 ICCES, vol.3, no.2, pp.87-92, 2007

Linear coupled thermoelastic analysis for 2-d orthotropic solids by MLPG

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Summary

In this paper, the Meshless Local Petrov-Galerkin (MLPG) method for two-dimensional (2-d), linear and transient coupled thermoelastic analysis in orthotropic solids is presented. To eliminate the time-dependence in the governing equations, the Laplace-transform technique is used. Local integral equations are derived for small circular sub-domains which surround nodal points distributed over the analyzed domain. As for the spatial variations of the displacements and temperature, they are approximated by the Moving Least-Squares (MLS) scheme.

Introduction

In coupled thermoelasticity, a temperature field arises from the strain rate, i.e. the thermoelastic dissipation. Several computational methods have been proposed over the past years to analyze thermoelastic problems, many of which have been directed to uncoupled problems in heat conduction. There are relatively few investigations successfully applied to coupled thermoelasticity. Domain-based techniques, such as the finite element method (FEM), have been developed and applied to thermoelasticity [1,2]. The boundary element method (BEM), a powerful alternative numerical tool, has also been successfully applied to coupled thermoelastic problems [3-6].

Although the FEM and BEM have been established as effective computational tools for engineering analysis, there is still a growing interest in the development of new advanced methods. In particular, meshless formulations are becoming popular due to their high adaptivity and low costs in data preparation. Several meshless methods have, hitherto, been proposed and some of them have been applied to thermoelastic problems [7-9].

The meshless local Petrov-Galerkin (MLPG) method is a fundamental base for the derivation of many meshless formulations, as the trial and test functions can be chosen from different functional spaces. In this paper, the MLPG method with a Heaviside step function as the test functions [10-12] is applied to solve two-dimensional transient coupled thermoelasticity problems. An inertial term exists in the equations of motion for transient thermoelasticity, and the second governing equation derived from energy balance has a diffusion character. To eliminate

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the time-dependence in both governing partial differential equations, the Laplace-transform technique is applied such that they are satisfied in the transformed domain in a weak-form on small arbitrary subdomains. Nodal points are introduced and distributed over the analyzed domain; each of them is surrounded by a small circle for simplicity, but without loss of generality. The resulting integral equations have a very simple nonsingular form. The spatial variations of the displacements and temperature are approximated by the Moving Least-Squares (MLS) scheme [10].

Local boundary integral equations and their numerical solution

Consider a homogeneous orthotropic solid. Equilibrium and thermal balance equations in transient coupled thermoelasticity [13] can be written as

$$\sigma_{ij,j}(\mathbf{x},\tau) - \rho \ddot{u}_i(\mathbf{x},\tau) + X_i(\mathbf{x},\tau) = 0, \tag{1}$$

$$\left[k_{ij}(\mathbf{x})\theta_{,j}(\mathbf{x},\tau)\right]_{,i} - \rho c \dot{\theta}(\mathbf{x},\tau) - \gamma_{ij}\theta_0 \dot{u}_{i,j}(\mathbf{x},\tau) + Q(\mathbf{x},\tau) = 0, \tag{2}$$

where σ_{ij}, τ, θ,θ_0,u_i,X_i and Q are the stresses, time, temperature difference, reference temperature, displacements, density of body force vector and density of heat sources, respectively. Also, ρ, k_{ij}, c and γ_{ij} are the mass density, thermal conductivity tensor, specific heat and stress-temperature modulus, respectively.

The relation between the stresses σ_{ij} and the strains ε_{ij}, when temperature changes are considered, is given by Duhamel-Neumann law as follows

$$\sigma_{ij}(\mathbf{x},\tau) = c_{ijkl}\varepsilon_{kl}(\mathbf{x},\tau) - \gamma_{ij}\theta(\mathbf{x},\tau), \tag{3}$$

where $c_{ijkl}$are the material stiffness coefficients. The stress-temperature modulus can be expressed through the stiffness coefficients and the coefficients of linear expansion α_{kl}

$$\gamma_{ij} = c_{ijkl}\alpha_{kl}\,. \tag{4}$$

For 2-d plane problems, equation (3) is frequently written in terms of the second-order tensor of elastic constants [14]. Under plane-strain condition, it has the following form

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & 0 \\ c_{12} & c_{22} & 0 \\ 0 & 0 & c_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix} - \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \end{bmatrix} \theta = \mathbf{C} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix} - \gamma\theta, \tag{5}$$

with $\gamma = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \end{bmatrix} = \begin{bmatrix} \gamma_{11} \\ \gamma_{22} \\ 0 \end{bmatrix}$.

The MLPG method constructs weak-forms of the above governing equations over the local arbitrary sub-domains such as Ω_s, which is a small region taken for each node inside the global domain [10]. In the Laplace-transformed domain, these equations can be converted to the following local boundary-domain integral equations [12]

$$\int_{L_s+\Gamma_{su}} \bar{t}_i(\mathbf{x},p)d\Gamma - \int_{\Omega_s} \rho p^2 \bar{u}_i(\mathbf{x},p)d\Omega = -\int_{\Gamma_{st}} \tilde{\bar{t}}_i(\mathbf{x},p)d\Gamma - \int_{\Omega_s} \bar{F}_i(\mathbf{x},p)d\Omega, \quad (6)$$

where $\bar{F}_i(\mathbf{x},p)$ is the re-defined body force and p is the Laplace-transform parameter.

Similarly, the local integral equation for equation (2) can be obtained as

$$\int_{L_s+\Gamma_{sp}} \bar{q}(\mathbf{x},p)d\Gamma - \int_{\Omega_s} \rho c p \bar{\theta}(x,p)d\Omega - \int_{\Omega_s} \gamma_{ij}\theta_0 p \bar{u}_{i,j}(\mathbf{x},p)d\Omega = -\int_{\Gamma_{sq}} \tilde{\bar{q}}(\mathbf{x},p)d\Gamma - \int_{\Omega_s} \bar{R}(\mathbf{x},p)d\Omega. \quad (7)$$

In equations (6) and (7), Γ_u,Γ_t,$\Gamma_p$and Γ_q are the parts of the global boundary with prescribed displacements $\tilde{u}_i(\mathbf{x},\tau)$, tractions $\tilde{t}_i(\mathbf{x},\tau)$, temperature $\tilde{\theta}(\mathbf{x},\tau)$ and heat flux $\tilde{q}(\mathbf{x},\tau)$, respectively. The trial functions are approximated by the Moving Least-Squares (MLS) method [10].

The Laplace-transforms of the displacements and the temperature can be written as

$$\bar{\mathbf{u}}^h(\mathbf{x},p) = \blacksquare^T(\mathbf{x}) \cdot \hat{\mathbf{u}}(p) = \sum_{a=1}^{n} \phi^a(\mathbf{x})\hat{\mathbf{u}}^a(p),$$

$$\bar{\theta}^h(\mathbf{x},p) = \sum_{a=1}^{n} \phi^a(\mathbf{x})\hat{\theta}^a(p), \quad (8)$$

where the nodal values $\hat{\mathbf{u}}^a(p)$ and $\hat{\theta}^a(p)$ are the fictitious parameters for the displacements and the temperature, respectively, and $\phi^a(\mathbf{x})$ is the shape function. The traction vector $\bar{t}_i(\mathbf{x},p)$ at a boundary point $\mathbf{x} \in \partial\Omega_s$ is approximated in terms of the same nodal values $\hat{\mathbf{u}}^a(p)$ and $\hat{\theta}^a(p)$ as

$$\bar{\mathbf{t}}^h(\mathbf{x},p) = \mathbf{N}(\mathbf{x})\mathbf{C}\sum_{a=1}^{n} \mathbf{B}^a(\mathbf{x})\hat{\mathbf{u}}^a(p) - \mathbf{N}(\mathbf{x})\gamma \sum_{a=1}^{n} \phi^a(\mathbf{x})\hat{\theta}^a(p), \quad (9)$$

where the matrix **N(x)** is related to the normal vector **n(x)** on $\partial\Omega_s$ and the matrix $\mathbf{B}^a$ is represented by the gradients of the shape functions. Similarly, the heat flux $\bar{q}(\mathbf{x},p)$ can be approximated by

$$\bar{q}^h(\mathbf{x},p) = k_{ij} n_i \sum_{a=1}^{n} \phi^a_{,j}(\mathbf{x})\hat{\theta}^a(p). \quad (10)$$

The MLS-approximations (9) and (10) are substituted into the local boundary-domain integral equations (6) and (7). This results in the discretized forms as given below; together with the boundary conditions, they represent the complete system of linear algebraic equations of the unknown nodal values of the displacements and temperature.

$$\sum_{a=1}^{n}\left(\int_{L_s+\Gamma_{su}}\mathbf{N}(\mathbf{x})\mathbf{C}\mathbf{B}^a(\mathbf{x})d\Gamma-\mathbf{I}\rho p^2\int_{\Omega_s}\phi^a(\mathbf{x})d\Omega\right)\hat{\mathbf{u}}^a(p)-\sum_{a=1}^{n}\left(\int_{L_s+\Gamma_{su}}\mathbf{N}(\mathbf{x})\gamma\phi^a(\mathbf{x})d\Gamma\right)\hat{\theta}^a(p)=$$
$$=-\int_{\Gamma_{st}}\tilde{\bar{\mathbf{t}}}(\mathbf{x},p)d\Gamma-\int_{\Omega_s}\bar{\mathbf{F}}(\mathbf{x},p)d\Omega, \tag{11}$$

$$\sum_{a=1}^{n}\left(\int_{L_s+\Gamma_{sp}}\mathbf{n}^T\mathbf{K}\mathbf{P}^a(\mathbf{x})d\Gamma-\int_{\Omega_s}\rho cp\phi^a(\mathbf{x})d\Gamma\right)\hat{\theta}^a(p)-\sum_{a=1}^{n}\left(\int_{\Omega_s}\theta_0 p\gamma^T\mathbf{B}^a(\mathbf{x})d\Gamma\right)\hat{\mathbf{u}}^a(p)=$$
$$=-\int_{\Gamma_{sq}}\tilde{\bar{q}}(\mathbf{x},p)d\Gamma-\int_{\Omega_s}\bar{R}(\mathbf{x},p)d\Omega \quad . \tag{12}$$

Numerical results

A unit square isotropic panel under a sudden heating on the top side is first analyzed (Fig. 1) with the following material constants are used: $k=1$, $\rho=1$, $c=1$, thermal expansion coefficient $\alpha=0.02$, Young's modulus $E=1$ and Poisson's ratio $\nu=0.3$.

The thermoelastic coupling parameter [3]

$$\delta=\frac{(1+\nu)\alpha^2E\theta_0}{(1-\nu)(1-2\nu)\rho c}=0.186$$

is considered; this corresponds to $\theta_0=100$ and the above-mentioned material constants. Plane strain conditions are assumed.

The coupling effect on the temperature at $x_2=0$ is shown in Fig. 2. It can be seen that the influence of the coupling on the temperature is weaker for small and large time instants. The strongest influence is at about $\tau=0.8$ for the material constants considered. A similar characteristic has also been observed for a suddenly heated half-space analyzed by Chen and Dargush [4].

Next, an orthotropic square panel is analyzed. The boundary conditions are the same as those shown in Fig.1. The following material constants are consid-

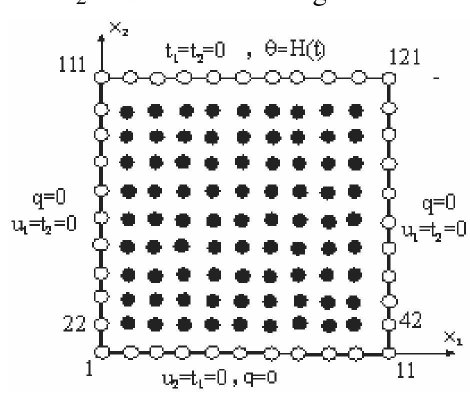

Figure 1: A suddenly heated unit square panel

ered: $k = 1$, $\rho = 1$, $c = 1$, $\alpha = 0.02$, Young's modulii $E_1 = 1$, $E_2 = 2E_1$ and Poisson's ratio $\nu = 0.3$. The stress component σ_{11} at the mid-side of the panel, $x_2 = 0.5$, is shown in Fig. 3. Here, σ_{11} is higher for the orthotropic panel than for the isotropic one. The influence of the coupling on σ_{11} due to the orthotropy of the material is evidently weak. The influence of the orthotropic mechanical properties on the mechanical stresses is much stronger than the mechanical-thermal coupling, at least in the cases considered here.

$x_2 = 0$

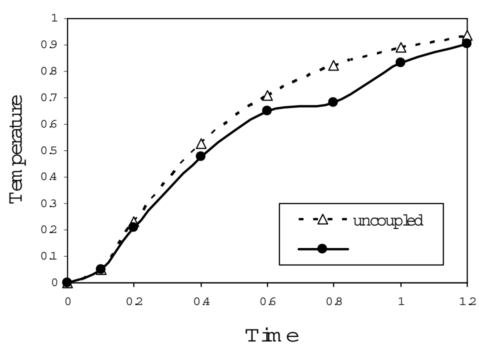

Figure 2: Coupling effect on the temporal variation of the temperature at $x_2 = 0$

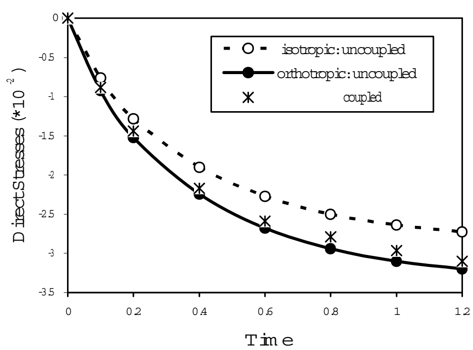

Figure 3: Temporal variation of the stress σ_{11} in the orthotropic square panel at $x_2 = 0.5$

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