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To the computing of point source-generated potential in multiply-connected regions of irregular shape

I.K. Lifanov[1], Y.A. Melnikov[2] and A.S. Nenashev[3]

Summary

Two different boundary integral equation method-based approaches are developed for computing potential fields generated by point sources in multiply-connected regions of irregular configuration. Numerical experiment is conducted to demonstrate the computational potential of the approaches.

Introduction

Of many areas of computational mechanics, the method of boundary integral equation could, probably, benefit most from the use of Green's functions. Compact and easily computable representations of Green's functions are known [1, 4, 5] for only regions of regular shape. The focus in this study is on the development of effective algorithms for computing Green's functions for a variety of boundary value problems for Laplace equation stated in regions of irregular configuration. Consider the homogeneous boundary value problem for Laplace equation

$$\nabla^2 u(M) = 0, \quad M \in \Omega \tag{1}$$

$$\alpha_j \frac{\partial u(M)}{\partial n_j} + \beta_j u(M) = 0, \ \ M \in \Gamma_j, \quad j = \overline{1,m} \tag{2}$$

$$u(M) = 0, \quad M \in L_i, \quad i = \overline{1,k} \tag{3}$$

stated in a multiply-connected region Ω which is bounded from outside by a piecewise smooth contour $\Gamma = \cup_{j-1}^{m} \Gamma_j$ and weakened with k apertures whose contours L_i represent smooth closed curves that do not overlap. The parameter n_j in (2) represents the normal to Γ_j.

Let $G_0(M;P)$ represent the Green's function to the problem in (1)-(2). We look for the Green's function $G(M,P)$ to the problem in eqns (1)-(3) in the form

$$G(M;P) = G_0(M;P) + g(M) \tag{4}$$

where $g(M)$ represents a harmonic function in Ω that satisfies the boundary conditions in (2) for any fixed location P^* of the source point P. This requires for $g(M)$ to compensate the component $G_0(M;P^*)$ on the contours L_i. In other words, $g(M)$ ought to be harmonic in Ω and to satisfy the boundary conditions

$$\alpha_j \frac{\partial g(M)}{\partial n_j} + \beta_j g(M) = 0, \ \ M \in \Gamma_j, \quad j = \overline{1,m} \tag{5}$$

[1]Institute of Computational Mathematics, Academy of Sciencies, Russia
[2]Middle Tennessee State University, USA
[3]Zhukovsky Air Force Engineering Academy, Russia

$$g(M) = -G_0(M;P^*), \quad M \in L_i, \quad i = \overline{1,k} \tag{6}$$

imposed on the outer and inner contours of Ω.

Two different numerical procedures are developed in this study for accurate computation of the regular component of the Green's function $G(M,P)$. Both procedures are based on different boundary integral equation method approaches. In what follows, we offer sketchy description of these approaches.

Method of hyper-singular integral equations

Within the scope of this approach, let the solution to the problem in eqns (5)-(6) be expressed as the modified potential

$$g(M) = \sum_{i=1}^{k} \int_{L_i} G_0(M;Q)\mu_i(Q)dL_i(Q), \quad M \in \Omega \tag{7}$$

The choice of $G_0(M;Q)$ for the kernel makes $g(M)$ harmonic in Ω and satisfying the conditions in (5). By satisfying the conditions of eqn (6), we obtain

$$\sum_{i=1}^{k} \int_{L_i} G_0(M;Q)\mu_i(Q)dL_i(Q) = -G_0(M;P^*), \quad M \in L_s, \quad s = \overline{1,k} \tag{8}$$

which is a system of integral equations of the first kind with logarithmic singularity that represents an ill-posed problem. To develop a stable numerical scheme for its solution, we apply the hyper-singular integral equation method-based algorithm [3]. In doing so, we introduce parametric equations for the curves L_j and differentiate the system in (8) with respect to the parameter

$$\sum_{i=1}^{k} \int_{L_i} \frac{d}{dt} G_0(M;Q)\mu_i(Q)dL_i(Q) = -\frac{d}{dt} G_0(M;P^*), \quad M \in L_s, \quad s = \overline{1,k} \tag{9}$$

This is a system of hyper-singular integral equations whose solution is not unique. A uniqueness condition that yields such a solution to the above system that represents the solution for (8) can be obtained by integrating (8) as

$$\sum_{i=1}^{k} \int_{L_i} \int_{L_j} G_0(M;Q)dL_j(M)\mu_i(Q)dL_i(Q)$$

$$= -\int_{L_j} G_0(M;P^*)dL_j(M), \quad M \in L_s, \quad s = \overline{1,k} \tag{10}$$

Table 1: Approximate solution to the validation problem obtained for a concentric circular ring, where L is a circle of radius $a<R$

Field	Exact value	$N=20$		$N=100$	
point, r	$G(r,0;0.5,0)$	$u(r,0)$	Abs. Error	$u(r,0)$	Abs. Error
1.0	0.199383	0.181690	0.017693	0.196039	0.003344
1.2	0.141219	0.140335	0.000884	0.141219	0.8e-10
1.4	0.096471	0.096407	0.000064	0.096471	0.2e-16
1.6	0.059634	0.059626	0.000008	0.059634	0.1e-16
1.8	0.027994	0.027992	0.000002	0.027994	0.4e-16
2.0	0.000000	0.000000	0.9e-17	0.000000	0.5e-17

Upon approximating the integral operators in (8)-(10) with finite sums, we obtain a well-posed system of linear algebraic equations whose solution gives an approximate solution to the system in (8).

A validation example was designed to check out the convergence rate of the algorithm and to estimate the actual accuracy level attained. A Dirichlet problem is considered of the type in eqns (1)-(3) as stated in a double connected region Ω whose outer contour Γ is a circle of radius R, while the inner contour L is a smooth closed curve. The classical Green's function $G_0(r,\phi;\rho^*,\psi^*)$ of the Dirichlet problem for the disk of radius R could represent the exact solution to this problem if the trace of $G_0(r,\phi;\rho^*,\psi^*)$ on L is imposed as the right-hand side of the condition on L, while the source point (ρ^*,ψ^*) is arbitrarily fixed in the region bounded with L.

Approximate values of the function $u(r,0)$ are exhibited in Table 1 revealing high accuracy level and convergence rate that are attained. The parameters in the setting are chosen as: $R=2$, $a=1$, $\rho^*=0.5$, and $\psi^*=0$. The integral operators in (7) and (8) have been approximated with the standard trapezoidal rule where N represents the uniform partition number on the interval $[0,2\pi]$.

Method of functional equations

We revisit now the boundary value problem in eqns (5)-(6), and develop another approach to its solution. Analogously to the procedure in the method of super-singular integral equations, the solution is expressed in terms of the Green's function $G_0(M;Q)$ of the boundary value problem in eqn (5). A modified single layer type potential is written, however, in a form suggested in [2], which is different of that in eqn (7). That is

$$g(M)=\sum_{i=1}^{k}\int_{\Lambda_i} G_0(M;Q)\mu_i(Q)d\Lambda_i(Q),\quad M\in\Omega \tag{11}$$

where Λ_i, $(i=\overline{1,n})$ represents a fictitious contour embedded in the simply-connected region bounded with L_i. Similarly to the potential in (7), the above representation

Table 2: Approximate solution obtained for an eccentric circular ring by the method of functional equations

Field point, r	Exact value $G(r,0;1.2,0)$	$N=10$		$N=20$	
		$u(r,0)$	Abs. Error	$u(r,0)$	Abs. Error
1.5	0.206787	0.206789	0.000001	0.206787	0.1e-07
1.6	0.152074	0.152107	0.000033	0.152082	0.000008
1.7	0.107102	0.107126	0.000024	0.107107	0000005
1.8	0.068030	0.068043	0.000013	0.068032	0.000002
1.9	0.032762	0.032767	0.000005	0.032763	0.000001
2.0	0.000000	0.000000	0.4e-09	0.000000	0.3e-09

is harmonic in Ω and exactly satisfies the conditions in (6). When the boundary conditions in (6) are satisfied, the following system of functional equations

$$\sum_{i=1}^{k}\int_{\Lambda_i} G_0(M;Q)\mu_i(Q)d\Lambda_i(Q) = -G_0(M;P^*), \quad M\in L_s, \;\; s=\overline{1,k} \tag{12}$$

arises in the density functions $\mu_i(Q)$ of the potential in (11).

Since the points M and Q belong to sets that never intersect, the system in (12) is regular and can, therefore, be directly attacked. One of the key components in the resolving algorithm is the search for geometric parameters of the fictitious contours Λ_i that regularize the solution procedure. A numerical experiment has been conducted to develop some common recommendations as to the shape and location of the fictitious contours Λ_i. A validation problem similar to that of the previous section was considered. The data in Table 2 reveal the accuracy level and the convergence rate that have been attained by the method of functional equation. The region Ω is an eccentric ring whose outer radius $R=2$, the circular aperture of radius $a=0.7$ is centered at $r_0=0.8,\ \phi_0=0$, and the source point is placed at $(1.2,0)$. The standard trapezoidal rule has been used to approximate the integral operators in eqns (11) and (12), with the number of partitions of the integration line denoted with N. Note that the procedure appears to be pretty accurate for a limited number of partitions.

Illustrative Example

Green's functions for a variety of multiply-connected regions of irregular shape have been computed. In this section, an illustrative example is presented for the following mixed boundary value problem

$$\left(\frac{\partial u}{\partial x}-\beta u\right)_{x=0} = u|_{y=0} = \left.\frac{\partial u}{\partial y}\right|_{y=b} = 0, \;\; |u|_{x=\infty}<\infty, \;\; \beta\geq 0 \tag{13}$$

$$u|_{L_1}=0, \;\; \left.\frac{\partial u}{\partial n}\right|_{L_2}=0 \tag{14}$$

stated for the Laplace equation in the triple connected region Ω representing the semi-strip $\{0<x<\infty,\ 0<y<b\}$ weakened with an elliptic L_1 and a "fat-circular" aperture L_2 centered at (x_{10},y_{10}) and (x_{20},y_{20}), respectively.

The Green's function to the above problem for a fixed location (ξ^*,η^*) of the source point can, similarly to the previous example, be found as of eqn (4) where the regular component $g(x,y)$ is, in this case, expressed as the modified potential

$$g(x,y)=\sum_{k=1}^{2}\int_{\Lambda_k} G_0(x,y;\xi,\eta)\mu_k(\xi,\eta)d\Lambda_k(\xi,\eta),\quad (x,y)\in\Omega \tag{15}$$

where $G_0(x,y;\xi,\eta)$ is the Green's function

$$G_0(x,y;\xi,\eta)=\frac{1}{2\pi}\ln\frac{|1-e^{p(z-\overline{\zeta})}||1-e^{p(z+\zeta)}|}{|1-e^{p(z-\zeta)}||1-e^{p(z+\overline{\zeta})}|}$$

$$+\frac{1}{2\pi}\ln\frac{|1+e^{p(z+\overline{\zeta})}||1+e^{p(z-\zeta)}|}{|1+e^{p(z-\overline{\zeta})}||1+e^{p(z+\zeta)}|}$$

$$-\frac{2\beta}{b}\sum_{n=1}^{\infty}\frac{e^{-\nu(x+\xi)}}{\nu(\beta+\nu)}\sin\nu y\sin\nu\eta,\quad p=\frac{\pi}{2b},\quad \nu=(2n-1)p$$

of the boundary value problem in eqn (13) for Laplace equation on the semi-strip $\{0<x<\infty,\ 0<y<b\}$. The above expression for $G_0(x,y;\xi,\eta)$ can be found in [4]. The fictitious contours Λ_1 and Λ_2 are chosen as circles

$$(x-x_{k0})^2+(y-y_{k0})^2=a_k^2,\quad k=1,2$$

concentric with L_k, with a_k representing regularizing parameters.

The potential in (15) represents a harmonic function in Ω that satisfies the conditions in (13). By satisfying the conditions in (14) one obtains the system of regular functional equations

$$-G_0(x,y;\xi^*,\eta^*)=\sum_{k=1}^{2}\int_{\Lambda_k} G_0(x,y;\xi,\eta)\mu_k(\xi,\eta)d\Lambda_k(\xi,\eta),\ \ (x,y)\in L_1 \tag{16}$$

$$-\frac{\partial G_0(x,y;\xi^*,\eta^*)}{\partial n}=\sum_{k=1}^{2}\int_{\Lambda_k}\frac{\partial G_0(x,y;\xi,\eta)}{\partial n}\mu_k(\xi,\eta)d\Lambda_k(\xi,\eta),\ (x,y)\in L_2 \tag{17}$$

in the densities $\mu_k(\xi,\eta)$ of the potential in (15). After the system in (16) and (17) is numerically solved and the densities $\mu_k(\xi,\eta)$ are substituted in (15), values of the Green's function $G(x,y;\xi^*,\eta^*)$ can be computed at any point $(x,y)\in\Omega$.

To illustrate the potential of the approach, we computed a profile of the Green's function $G(x,y;\xi^*,\eta^*)$ by the algorithm that has just been described. The standard trapezoidal rule with uniform partition of the contour of integration appeared to be sufficient for high accuracy level.

A numerical experiment has been conducted to find radii and locations of the fictitious contours Λ_1 and Λ_2 for a variety of L_1 and L_2 shapes as well as optimal values of other computational parameters in the algorithm. We have experimented, in particular, with the partition number N. It has been found out that accurate enough results can potentially be attained with modest values of $N \leq 10$. Note that elliptic aperture may require different of a circle shape of the fictitious contour Λ_1 for "flat" ellipses with big eccentricity. Co-focal to L_1 ellipse Λ_1 better serves, in fact, in such cases compared to the circular shape of Λ_1.

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