathbb{R}^+)$ satisfies the condition $p_0 \le p(t) \le p_1 < \infty$ and $\alpha \le p(t) + tp'(t) \le \beta$ for constants $p_0, p_1, \alpha, \beta > 0$ (see [3]) and n denotes the unit normal on Γ_0 defined almost everywhere pointing from Ω into Ω^c.

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The interface problems widely occur in fluid mechanics and elasticity. The standard techniques to deal with the interface problems such as the finite element method will meet some difficulty, the computing costs could be very high for unbounded problems. The coupling methods of boundary element method and finite element method permit us to combine the advantages of boundary elements for treating domains extended to infinity with those of finite elements in treating the nonhomogeneity and nonlinearity of equations in some bounded domains. The standard procedure of coupling finite element and boundary element method is described as follows. First, the domain is divided into two subregions, a bounded inner region and an unbounded outer one, by introducing an auxiliary common boundary. Next, the problem is reduced to an equivalent one in the bounded region. There are many different ways of the boundary reduction (see [4][5]).

The natural boundary integral method and its coupling with the finite element method, which is also known as the exact artificial boundary condition method, are suggested and developed by K. Feng, D. Yu and H. Han. And a very similar method, so-called DtN method, has also been devised by J.B. Keller and D. Givoli. The idea of the natural boundary reduction, i.e., the DtN map is described as follows. By using the exact artificial boundary condition, we reduce the problem in unbounded domain into an bounded problem with a hyper-singular integral equation on the artificial boundary. It is fully compatible with the variational principle in the domain, which will avoid to directly calculate the singular integration when we choose some special auxiliary boundary (e.g., circle, ellipse, spherical surface or ellipsoid surface, etc), and the boundary elements are also fully compatible with the domain elements. This coupling is natural and direct. It is highly flexible and good for complex and large-scale problems. The theory and some fast numerical methods are developed.

In this paper, we apply the coupling of natural boundary element method and finite element method to solve a three-dimensional nonlinear interface problem. The two equations are coupled by interface conditions on interface boundary. In Sect.2, we introduce a spherical surface as the auxiliary boundary, describe the equivalent coupled variational problem, obtain existence and uniqueness of the solution. In Sect.3 we discuss the well-posedness of the solution of the concerned discrete problem and obtain the error estimate of its approximate solution. Two numerical examples are presented to demonstrate the effectiveness of this method.

Variational formulation and well-posedness

Let $H^s(\Omega), H^s(\Gamma_0)$ and $H^{-1/2}(\Gamma_0)$ denote the usual Sobolev spaces and

$$H^1_{loc}(\Omega^c) = \{v : v|_O \in H^1(O) \text{ for any } O = \Omega^c \cap B \text{ with } \Omega \subset\subset B \subset\subset \mathbb{R}^3\},$$

where B is any ball. Define the set of admissible functions

$$\hat{C} := \{(v_1, v_2) \in H^1(\Omega) \times H^1_{loc}(\Omega^c) : v_1|_{\Gamma_0} = v_2|_{\Gamma_0} + u_0, \text{ and } v_2 \text{ satisfies (4)}\}. \quad (5)$$

and the set of trial functions

$$\hat{C}^* := \{(v_1, v_2) \in H^1(\Omega) \times H^1_{loc}(\Omega^c) : v_1|_{\Gamma_0} = v_2|_{\Gamma_0}, \text{ and } v_2 \text{ satisfies (4)}\}. \quad (6)$$

Obviously, $\hat{C}$ is a nonempty convex set.

The weak form of problem (1)-(4) is to find $u = (u_1, u_2) \in \hat{C}$ such that

$$\int_\Omega (p(|\nabla u_1|)\nabla u_1 \cdot \nabla v_1 + u_1 v_1)\, dx + \int_{\Omega^c} \nabla u_2 \cdot \nabla v_2 dx = L(v), \quad \forall v = (v_1, v_2) \in \hat{C}^*, \quad (7)$$

where $L : H^1(\Omega) \times H^1_{loc}(\Omega^c) \to \mathbb{R}$ is the bounded linear functional

$$L(v) := \int_\Omega f v_1 dx + \int_{\Gamma_0} t_0 v_2 ds, \quad \forall v = (v_1, v_2) \in \hat{C}^*. \quad (8)$$

Let $\Gamma := \{(r, \theta, \varphi) : r = R, \theta \in [0, \pi], \varphi \in [0, 2\pi)\}$ such that the ball $\Omega_0 := \{(r, \theta, \varphi) : r < R, \theta \in [0, \pi], \varphi \in [0, 2\pi)\} \supset \Omega$. Set $\Omega_1 = \Omega^c \cap \Omega_0$, $\Omega_2 = \mathbb{R}^2 \backslash \overline{\Omega_0}$, and $u_{21} = u_2|_{\Omega_1}$, $u_{22} = u_2|_{\Omega_2}$. According to natural boundary reduction principle [1], if $u_2 \in D_1 := \{v \in H^1_{loc}(\Omega_2) : v \text{ satisfy (4)}\}$, we have

$$\mathscr{K}(u_2) = -\frac{\partial u_2}{\partial r}|_\Gamma = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{(n+1)}{R} U_{nm} Y_{nm}(\theta, \varphi), \quad (9)$$

$$< \mathscr{K}(u_2), v >_\Gamma = \int_{\Omega_2} \nabla u_2 \cdot \nabla v dx = R \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (n+1) U_{nm} V^*_{nm}, \quad \forall v \in D_1, \quad (10)$$

where the operator $\mathscr{K} : H^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)$ (so-called DtN map) is linear, coercive and continuous, $U_{nm} = \int_0^\pi \int_0^{2\pi} u_2|_\Gamma Y^*_{nm}(\theta, \varphi) \sin\theta d\theta d\varphi$, $V^*_{nm} = \int_0^\pi \int_0^{2\pi} v|_\Gamma Y_{nm}(\theta, \varphi) \sin\theta d\theta d\varphi$, Y_{nm} is spherical harmonic functions and Y^*_{nm} is the conjugate complex of Y_{nm}.

Let

$$V = \{(v_1, v_2) \in H^1(\Omega) \times H^1(\Omega_1) : v_1 = v_2 + u_0 \text{ on } \Gamma_0\}$$

and

$$V^* = \{(v_1, v_2) \in H^1(\Omega) \times H^1(\Omega_1) : v_1 = v_2 \text{ on } \Gamma_0\}.$$

We define the norm

$$\|v\|_V = \left(\|v_1\|^2_{H^1(\Omega)} + \|v_2\|^2_{H^1(\Omega_1)}\right)^{\frac{1}{2}}, \quad \forall v = (v_1, v_2) \in H^1(\Omega) \times H^1(\Omega_1).$$

Then the variational form (7) is equivalent to the following variational form: find $u=(u_1,u_2)\in V$ such that

$$B(u;v)=L(v), \quad \forall v=(v_1,v_2)\in V^*, \tag{11}$$

where $B(u;v)=\int_\Omega (p(|\nabla u_1|)\nabla u_1\cdot\nabla v_1+u_1v_1)\,dx+\int_{\Omega_1}\nabla u_2\cdot\nabla v_2dx+<\mathscr{K}(u_2), v_2>_\Gamma$.

In order to derive existence and uniqueness of solution of the original interface problem, we must find a functional whose variational form gives (11). Define $\Phi : V\to\mathbb{R}$ by

$$\Phi(u):=\int_\Omega (g(|\nabla u_1|)+\frac{1}{2}|u_1|^2)dx+\frac{1}{2}\int_{\Omega_1}|\nabla u_2|^2dx+\frac{1}{2}<\mathscr{K}(u_2),u_2>_\Gamma-L(u), \tag{12}$$

where the functional $g(t)$ is given by $p(t)$ through

$$g\ :\ [0,\infty)\to[0,\infty), \quad t\longmapsto g(t)=\int_0^t sp(s)ds.$$

The coupled minimization problem is to find $u=(u_1,u_2)\in V$ such that

$$\Phi(u)=\inf_{v\in V}\Phi(v). \tag{13}$$

Lemma 1 *For any $u\in V$ and $v\in V^*$, the following conclusions hold.*
1). The Gateaux derivative of Φ is

$$D\Phi(u;v)=B(u;v)-L(v). \tag{14}$$

2). $D\Phi$ is strongly monotone and Lipschitz-continuous for bounded arguments with respect to the norm $\|\cdot\|_V$.
3). The weak form of the Euler equation to the variational problem of Φ coincides with the weak form (11).

Theorem 1 *The functional Φ has a unique minimizer u on V and u is also a unique solution of the variational problem (11).*

Discrete problem and error estimate

To describe a discrete form of (13), we divide Ω and Ω_1 into some regular quasiuniform triangles with diameter h, such that the nodes on Γ_0 are matching (i.e., coincident) and these triangles nearby Γ are curved. The conforming linear finite element spaces associated with Ω and Ω_1 are denoted by $V_h(\Omega)$ and $V_h(\Omega_1)$ respectively. Usually, the curved triangles are approximated by the straight edge triangles which have the same nodes with the curved triangles. This method generates only small error. Letting $\mathcal{N}_h$ denote the set of nodes in the space $V_h(\Omega)\times V_h(\Omega_1)$, we let

$$U_h=\{v^h=(v_1^h,v_2^h)\in V_h(\Omega)\times V_h(\Omega_1):\forall b\in\mathcal{N}_h\cap\Gamma_0, v_1^h(b)=v_2^h(b)+u_0(b)\}, \quad (15)$$

$$U_h^*=\{v^h=(v_1^h,v_2^h)\in V_h(\Omega)\times V_h(\Omega_1):\forall b\in\mathcal{N}_h\cap\Gamma_0, v_1^h(b)=v_2^h(b)\}. \quad (16)$$

Then $U_h^*\subset V^*$. The discrete variational problem is to find $u^h=(u_1^h,u_2^h)\in U_h$ such that

$$\Phi(u_1^h,u_2^h)=\inf_{v^h\in U_h}\Phi(v^h). \quad (17)$$

Theorem 2 *The minimization problem (17) has one and only one solution. Its solution $u_h=(u_1^h,u_2^h)\in U_h$ is also the unique solution of the variational equation*

$$B(u^h;v^h)=L(v^h),\forall v_h\in U_h^*. \quad (18)$$

Theorem 3 *Let there be given a family of linear finite element spaces and the interpolation error be*

$$|v-\Pi_h v|_{H^1(\Omega)}\le Ch|v|_{H^2(\Omega)},\quad \forall v\in H^2(\Omega),$$

and

$$|v-\Pi_h v|_{H^1(\Omega_1)}\le Ch|v|_{H^2(\Omega_1)},\quad \forall v\in H^2(\Omega_1).$$

Then, if the solution $u=(u_1,u_2)\in V$ of the minimization problem (13) is in the space $H^2(\Omega)\times H^2(\Omega_1)$, there is a positive constant C independent of h such that

$$\|u-u^h\|_V\le Ch\|u\|_{H^2(\Omega)\times H^2(\Omega_1)}. \quad (19)$$

Numerical Examples

In practical computation, we first do harmonic extension of u_0^h to Ω_1. Let $v_0^h\in V_h(\Omega_1)$ be the weak solution of the following problem

$$\begin{cases}\Delta v_0^h=0, & in\ \Omega_1,\\ v_0^h=u_0, & on\ \Gamma_0,\\ v_0^h=0, & on\ \Gamma.\end{cases}$$

Set $v^h=u_2^h+v_0^h$ and define

$$\tilde{L}(v^h)=L(v^h)+\int_{\Omega_1}(\nabla v_2^h)\cdot\nabla v_0^h dx.$$

Obviously, $\tilde{L}(v^h)$ is still a linear bounded functional on U_h^*. Then, the functional Φ may be written in the form

$$\Psi(u^h) := \int_\Omega \left[g(|\nabla u_1^h|) + \frac{1}{2}|u_1^h|^2 \right] dx + \frac{1}{2}\int_{\Omega_1} |\nabla u_2^h|^2 dx + \frac{1}{2} < \mathscr{K}(u_2^h), u_2^h >_\Gamma + \frac{1}{2}\int_{\Omega_1} |\nabla v_0^h|^2 dx + \int_{\Gamma_0} t_0 v_0^h ds - \tilde{L}(u^h).$$

Let

$$\Psi_0(u^h) := \int_\Omega \left[g(|\nabla u_1^h|) + \frac{1}{2}|u_1^h|^2 \right] dx + \frac{1}{2}\int_{\Omega_1} |\nabla u_2^h|^2 dx + \frac{1}{2} < \mathscr{K}(u_2^h), u_2^h >_\Gamma - \tilde{L}(u^h).$$

The coupled minimization problem is to find $u^h = (u_1^h, w^h) \in U_h^*$ such that

$$\Psi_0(u^h) = \inf_{v^h \in U_h^*} \Psi_0(v^h). \tag{20}$$

Thus, $u^h = (u_1^h, u_2^h) \in U_h$ minimizes Φ in U_h if only if $(u_1^h, u_2^h + v_0^h)$ minimizes Ψ_0 in U_h^*.

To solve the nonlinear problem (20), we apply Newton iterations. The series $\sum_{n=0}^\infty \sum_{m=-n}^n$ in (10) is usually replaced with $\sum_{n=0}^N \sum_{m=-n}^n$. In practical computation, N is very small. Let $e_1(h,N) = \|u_N^h - u\|_{H^1(\Omega)}$, $e_0(h,N) = \|u_N^h - u\|_{L^2(\Omega)}$ and $e_\infty(h,N) = \|u_N^h - u\|_{L^\infty(\Omega)}$ where u_N^h is approximate solution after truncating the infinite series of (10). *iters* is the times of Newton iterations.

Example 1 *Let* $\Omega = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : |x_i| < 1, i = 1,2,3\}$*, the auxiliary boundary* $\Gamma = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : r = \sqrt{x_1^2+x_2^2+x_3^2} = 2\}$. $p(t) = 1 + e^{-t^2}$*,* $f = e^{-r^2}(2r^2 - 3) - 3 + r^2/2$*,* $u_0 = r^2/2 - 1/r$ *and*

$$t_0 = \begin{cases} |x_1|(1+e^{-r^2}+1/r^3), & |x_1| = 1, \\ |x_2|(1+e^{-r^2}+1/r^3), & |x_2| = 1, \\ |x_3|(1+e^{-r^2}+1/r^3), & |x_3| = 1. \end{cases}$$

Then the exact solution of problem (1)-(4) is $u = (r^2/2, 1/r)$.

In Table 1, we present experimental rates of convergence for the L^2-errors and H^1-errors in the relevant domains. The numerical results are in agreement with theoretical analysis and the numbers of iterations of Newton method are very small.

Table 1: N=20, $h_1 = 0.0208$, $h_2 = 0.0111$

mesh size	domain	node	$e_1(h,N)$	ratio	$e_0(h,N)$	ratio	$e_\infty(h,N)$	iters
h	Ω	125	2.3313e-1	–	4.4980e-2	–	5.6569e-2	4
$h_1/2$	Ω	729	1.1038e-1	2.1121	1.5017e-2	2.9952	2.9795e-2	5
$h_1/4$	Ω	4913	4.8815e-2	2.2614	4.6182e-3	3.2517	1.7115e-2	5
h_1	Ω_1	490	1.9565e-1	–	4.9411e-2	–	4.6950e-2	4
$h_2/2$	Ω_1	3474	9.7283e-2	2.0111	1.5609e-2	3.1656	2.9795e-2	5
$h_2/4$	Ω_1	26146	4.7756e-2	2.0371	5.6206e-3	2.7771	1.7115e-2	5

References

1. J.M. Wu and D.H. Yu (1999): “The natural integral equations of 3-D harmonic problems and their numerical solutions”, *Chinese J. Numer. Math. & Appl.*, Vol. 21, pp. 73-85.
2. D.H. Yu (2002): *Natural Boundary Integral Method and Its Applications*, Kluwer Academic Publishers, 2002.
3. E.P. Stephan (1992): “ Coupling of finite elements and boundary elements for some nonlinear interface problems”, *Comput. Methods Appl. Mech. Engrg.* , Vol. 101, pp. 61-72.
4. M. Costabel (1987): “ Symmetric methods for the coupling of finite elements and boundary elements”, in *Boundary Elements IV*,vol.1, Comput. Mech. , Southampton, pp. 411-420.
5. H.D. Han (1987/1990): “A new class of variational formulations for the coupling of finite and boundary element methods”, Preprint, University of Maryland, College Park,1987; *J. Comput. Math.* , Vol. 8, pp. 223-232.
6. G.N.Gatica and G.C.Hsiao (1992): “ On the coupled BEM and FEM for a nonlinear exterior Dirichlet problem in $\mathbb{R}^2$”, *Numer. Math.* , Vol. 61, pp. 171-214.