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A Numerical Method Based On Element Free Galerkin Method For Lower Bound Limit Analysis

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Summary

A solution procedure for lower bound limit analysis is presented making use of the element free Galerkin (EFG) method rather than of the traditional numerical methods such as finite element method and boundary element method. A reduced basis technique is adopted to solve the mathematical programming iteratively in a sequence of reduced self-equilibrium stress subspaces with very low dimensions. Numerical example in this paper shows that it is feasible and efficient to solve the problems of limit analysis by using the EFG method.

Introduction

The design of engineering structures subjected to external loads demands a realistic assessment of the limit load-carrying capacities, which is a basic requirement for an economical design. In comparison with elasto-plastic analysis, linear elastic analysis always gives conservative results of engineering problems so that the load-carrying capacities of the structures cannot be brought into play effectively. So, elasto-plastic analysis method is more and more widely applied to engineering problems.

It is worth noting that, to many practical engineering problems, only limit load and collapse mode are needed. This fact suggests that limit analysis, intended to determine the load-carrying capacities, is more practical than the elasto-plastic incremental analysis. Up to now, most of numerical methods for solving limit analysis problems are based on traditional numerical methods such as finite element method [1] and boundary element method [2]. In addition, The element free Galerkin (EFG) method [3,4] has achieved remarkable progress in recent years and offers tremendous potential in industrial applications because it requires only nodal data. Therefore, it is feasible and reasonable to carry out the fictitious elastic stress field analysis and the elasto-plastic equilibrium iteration by the EFG method.

In this study, a solution procedure is established by using the EFG method based on the static theorem of limit analysis. A reduced basis technique is adopted to solve the mathematical programming iteratively in a sequence of reduced self-equilibrium stress subspaces with very low dimensions. A numerical example is solved and comparisons with other available solutions are made.

Lower bound theorem of limit analysis

A load set does not exceed the carrying capacity (i.e., the load factor β is not greater than the safety factor β^s, $\beta \le \beta^s$) if, only if, there exists a stress field that

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simultaneously satisfies equilibrium with the loads and complies with the yield conditions of the material. Precisely,

$$\beta^s = \max \beta \tag{1a}$$

$$\text{s.t. } \varphi[\beta \sigma_{ij}^{\mathrm{E}}(\mathbf{x}) + \rho_{ij}(\mathbf{x})] \leq 0 \; \forall \mathbf{x} \in \Omega \tag{1b}$$

$$\rho_{ij,j} = 0 \; \forall \mathbf{x} \in \Omega, \tag{1c}$$

$$\rho_{ij} n_j = 0 \; \forall \mathbf{x} \in \Gamma_t. \tag{1d}$$

Here β is the load factor, $\sigma_{ij}^{E}(\mathbf{x})$ the fictitious elastic stress field under the basic load, $\rho_{ij}(\mathbf{x})$ the self-equilibrium stress field and $\varphi[\cdot]$ is the yield function. The constraint condition (1b) denotes the yield function, conditions (1c) and (1d) represent the relations that self-equilibrium stress field $\rho_{ij}(\mathbf{x})$ must satisfy within the domain Ω and on its boundary Γ_t, respectively.

The element free Galerkin method

The EFG method has been found to be attractive, mainly due to the possibility of overcoming the drawbacks of mesh-based methods, such as the labor-intensive process of mesh-generation, locking, poor derivative solutions, etc. Applications of the EFG method are presented for solving a wide variety of academic and engineering problems [4]. The moving least squares (MLS) approximation employed to appropriate $u(\mathbf{x})$ in the EFG method can be written as

$$\Phi^T(\mathbf{x}) = \mathbf{p}^T(\mathbf{x})\mathbf{A}^{-1}(\mathbf{x})\mathbf{B}(\mathbf{x}) \tag{2}$$

with the matrices$\mathbf{A}(\mathbf{x})$and$\mathbf{B}(\mathbf{x})$being defined by

$$\mathbf{A}(\mathbf{x}) = \sum_{i=1}^{n} w_i(\mathbf{x})\mathbf{p}(\mathbf{x}_i)\mathbf{p}^T(\mathbf{x}_i) \tag{3}$$

$$\mathbf{B}(\mathbf{x}) = \left[\; w_1(\mathbf{x})\mathbf{p}(\mathbf{x}_1), \quad w_2(\mathbf{x})\mathbf{p}(\mathbf{x}_2), \quad \cdots, \quad w_n(\mathbf{x})\mathbf{p}(\mathbf{x}_n) \; \right] \tag{4}$$

where $\mathbf{p}^T(\mathbf{x})$ is a complete monomial basis function, $w_i(\mathbf{x})$ is the weight function associated with node $\mathbf{x}_i$. The commonly used quartic spline function is chosen to approximate $u(\mathbf{x})$ in this paper.

In the elasto-plastic incremental analysis, the load level when plastic state beginning is first calculated, and this step is completely elastic. After that, plastic will happen and an incremental loading scheme is adopted with equilibrium iterations performed for each increment. A step-by-step description of the computational procedure that is employed for each load increment is as follows:

1. The incremental displacement $\Delta\hat{\mathbf{u}}^{(n)}$ for current iteration at loading step $t+\Delta t$ can be obtained by solving the incremental form of the equilibrium equation given as

$$\mathbf{K}_{ep}\Delta\hat{\mathbf{u}}^{(n)} = \Delta\mathbf{Q}^{(n)} \ (n = 0, 1, 2, \cdots) \tag{5}$$

where

$$\mathbf{K}_{ep} = \int_{\Omega} \mathbf{B}^T \mathbf{D}_{ep} \mathbf{B} \mathrm{d}\Omega \tag{6}$$

$$\Delta\mathbf{Q}^{(n)} = \int_{\Omega} \Phi^{T\,t+\Delta t}\bar{\mathbf{F}}\mathrm{d}\Omega + \int_{\Gamma_t} \Phi^{T\,t+\Delta t}\bar{\mathbf{T}}\mathrm{d}\Gamma - \int_{\Omega} \mathbf{B}^{T\,t+\Delta t}\sigma^{(n)}\mathrm{d}\Omega \tag{7}$$

Here, $\mathbf{D}_{ep}$ is the elasto-plastic matrix. It should be mentioned that the essential boundary conditions are imposed by penalty method in this paper because the MLS shape functions do not, in general, satisfy the Kronecker delta condition. Then the incremental strain $\Delta\varepsilon^{(n)}$ is computed using the strain-displacement matrix $\mathbf{B}$.

2. The incremental stress $\Delta\sigma^{(n)}$ for current iteration at loading step $t+\Delta t$ can be given by [5]

$$\Delta\sigma^{(n)} = m\mathbf{D}_e\Delta\varepsilon^{(n)} + \int_0^{(1-m)\Delta\varepsilon^{(n)}} \mathbf{D}_{ep}\mathrm{d}\varepsilon \tag{8}$$

where $\mathbf{D}_e$ is the elastic matrix. In this paper, the tangent predictor-radial return algorithm is employed to perform the integration in Eq.(8). Finally, the stress state can be obtained as

$$^{t+\Delta t}\sigma^{(n+1)} = {}^{t+\Delta t}\sigma^{(n)} + \Delta\sigma^{(n)} \tag{9}$$

3. Check the solutions obtained against a selected tolerance to see if a convergence has occurred. If the solutions of this step touch the convergence condition, go to next loading step; else go to next iteration step of current loading.

Numerical implementation

According to the reduced-basis technique [6], the resulting mathematical programming of the discretized body is as follows

$$\beta^s = \max\beta \tag{10a}$$

$$\text{s.t.}\ \ \varphi[\beta\sigma_i^E + C_1\rho_i^1 + C_2\rho_i^2 + \cdots + C_R\rho_i^R] \leq 0;\ i = 1 - NG \tag{10b}$$

Here, R is the number of basis vectors, $\rho_i^1, \rho_i^2, \cdots, \rho_i^R$ are the selected self-equilibrium stress basis vectors, and $C_1 - C_R$ are the parameters to be determined, NG is the total number of Gaussian points of the discretized body.

The whole process of solving this problem can be divided into some sub-problem. Through computing the equivalent stress at every Gaussian integration point, we can get the elastic limit load amplifier β^E of the body. Then adding a load increment $\Delta\beta^1$ to the body and the body will yield further. After performing $R+1$ different equilibrium iterations based on the initial stress method, the R basis vectors can be obtained as

$$\rho_i^{q(1)} = \sigma_i^{(q-1)} + \mathbf{D}_e \Delta\varepsilon_i^{(q)} - \sigma_i^{(q)} - \mathbf{D}_e \Delta\varepsilon_i^{(q+1)} \; i = 1 - NG, q = 1 - R \tag{11}$$

Applying the Complex method to solve the nonlinear programming problem, the first approximate solution $\beta_{\max}^{(1)}$ and the corresponding self-equilibrium stress field $\rho_i^{(1)}$ can be obtained. Then adding the second load increment $\Delta\beta^2$ to the body, we can get a group of new basis vectors (i.e. $\rho_i^{1(2)}, \rho_i^{2(2)}, \cdots \rho_i^{R-1(2)}$) in the same way. Then take the self-equilibrium stress field $\rho_i^{(1)}$ of last solution as one basis vector $\rho_i^{R(2)}$ to supplement the above new basis vectors. Through the Complex method, we also can get the second approximate solution $\beta_{\max}^{(2)}$ and the corresponding self-equilibrium stress field $\rho_i^{(2)}$. Repeating the above solving process, the computation will be terminated until the following convergence criterion is satisfied

$$\frac{\beta_{\max}^{(n)} - \beta_{\max}^{(n-1)}}{\beta_{\max}^{(n-1)}} \leq \varepsilon, \; n \geq 2 \tag{12}$$

Here ε is a given error tolerance. Our numerical experiences show that, in general, when $n \geq 5$, $\beta_{\max}^{(n)}$ is already a very good approximate solution to the actual limit load factor. According to the numerical experiment, the value of R can be chosen between 4 and 6, in general.

Numerical example

A classical problem in numerical limit analysis has been chosen to demonstrate the accuracy and computational effectiveness of the proposed method. This is a square plate with a central circular hole subjected to biaxial uniform loads P_1 and P_2, as shown in Fig.1. The ratio between the diameter of the hole and the length of the plate is 0.2. Let the yield stress $\sigma_s = 200$MPa, Young's modulus $E = 2.1 \times 10^5$MPa, Poisson's ratio $v = 0.3$.

During the computation, a discretization with 289 EFG nodes is adopted and the quadratic basis functions are used. The support radius r_i is set to be αd_i^9, where α taken as 3.0 in this example is a scaling factor and d_i^9 is the distance to the ninth closet neighboring node from node i. In Fig.2, the numerical results by the present method show reasonable agreement with the lower bound of [7] and slightly lower than the upper bound of [8].

Conclusions

A numerical solution procedure is proposed for lower bound limit analysis by the EFG method. A reduced basis technique and the Complex method are adopted

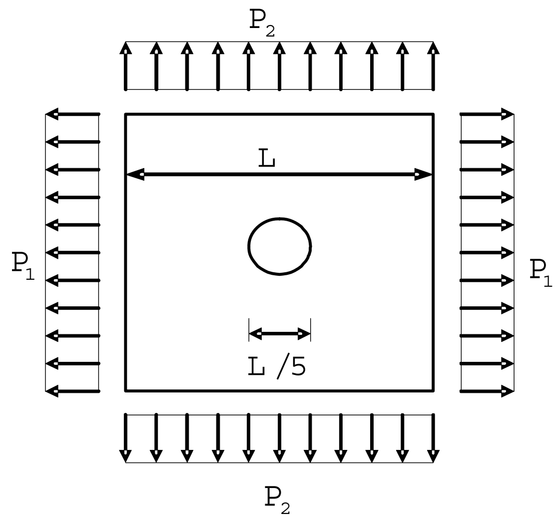

Figure 1: A square plate with a central circular hole

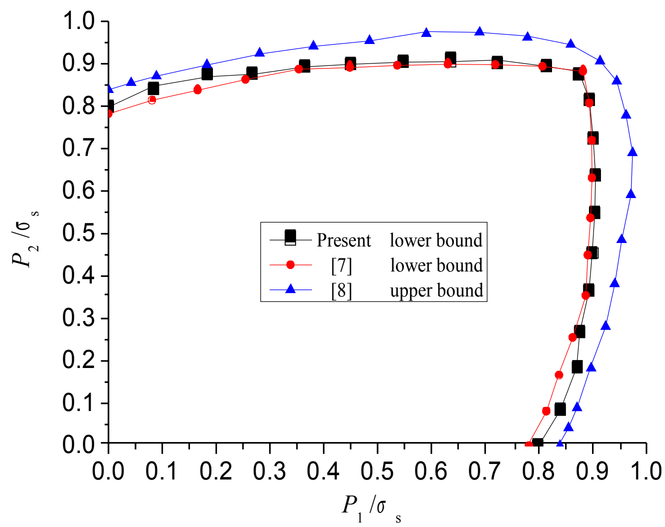

Figure 2: The limit load domains of the plate computed by different methods

herein. By doing these, the proposed numerical method yields good results and reduces the computational cost. A numerical example is given to demonstrate the efficiency and accuracy of the present method. The extension of present numerical procedure to shakedown analysis is in progress.

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