Copyright © 2008 ICCES ICCES, vol.5, no.3, pp.157-162

Dynamics of Laminated Composite Beams-Instability Behavior

Alireza Fereidooni[1], Kamran Behdinan[1] and Zouheir Fawaz[1]

Summary

In this paper, dynamics instability of laminated composite beams subjected to axial and harmonically varying time loads are studied. The equations of motion are derived in integral form using the principle of virtual work and the first order shear deformation theory. A five node twenty-two degrees of freedom beam element has been developed to discretize these equations. The regions of dynamic instability of the beam are determined by solving the obtained Mathieu form differential equations. The effects of non-conservative loads and shear stiffness parameters on dynamic instability of the beam are studied.

Introduction

Laminated composites are increasingly being used in the design of load-carrying lightweight structures, where high strength and stiffness to weight ratios are desired. The dynamic analysis of laminated composite structures is important to be investigated when such structures are subjected to varying time loads. Most of the studies in this field are associated to laminated composite plates analysis. Extensive efforts in this area have been done by Reddy [1]. In this study, the dynamics of laminated composite beams is investigated, which has not been studied as comprehensively as laminated composite plates.

The dynamic instability of structures occurs because of parametric resonance. The analytical dynamic instability analysis of the beams subjected to varying time loads has been studied extensively by Bolotin [2]. The finite element method and numerical simulation have been widely used by researchers to study the dynamic analysis of laminated beams. Currently the demand for developing of beam elements and implementation of numerical tools to predict the response of such structures increases. Regular beam models can be used for moderately thick beams. But for slender beams that the length to thickness ratio is extremely high and geometrically nonlinear analysis is required, convergence may become very poor using such models. Also when the beam is shear deformable with small strains and large deformation, developing a model that can take in account the various coupling effects, such as stretching-bending coupling is important in dynamic analysis.

In this study, a new beam model is developed to descritize the equations of motion, which can acquire accurate results with faster convergence and less computation time. The model is a straight beam element with five nodes and twenty-two degrees of freedom with considering transverse bending, stretching and twisting coupling effects.

[1]Department of Aerospace Engineering, Ryerson University, Toronto, ON, M5B 2K3, Canada

Formulation

Consider a laminated prismatic composite four layer beam with uniform thickness and coordinate systems as shown in Figure 1 is subjected to an axial harmonic varying time load.

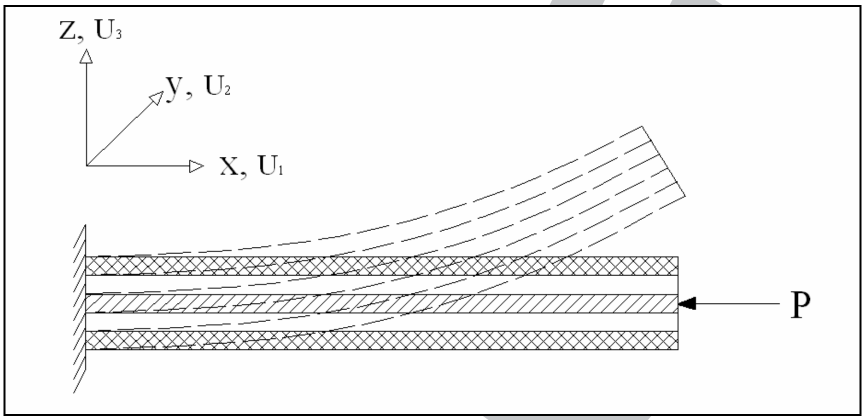

Figure 1: The laminated composite beam geometry and local coordinate system.

The constitutive stress-strain equations for the beam are

$$\mathbf{S} = \mathbf{E}\varepsilon \tag{1}$$

where **S** is the stress resultant matrix, ε the strains matrix, $\boldsymbol{E}$ the laminate stiffness matrix and the strains for each point of the beam in terms of the mid-surface displacements (u_1, u_2, u_3), and rotations(ϕ_1, ϕ_2, ϕ_3) based on the generalized displacement vector $\mathbf{U}^t = \left\{ \begin{array}{cccccc} u_1, & u_2, & u_3, & \phi_1, & \phi_2, & \phi_3, \end{array} \right\}$ are defined as follows;

$$\varepsilon = \begin{bmatrix} \frac{\partial u_1}{\partial x} + y\frac{\partial \phi_3}{\partial x} + \frac{1}{2}[(\frac{\partial u_3}{\partial x})^2 + 2y\frac{\partial u_3}{\partial x}.\frac{\partial \phi_1}{\partial x} + (y\frac{\partial \phi_1}{\partial x})^2] \\ \frac{1}{2}\phi_1^2 \\ \frac{\partial u_2}{\partial x} + \phi_3 + (\frac{\partial u_3}{\partial x} + y\frac{\partial \phi_1}{\partial x}).\phi_1 \\ \frac{\partial \phi_1}{\partial x} \\ \frac{\partial \phi_2}{\partial y} \\ \frac{\partial \phi_1}{\partial y} + \frac{\partial \phi_2}{\partial x} \\ \frac{\partial u_3}{\partial x} + y\frac{\partial \phi_1}{\partial x} + \phi_2 \\ \frac{\partial u_3}{\partial y} + y\frac{\partial \phi_1}{\partial x} + \phi_1 \end{bmatrix} \tag{2}$$

Equations of motion

The governing equations of motion corresponding to the constitutive Eq.(1) are derived using the dynamic version of principle of virtual work;

$$\int_{0}^{T} (\delta W_I - \delta W_E - \delta K)dt = 0 \tag{3}$$

The δW_E, δW_I, δK are the virtual work done by external forces, the virtual work done by internal forces, and the virtual kinetic energy respectively and defined as follow;

$$\delta W_I = \int_{0}^{l} \int_{-b/2}^{b/2} \left(\delta \varepsilon^t \mathbf{S}\right) dydx \tag{4}$$

$$\delta W_E = \int_{0}^{l} \int_{-b/2}^{b/2} f_i.\delta U_j dydx \tag{5}$$

$$\int_{0}^{T} \delta K dt = \int_{0}^{T} \int_{0}^{l} \delta \mathbf{U}^t \mathbf{M} \ddot{\mathbf{U}} dxdt \tag{6}$$

where $f_i$are surface forces per unit area acting on the beam and$\delta U_i$are virtual displacements and **M** represents the mass matrix. It is perceptible that all terms of the integral form of the equations of motion Eq.(3) are displacement dependents and can be discretized through a well established beam finite element model.

Finite Element Model

A five-node twenty-two degrees of freedom beam element, Figure 2, based on the first order shear deformation theory is developed to discretize the integral form of equations of motion. The effects of bending-stretching, shear-stretching, bending-twisting couplings, transverse shear deformation, and continuity have been considered to define optimum degrees of freedom for each node to acquire fast and accurate results.

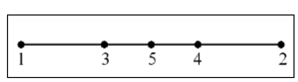

Figure 2: A proposed beam element.

The nodal displacement$\mathbf{U}^e$ can be expressed as the generalized global displacement **U** and shape functions N as defined with the following equation;

$$\mathbf{U} = \mathbf{N}\mathbf{U}^e \tag{7}$$

The shape functions matrix **N** comply with Lagrangian cubic and quadratic interpolation polynomials. All the nodal displacements and rotations are measured at the mid-surface and expressed as;

$$\mathbf{U}^e = \mathbf{U}^e_{1,2} + \mathbf{U}^e_{3,4,5} \tag{8}$$

for end nodes 1 and 2: $\mathbf{U}^e_{1,2} = \{u_{11}, u_{12}, u_{13}, \phi_{11}, \phi_{12}, \phi_{13}, u_{21}, u_{22}, u_{23}, \phi_{21}, \phi_{22}, \phi_{23}\}$.
for nodes 3,4, and 5: $\mathbf{U}^e_{3,4,5} = \{u_{32}, u_{33}, \phi_{31}, u_{42}, u_{43}, \phi_{41}, u_{52}, u_{53}, \phi_{52}, \phi_{53}\}$

Nodal displacements are a combination of the axial displacements (u_{11}, u_{21}), the lateral displacements $(u_{12}, u_{22}, u_{32}, u_{42}, u_{52})$, the transverse displacements $(u_{13}, u_{23}, u_{33}, u_{43}, u_{53})$, and the rotations $(\phi_{11}, \phi_{12}, \phi_{21}, \phi_{22}, \phi_{13}, \phi_{23}, \phi_{31}, \phi_{41}, \phi_{52}, \phi_{53})$.

Substituting equations (4)-(7) into equation (3), the element dynamic equations of motion in matrix form are obtained as follows;

$$\mathbf{M}^e \ddot{\mathbf{U}}^e_t + (\mathbf{K}^e_e + \mathbf{K}^e_G + \mathbf{K}^e_L)\, \mathbf{U}^e_t = \mathbf{F}^e \tag{9}$$

The element mass matrix$\mathbf{M}^e$, and the elastic stiffness matrix$\mathbf{K}^e_e$, the geometric stiffness matrix$\mathbf{K}^e_G$, the loading stiffness matrix$\mathbf{K}^e_L$, and the total external nodal force $\mathbf{F^e}$are given as;

$$\mathbf{M}^e = \rho \iiint_V \mathbf{N}^t \mathbf{N} dV \tag{10}$$

$$\mathbf{K}^e_e = \int_0^{l_e} \int_{-b/2}^{b/2} \mathbf{B}^t \mathbf{D} \mathbf{B} dydx \tag{11}$$

$$\mathbf{K}^e_G = \int_0^{l_e} \int_{-b/2}^{b/2} \left\{ \frac{\partial \varepsilon^t}{\partial \mathbf{U}^e} \mathbf{S} \right\} dydx \tag{12}$$

$$\mathbf{K}^e_L = \frac{\partial \mathbf{F}^e}{\partial \mathbf{U}^e} \tag{13}$$

$$\mathbf{F^e} = \int_0^{l_e} \int_{-b/2}^{b/2} \mathbf{N^T} f_i dydx \tag{14}$$

If the external force is in the parametric form of $p = p_0 + p_t \cos\theta t$, where p_0 is static component of the load, p_t is dynamic component of the load, θ is loading frequency, and t is time, the loading stiffness matrix yields to the static and dynamic stiffness matrices as follows;

$$\mathbf{K}_L = \mathbf{K}^0_L + \mathbf{K}^t_L \cos\theta t \tag{15}$$

A prismatic beam under an axial load may undergo flexural buckling. The buckling load of the beam without shear deformation in account herein is defined as

$p_{cr}^{w} = \frac{k^2\pi^2 Ebh^3}{12l^2}$ and with shear deformation as $\frac{1}{p_{cr}} = \frac{1}{p_{cr}^{w}} + \frac{1}{\hat{S}}$, where $\hat{S}$ is shear stiffness of the beam. The first natural frequency of the beam without shear deformation in account herein is defined as $(\omega_n^w)^2 = \frac{Ebh^3}{12I_0}\left(\frac{\pi}{2l}\right)^4$ and with shear deformation as $\frac{1}{\omega_n^2} = \left(\frac{2l}{\pi}\right)^4 \frac{12I_0}{Ebh^3} + \left(\frac{2l}{\pi}\right)^2 \frac{I_0}{\hat{S}}$, where $I_0 = b\sum_{k=1}^{N} \rho^k (z_{k+1} - z_k)$ and N = number of layers. The direction of the external forces applied on the surface of the beam can be conservative P_c or changes with the beam deformation and to be non-conservativeP_{nc}.

Then the global dynamic equation of motion of the beam becomes;

$$\mathbf{M\ddot{U}}_t + \left(\mathbf{K}_e + \mathbf{K}_G - \mathbf{K}_L^0 - \mathbf{K}_L^t \cos\theta t\right)\mathbf{U}_t = \mathbf{F} \tag{16}$$

Stability Analysis

Equation (16) is a set of coupled Mathieu equations which govern the motion of the beam with periodic solutions. The dynamic instability analysis is essentially about the determination of the boundaries of the dynamic instability regions. Using the approach defined by Bolotin [2], the characteristic equation of the system are obtained as,

$$\begin{vmatrix} \mathbf{K}_e + \mathbf{K}_G - \mathbf{K}_L^0 \pm \frac{\mathbf{K}_L^t}{2} - \frac{\mathbf{M}\theta^2}{4} & -\frac{\mathbf{K}_L^t}{2} & 0 & . \\ -\frac{\mathbf{K}_L^t}{2} & \mathbf{K}_e + \mathbf{K}_G - \mathbf{K}_L^0 - \frac{9\mathbf{M}\theta^2}{4} & -\frac{\mathbf{K}_L^t}{2} & . \\ 0 & -\frac{\mathbf{K}_L^t}{2} & \mathbf{K}_e + \mathbf{K}_G - \mathbf{K}_L^0 - \frac{25\mathbf{M}\theta^2}{4} & . \\ . & . & . & . \end{vmatrix} = \mathbf{0} \tag{17}$$

Expansion of the above determinant in second order form yields the equations of the boundary of principal instability regions of the system. Consider a cross ply $0°/90°/90°/0°$ laminated beam with equal thickness for each lamina. The material and geometry properties of the beam are defined as,

$E_{xx} = 129.20708$ GPa , $E_{yy} = 9.42512$ GPa , G_{xy}=5.15658 GPa , G_{xz}=4.30530 GPa ,

G_{yz}=2.54139 GPa,$\nu_{xy} = \nu_{xz} = 0.3$, $\nu_{yz} = 0.218837$, $\rho = 1550.0666$ Kg-m^3, $b_t = 0.0127$ m, $l = 0.1905$ m. The first principal dynamic regions of instability of the beams with length to height ratio$\frac{l}{h} = 10$ are determined by solving the equation (17) and shown in Figure 3.

Conclusion

In this paper, dynamic stability analysis of laminated composite beams under varying time loading was studied. A five node twenty degrees of freedom beam model was developed to descritize the governing equations of motion. The matrix form of the equations of motion was solved using the symbolic computation to determine the principal instability regions of the beam subjected to conservative and non-conservative loading. The results show:

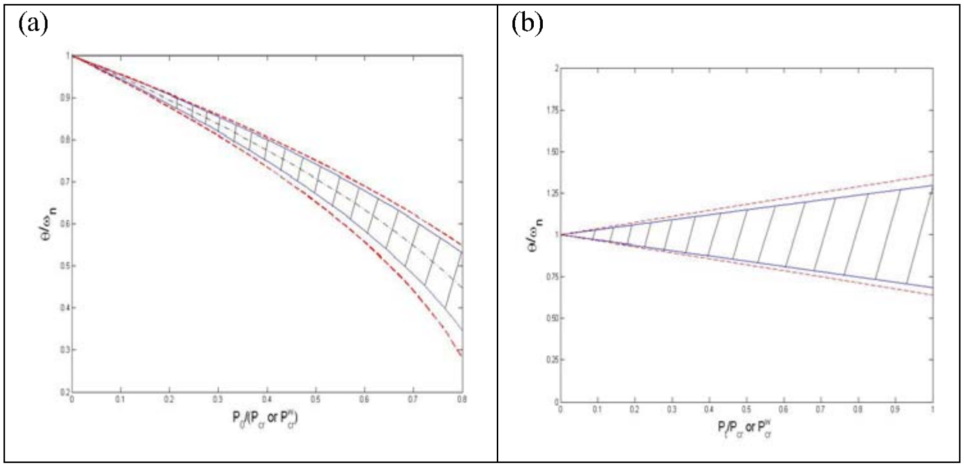

Figure 3: Dynamic principal instability regions of a cantilever cross-ply laminated beams without shear stiffness (crosshatched region) and with shear stiffness (dash lines) subjected to (a) conservative loads (b) nonconservative loads.

- The regions of dynamic instability of the shear un-deformable beams trends to narrowing.
- The lower bound position of the shear deformable beams changes faster than upper bound.
- The instability zones of the shear un-deformable shift towards lower excitation frequencies.
- The instability region of the beam subjected to the nonconservative load doesn't intersect the axis of loading.
- The region of instability for the beam subjected to the nonconservative loading is enlarged in compare to conservative loading system.

References

1. Reddy, J.N. (1997): *Mechanics of Laminated Composite Plates: Theory and Analysis*, CRC Press, Boca Raton, USA.
2. Bolotin, V.V. (1964): *The Dynamic Stability of Elastic Systems*, Holden-Day, San Francisco, USA.