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Elastic-plastic constitutive equation taking account of particle size

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Summary

Composite materials have complicated microstructures. These structures affect macroscopic deformation. In this study, we focus on the particle size effect in composite materials. For this purpose, we derived an elastic-plastic constitutive equation considering size effect by using Eshelby's theory and the strain gradient theory of plasticity. We performed a homogenized finite element analysis using the elastic-plastic constitutive equation. The results obtained from the analysis show increase of the strength with decrease of the particle size in composite materials.

Introduction

Composite materials have inhomogeneous microstructures, and the inhomogeneity affects mechanical properties of composites. The finite element method can be used to clarify how the macroscopic behaviors of solid structures are influenced by the microstructures. In such a case, if the whole structure including the microstructure is modeled by the finite elements, an enormous number of finite elements and enormous amount of computational time are required. To overcome such difficulties, various studies have been performed on the macroscopic constitutive equation for particle-dispersed composites in order to predict their mechanical behaviors. Among them, the Eshelby′s equivalent inclusion method [1] has been used for predicting mechanical behaviors of particle-dispersed composites. For example, Mori and Tanaka [2] developed the mean field theory based on the Eshelby′s equivalent inclusion method. They assumed that stress and strain are uniform in each phase of a composite, and derived the elastic constitutive equation of the composite. On the other hand, Tandon and Weng [3] extended the Mori-Tanaka′s theory to an elastic-plastic constitutive equation for a particle-dispersed composite.

We focus on the particle size effect in particle-dispersed composites. Experimental results [4][5] show that strength of the composite has dependence on particle size. From the view point of material design, it is important to clarify how much particle size affects the macroscopic deformation. Zbib[6] presented a finite element formulation for the strain gradient theory of plasticity which make it possible to consider the particle size effect on the strength of particle-dispersed composites. When we carry out the finite element analysis of structures made of composite materials, micro-structures of composite materials are often homogenized to reduce

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the total degrees-of-freedom of a finite element model. Without such homogenization, it is practically impossible to apply the strain gradient theory of plasticity to the finite element stress analysis of composite materials.

In this study, we tried to combine the equivalent inclusion method (Eshelby′s theory) with the strain gradient theory of plasticity to consider the particle size effect. In Mori-Tanaka′s model and Tandon-Weng′s model, the stress and strain distributions are not taken into account in deriving the elastic-plastic constitutive equation. So we developed a new model to take account of the strain distribution around particles in the macroscopic constitutive equation. Then we combined this model with the strain gradient plastic theory. We performed a homogenized finite element analysis using the elastic-plastic constitutive by base on the Eshelby′s theory and the strain gradient theory of plasticity newly derived in this paper.

Eshelby's theory

We assume that a single particle or inclusion exists in uniform infinite matrix. According to the Eshelby′s equivalent inclusion method, we can evaluate stresses and strains in the inclusion and matrix under the uniform loading, assuming that the real inclusion is replaced by the virtual inclusion or the equivalent inclusion with the same material of the matrix and a certain arbitrary eigenstrain. The total strain of the equivalent inclusion is given as follows:

$$\varepsilon_2 = \varepsilon_2^e + \varepsilon_2^* \tag{1}$$

where ε^* is the eigenstrain, and the subscripts 0, 1 and 2 indicate the matrix at the infinite location, the matrix and the inclusion, respectively. We assume that a composite is subjected to a uniform strain ε_0 at the infinite location. Here, we define the strain difference between ε_1 and ε_0 as ε_0^c, and that between ε_2 and ε_2 as ε_2^c, respectively.

$$\varepsilon_1(x) = \varepsilon_0 + \varepsilon_1^c(x), \qquad \varepsilon_2 = \varepsilon_0 + \varepsilon_2^c \tag{2}$$

In Eq.(2), $\varepsilon_1(x)$ and $\varepsilon_1^c(x)$ indicate the functions of position. On the other hand, ε_2 and ε_2^c in the inclusion are assumed to be constant. We can obtain $\varepsilon_1^c(x)$ and ε_2^c from the Eshelby tensors ($S_{out}(x)$,S_{in}) and eigenstrain.

$$\varepsilon_1^c(x) = S_{out}(x) : \varepsilon^*, \qquad \varepsilon_2^c = S_{in} : \varepsilon^* \tag{3}$$

The eigenstrain given by Eq.(3) is arbitrary. We consider equivalent condition for the stress between the real inclusion and equivalent inclusion, and obtain the following equation.

$$\sigma_2 = \sigma_{eqv} = D_1^e : (\varepsilon_2 - \varepsilon^*), \qquad \sigma_2 = D_2^e : \varepsilon_2 \tag{4}$$

where D_1^e and D_2^e are the elastic matrix for the matrix material and that for the inclusion, respectively. Using Eqs.(2)-(4), we obtain the eigenstrain as a function of ε_0 that satisfies the equivalent condition.

$$\varepsilon^* = A_0 : \varepsilon_0 \tag{5}$$

$$A_0 = \left[\left(I \otimes I - (D_1^e)^{-1} : D_2^e \right)^{-1} - S_{in} \right]^{-1} \tag{6}$$

where I denotes the unit matrix. By substituting Eq.(5) into Eq.(3) and using Eq.(2), we can obtain the strains in the matrix and the inclusion, respectively, as follows:

$$\varepsilon_1(x) = (I \otimes I + S_{out}(x) : A_0) : \varepsilon_0 \tag{7}$$

$$\varepsilon_2 = (I \otimes I + S_{in} : A_0) : \varepsilon_0 \tag{8}$$

As shown in Eqs.(7) and (8), both the strains in the matrix and the inclusion are calculated from the strain at the infinite location and the several material properties included in D_1^e and D_2^e. Then the stresses in the matrix and the inclusion are written as

$$\sigma_1(x) = D_1^e : (I \otimes I + S_{out}(x) : A_0) : \varepsilon_0 \tag{9}$$

$$\sigma_2 = D_2^e : (I \otimes I + S_{in} : A_0) : \varepsilon_0 \tag{10}$$

Macroscopic constitutive law

In this section, we derive a macroscopic constitutive equation from the equations shown in the previous section. Fig. 1 shows a unit cell in the present model consisting of a lot of background cells for numerical integration, which will be mentioned later. For the unit cell, we define the average strain of the matrix $\overline{\varepsilon}_1$ and that of the inclusion $\overline{\varepsilon}_2$ as follows:

$$\overline{\varepsilon}_1 = \frac{1}{V_1} \int_{V_1} \varepsilon_1 dV, \qquad \overline{\varepsilon}_2 = \frac{1}{V_2} \int_{V_2} \varepsilon_2 dV \tag{11}$$

where V_1, V_2 and V are the volume of matrix, that of inclusion and the overall volume, respectively. We assume that the average strain of the overall volume $\overline{\varepsilon}$ is given as:

$$\overline{\varepsilon} = (1 - f)\overline{\varepsilon}_1 + f\overline{\varepsilon}_2, \tag{12}$$

where f is the volume fraction. By substituting Eqs.(7), (8) and (11) into Eq.(12), $\overline{\varepsilon}$ is written as a function of ε_0 as follows:

$$\overline{\varepsilon} = \alpha : \varepsilon_0, \tag{13}$$

$$\alpha = \frac{1}{V} \left[\int_{V_1} I \otimes I + S_{out}(x) : A_0 dV + \int_{V_2} I \otimes I + S_{in} : A_0 dV \right] \tag{14}$$

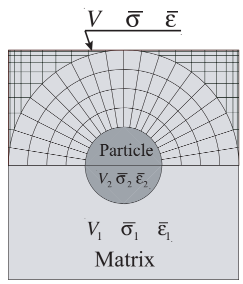


Figure 1: A unit cell in the present model

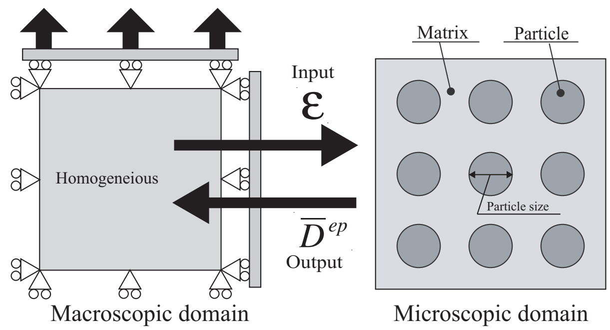


Figure 2: Numerical model

Similarly, the average stress in the matrix $\overline{\sigma}_1$, that in the inclusion $\overline{\sigma}_2$ and that of the overall volume $\overline{\sigma}$ are written as:

$$\overline{\sigma}_1 = \frac{1}{V_1}\int_{V_1} \sigma_1 dV, \qquad \overline{\sigma}_2 = \frac{1}{V_2}\int_{V_2} \sigma_2 dV, \tag{15}$$

$$\overline{\sigma} = \beta : \varepsilon_0, \tag{16}$$

$$\beta = \frac{1}{V}\left[\int_{V_1} D_1^e \left(I \otimes I + S_{out}(x) : A_0\right) dV + \int_{V_2} D_2^e \left(I \otimes I + S_{in} : A_0\right) dV\right], \tag{17}$$

The average stress $\overline{\sigma}$ is written as a function of ε_0 as well as the average strain $\overline{\varepsilon}$. α and β are 4th order tensors that relate the average strain and the average stress with the strain at the infinite location. Finally, we obtain the relationship between the average stress and the average strain by eliminating ε_0 from Eq.(13) and Eq.(16) as follows:

$$\overline{\sigma} = \overline{D}^e : \overline{\varepsilon} = \beta : \alpha^{-1} : \overline{\varepsilon} \tag{18}$$

We can regard the above equation as a constitutive equation for a particle-dispersed composite. When the matrix material is in a plastic state and the inclusion remains elastic, a constitutive equation for an elastic-plastic problem can be obtained by changing the elastic matrix of the matrix material D_1^e in Eqs.(6) and (17) to the elastic-plastic matrix of the matrix material D_1^{ep} and revising Eq.(18) to an incremental form. Conclusively, a constitutive equation for an elastic-plastic problem of a particle-dispersed composite is given as follows:

$$d\overline{\sigma} = \overline{D}^{ep} : d\overline{\varepsilon} = \beta : \alpha^{-1} : d\overline{\varepsilon}, \tag{19}$$

$$\alpha = \frac{1}{V}\left[\int_{V_1} I \otimes I + S_{out}(x) : A_0 dV + \int_{V_2} I \otimes I + S_{in} : A_0 dV\right], \tag{20}$$

$$\beta = \frac{1}{V}\left[\int_{V_1} D_1^{ep} \left(I \otimes I + S_{out}(x) : A_0\right) dV + \int_{V_2} D_2^e \left(I \otimes I + S_{in} : A_0\right) dV\right], \tag{21}$$

where

$$A_0 = \left[\left(I \otimes I - \left(D_1^{ep} \right)^{-1} : D_2^e \right)^{-1} - S_{in} \right]^{-1} \tag{22}$$

Numerical integration is required to calculate α and β both for an elastic problem and for an elastic-plastic problem. Fig. 1 also shows the background cells for the numerical integration. Due to the symmetry, the numerical integration is performed for a half region. When the matrix material is in a plastic state and the inclusion remains elastic, the incremental strain and incremental stress in the matrix material are given as follows, by replacing D_1^e in Eqs.(6) and (9) with :

$$d\varepsilon_1(x) = (I \otimes I + S_{out}(x) : A_0) : d\varepsilon_0, \tag{23}$$

$$d\sigma_1(x) = D_1^{ep} : (I \otimes I + S_{out}(x) : A_0) : d\varepsilon_0. \tag{24}$$

Table 1: Material paramaters

	E [GPa]	σ_y [MPa]	ν	B [MPa]	n
Matrix	30.0	60.0	0.33	135.02	7.79
Particle	130.0	-	0.33	-	-

Strain gradient theory of plasticity

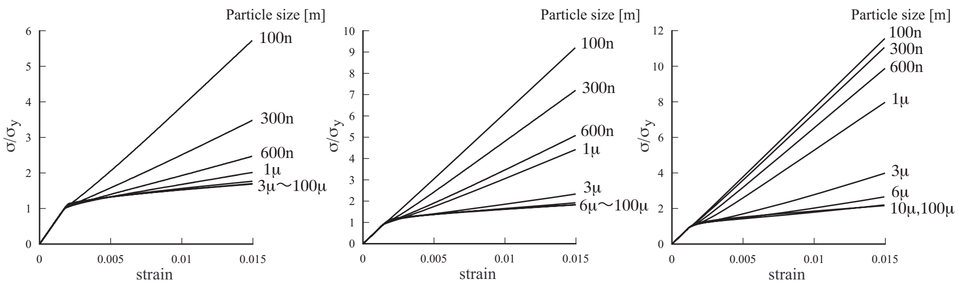
Zbib and Aifantis [7] proposed an elastic-plastic constitutive equation including strain gradient in 1988. Based on the strain gradient theory of plasticity, the gradient-dependent yield stress τ is given by

$$\tau = \kappa(\gamma) - c\nabla^2\gamma \tag{25}$$

where κ is the conventional strain hardening term, and $\nabla^2\gamma$ the strain gradient term which leads to allowing to include the length scale into the plastic constitutive law. We used Eq.(25) as a plastic potential and devided a microscopic constitutive equation D_1^{ep} which is Prandtl-Reuss equation including strain gradient term in Eqs.(21)-(22). Finally, we can incorporate the strain gradient into macroscopic constitutive equation (Eqs.(19)).

Analysis and results

We performed a homogenized finite element analysis using the elastic-plastic constitutive equation based on the Eshelby′s theory and the strain gradient theory of plasticity. In this analysis, we consider two domains, a macroscopic domain and microscopic domain. Fig. 2 shows the concept of this analysis. We apply the usual finite element method to the macroscopic domain with a homogeneous material subjected to uniaxial tensile loading. In the macroscopic analysis, we employ the elastic-plastic constitutive equiation derived in the present paper. In



(a) volume fraction 3 % (b) volume fraction 12 % (c) volume fraction 28 %

Figure 3: Stress-strain curve depending on particle size

the microscopic domain, we calculate the strain and stress distributions around a particle with the use of strain obtained from the macroscopic analysis. The material properties used in the present study are given in Table 1, where E,σ_y and ν denote Young's modulus, yield stress and Poisson's ratio, respectively, and B and n are the coefficients of Ramberg-Osgood relation given by

$$\varepsilon = (\sigma/E) + (\sigma/B)^n \tag{26}$$

Additionally, c the coefficient of the strain gradient term in Eq.(25) is 1.0×10^{-3} [N]. The results of the stress-strain relation at the loading location are shown in Fig. 3 for the volume fraction of the particle f of 3%, 12% and 28%. We considered eight cases of particle size, 100nm, 300nm, 600nm, 1μm, 3μm, 6μm, 10μm, and 100μm. When the particle size is big, the stress-strain curves are almost independent of the particle size. As the particle size becomes smaller, the tangent modulus in a plastic range becomes larger. This fact agrees with the experimental results [4] that a particle-disparsed composite with smaller particle size has a larger mechanical strength.

Summary

In the present study, we derived an elastic-plastic constitutive equation based on Eshelby′s theory and the strain gradient theory of plasticity. Using this constitutive equation, we performed a homogenized finite element analysis of a particle-dispersed composite material. As a results, we obtained the effect of particle size on the strength of a composite material, which agrees with experimental results.

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