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Modified Lattice Model for Mode-I Fracture Analysis of Notched Plain Concrete Beam using Probabilistic Approach

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Summary

A modified lattice model using finite element method has been developed to study the mode-I fracture analysis of heterogeneous materials like concrete. In this model, the truss members always join at points where aggregates are located which are modeled as plane stress triangular elements. The truss members are given the properties of cement mortar matrix randomly, so as to represent the randomness of strength in concrete. It is widely accepted that the fracture of concrete structures should not be based on strength criterion alone, but should be coupled with energy criterion. Here, by incorporating the strain softening through a parameter 'α', the energy concept is introduced. The softening branch of load-displacement curves was successfully obtained. From the sensitivity study, it was observed that the maximum load of a beam is most sensitive to the tensile strength of mortar. It is seen that by varying the values of properties of mortar according to a normal random distribution, better results can be obtained for load-displacement diagram.

Introduction

Concrete is a highly heterogeneous material. Its properties vary widely from point to point, due to the presence of high strength aggregates, medium strength mortar and weak aggregate mortar interfaces. Further, voids are also present which act as stress raisers. Cracks generally propagate in a direction perpendicular to the maximum tensile stress. Due to high heterogeneity in concrete, they also follow the weakest links in the material. The heterogeneous nature of concrete, makes the crack tortuous, leading its way through weak bonds, voids, mortar and getting arrested on encountering a hard aggregate forming crack face bridges. Experimental study on the tortuosity of crack, crack face bridging etc. was reported by many researchers [1]. Hence, it would be far from reality to model this heterogeneous concrete as a homogeneous one as done in the numerical models presented in literature.

Models considering concrete to be heterogeneous material

Some of the models considering concrete as heterogeneous material can be found in the literature [2-6]. A lattice model, preferred for studying fracture of materials where disorder is important has been applied to study fracture of concrete [7,8]. In earlier models the material was modeled as a plane pin-jointed frame. Certain number of joints was fixed and the coordinates of these joints were selected

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randomly. The members connecting the joints are based on the distance between two joints. The properties of the various members like, Young's modulus E, and tensile strength σ_t were allocated on a random basis. The members are assumed to fail as soon as the stresses in those members equal to its tensile strength. The stiffness is reduced to zero for such members. The model is implemented by analyzing for a succession of small increments of boundary displacement and the corresponding load were obtained. This is continued until the load reaches zero. One of the principal difficulties in using the model arises from the need to limit the size of the sample in order to economize on the computational effort. The softening of the individual members has not been implemented. In lattice models the continuum is discretised such that the grid is made finer in regions of higher stress as often done for finite element methods. In lattice models regular triangular lattice was used for simulating fracture in concrete. Two approaches have been followed to introduce heterogeneity. The first approach used for introducing the heterogeneity is generating the grain structure of concrete [9]. Using a probability distribution function, circles of different diameter at different locations are generated. These circles projected on top of the generated grain structure, represent aggregates. The truss elements situated inside these aggregates are given the properties of aggregates. Different material properties are assigned to the respective bar elements in matrix, aggregates and bond zone.

The second approach of implementing disorder is to specify a statistical distribution of material properties [10,11]. Linear finite element analysis is performed using displacement control method until the load reduces to zero. The model was applied to single edged notched specimens to predict the fracture behavior of plain concrete. But the influence of strain softening on the fracture behavior of plain concrete beam for matrix elements was not implemented [12].

In the modified lattice model presented here, the concrete material is simulated using a combination of constant strain triangles and truss elements. The truss members are given the properties of the matrix randomly so as to represent the randomness of strength in concrete. It is widely accepted that the fracture of concrete material should not be based on strength criterion alone, but should be coupled with energy criterion. Hence, by incorporating the strain softening through a parameter α, the energy concept are introduced in the present model [13,14]. The model is validated with some of the published experimental results [15,16]. The softening branch of load-displacement curves has been successfully obtained using the present model.

Description of the modified lattice model

In this model the concrete is simulated using a combination of constant strain triangles and truss members. The constant strain triangular elements are used to represent the stiffer aggregates and the truss elements are used to represent the softer matrix. Hence in the present model simple elements have been chosen to model the concrete, which is a highly heterogeneous material. The positions of the coarse aggregates are chosen randomly. Pairs of pseudo-random numbers following uniform distribution have been generated. These pairs of random numbers represent the position coordinates of coarse aggregates. These are modeled using constant strain triangle (CST) elements. The domain of these random numbers is the boundary of the specimen analyzed. Even though the size of these CST's can be chosen randomly according to any probability distribution function, for the sake of simplicity, constant size for CST's has been assumed for validating the present model with experimental results.

The number of the aggregates 'n', required for a particular specimen is obtained by keeping the ratio of the total volume of the coarse aggregates to the total volume of the concrete as 0.75. In this process, it is also seen that no two aggregates completely overlap each other. At this stage, the chosen domain is filled with randomly chosen position of aggregates, resulting in 3n nodes (3 nodes per CST). Some nodes are also introduced on the boundary of the domain. Nodes are also introduced inside the domain at places where CST's are far apart. This fixes the total number of nodal points for the lattice model. The model is set ready for the generation of truss members which represent the matrix. Each node is taken in turn, and the distance 'd' to all the other nodes is calculated. If this distance is equal to or less than a chosen distance called threshold distance 'd_o', a member of length 'd' equal to the distance between the nodes is generated. Repetition of elements between the two same nodes is avoided. The various material properties like the Young's modulus E, tensile strength σ_t, and the softening slope parameter α can be ascribed randomly following any probabilistic distribution to the various matrix element and the aggregates. From the material parameters as shown in fig.1, ultimate strain ε_u can be determined.

The Poisson's ratio is taken as 0.1. Now the specimen is subjected to a small prescribed displacement and analyzed. At the end of the analysis, the load required to cause the displacement is noted. The stress in all the elements is monitored at the end of the cycle. If the maximum principal stress in none of the elements has not yet attained their respective tensile strengths, the prescribed displacement is given a small increment and the analysis is repeated. If the maximum principal stress in any of the elements has exceeded the tensile strength of that element, then Young's modulus and the tensile strength are suitably modified as shown in fig. 2. At some prescribed displacement 'd', the maximum principal stress σ_1, in an element 'e' could be greater than its tensile strength σ_t. The stress σ_1 is obtained on the basis of initial Young's modulus, E. In reality, the stress in that element cannot go beyond its limiting tensile strength, σ_t. After reaching σ_t, it follows the softening slope of the stress-strain curve, where the Young's modulus is constantly changing.

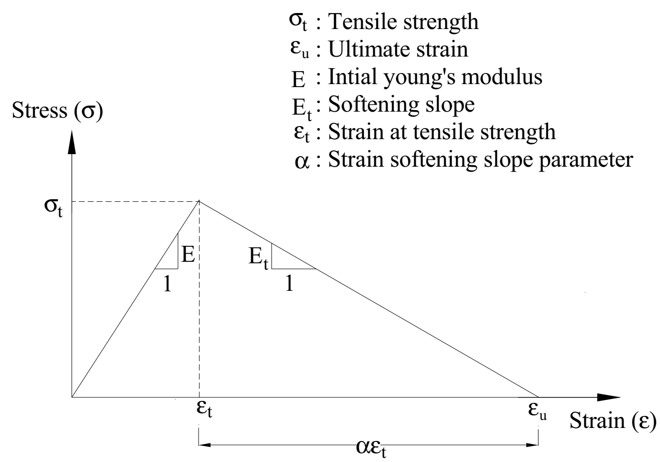

Figure 1: Stress-strain curve of concrete with input details.

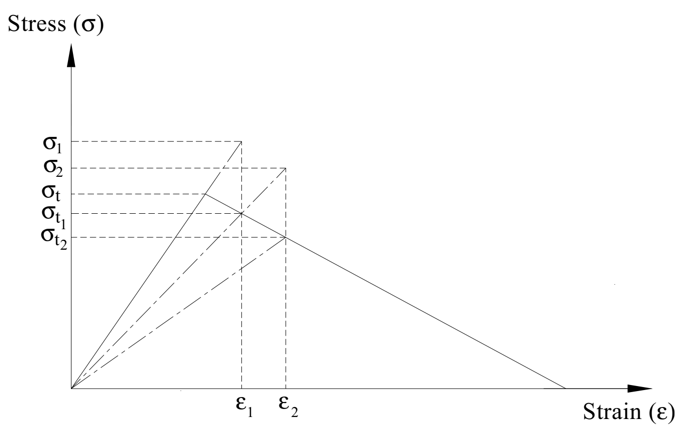

Figure 2: Modification procedure for material properties E and σ_t of an element when the maximum principal stress exceeds σ_t.

Modeling this changing Young's modulus exactly is extremely tedious and time consuming. The present model makes every element to follow the softening slope with a little approximation. The strain corresponding to the stress σ_t based on the initial Young's modulus is ε_1. But according to the softening slope, the stress bearing capacity of that element is σ_1 at that particular strain ε_1.The corresponding Young's modulus also changes to E^1.Thus, these element properties namely Young's modulus and tensile strength are modified to E^1 and σ_{t1} respectively during the increased prescribed displacements and analysis is repeated. If the stress attained during this step is σ_2 based on Young's modulus E^1, the strain ε_2 corresponding to that stress is calculated and the Young's modulus and tensile strength are further modified to E^{11} and σ_{t2}. The analysis procedure is repeated again with small increments of prescribed displacements and changing the material properties at each stage suitably. The element stiffness is made zero as soon as the strain in that element crosses the ultimate strain ε_u of that element. The analysis is continued until final fracture occurs or the load carrying capacity of the beam reduces to zero. The model described above is applied to some of the three-point bend specimen tested by many researchers. Since the three-point bend specimen is symmetric, only half part of the specimen is modeled for the present analysis. As shown in fig. 3, only the central part of the beam around the notch tip is modeled as a heterogeneous material. It is only the region around the crack tip is subjected to the fracture process developing micro-cracks and process zone. Elsewhere, the beam is normally not subjected to the fracture process. Hence, the remainder of the specimen is modeled using plane stress elements. The heterogeneity is modeled in the zone near the crack tip whose size is chosen as equal to the depth of the beam for convenience. Modeling the outer parts of the specimen using plane stress elements is also necessary for including the exact boundary conditions of the experiment. This also saves computer execution time and memory space.

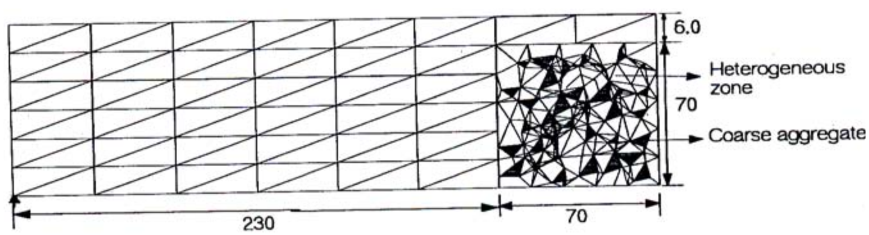

Figure 3: Modeling half part of the three point bend specimen (Nallathambi et al. [1984] (600×76×80 mm) and the zone of heterogeneity

Application of the present model to a notched plain concrete beam

Three point bend specimen tested in references [15,16] is analyzed by the present model. The specimen whose dimensions are shown in Fig. 3 is symmetric. Hence, only half beam is modeled, taking advantage of the symmetry. The zone where the heterogeneity of concrete is modeled is shown in fig. 3. A thin layer at the top of the heterogeneous zone is modeled as homogeneous continuum with plane stress elements for the application of boundary conditions. The width of the heterogeneous zone is approximately taken as equal to the depth of the beam. The area of this zone is calculated as 4900 mm^2 and the area of the aggregates is calculated as 75% of the total area of the heterogeneous zone which is equal to 3675 mm^2.

Taking the average size of the aggregates as 75 mm^2, which are modeled as constant strain triangles, the number of aggregates which make the above area is calculated (here n = 49). Here all the aggregates are assumed to be of the same size for simplicity. Then an approximate threshold distance (d_o = 12 mm) is assumed, to connect the truss members in the heterogeneous zone, which results in a certain number of truss members (n_s).

This modeled beam is now checked for equivalent stiffness. All the elements are given the same material properties E = 10000 N/mm^2, ν = 0.1, and thickness = 1.0 mm. A point load of P = 100N is applied at the tip treating it as cantilever beam and the deflection is noted at the same point. With the threshold distance 'd_o'= 12mm, 967 truss members are established. The deflection from the analysis is found to be 0.7844 mm as compared to the theoretical deflection of 0.7455 mm. Since the deflection is found to be more the stiffness is increased by increasing the number of truss members. It was found by trial with threshold distance 'd_o' = 17 mm, 1590 truss members are established for which deflection obtained from the analysis (0.7491 mm) coincides with the theoretical one with permissible error. Here the material properties of the truss elements representing mortar and CST's representing aggregates are varied randomly according to some probability distribution law. The beam is subjected to an incremental prescribed displacements and stiffness of an element is modified as and when required. The load P for each prescribed displacement is noted. The analysis is stopped when the load P reaches zero which means that the specimen has failed completely.

Discussion of Results from the modified lattice model

The present model is applied to some of the three-point bend specimens tested experimentally [15,16]. The most striking feature of the present model is that it is able to simulate the post-peak softening behavior of the load-deflection curves. The three-point bend specimens when tested under strain controlled conditions, show softening behavior beyond peak load where, the load decreases as the displacement increases. The model appears realistic, since the concrete is not modeled as a homogeneous continuum but modeled as a heterogeneous continuum.

By incorporating the softening parameter α into the model, energy criterion is introduced for the fracture analysis of the concrete. This parameter α is a size dependent parameter, which is based on the concepts of fracture energy G_f. The trend of variation of softening slopes which give the concrete maximum load of the various beams, represent the size effect which is prominent in concrete structures. The area under the stress-strain curve is a measure of the fracture energy G_f, which is the energy consumed in the formation and opening of all micro-cracks per unit area of crack plane. For a given value of E and σ_t as the value of the α increases, the area under the stress-strain curve increases and hence the fracture energy, G_f increases. The fracture energy G_f decreases as the value of α decreases. The fracture energy G_f also depends on initial Young's modulus E and tensile strength σ_t.

Basically, the mortar and aggregates are given different material properties. Later, three cases are analyzed, wherein the values of the material properties of mortar are assumed to vary in the specimen spatially. This is modeled by ascribing the values of the properties of various truss members representing mortar according to some probability distribution law. The three different distributions are constant distribution, uniform random distribution and normally random distribution. Due to lack of information on the values of individual material properties, the beams were analyzed assuming average values of mortar and aggregates. The part of the beam where it is modeled homogeneously, the properties of concrete mentioned in the literature are used. Sensitivity study is also done with regard to the effect of the variation of Young's modulus, tensile strength and the softening slope of the mortar on the maximum load of the beam.

Load-displacement curves from modified lattice model

The average values of the properties of mortar are assumed as E = 25000 N/mm^2, σ_t = 3.0 to 4.0 N/mm^2 and α = 5.0. The portion of the beam where concrete is modeled as a homogeneous material, the values of the properties mentioned in the literature are assumed. The central deflection is plotted against the corresponding load for the various specimens analyzed using the present model. The load-displacement curves are found to be very close to the experimentally obtained load-displacement curves.

Initially, all the truss elements representing mortar are given the same values of the properties, Young's modulus E, tensile strength σ_t and softening slope parameter α and all the CST's represents the aggregates are given a different set of values of the above mentioned properties. The part of the beam outside the heterogeneous zone is given the values of the properties of homogeneous concrete mentioned in the literature. The value of the softening slope parameter α of the truss members representing mortar in the heterogeneous zone are suitably varied, such that the value of the maximum load obtained from the present analysis is closer to the experimental value. A typical plot of the load-displacement curves obtained from the present analysis are shown in figs. 4 and 5. The softening response of the beams beyond peak load, observed in experiments is seen in the present model.

Two beams are analyzed to study the variation in the load-displacement diagrams when the values of properties of the mortar are assumed to vary spatially in the heterogeneous zone, according to three different distributions. Initially, the

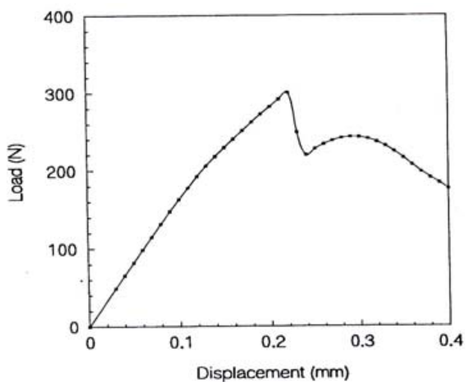

Figure 4: Predicted load-displacement diagram using modified lattice model for a beam tested by Gjorv et al. [1977], (550×50×50 mm, a = 15.0 mm, P_{exp} = 290 N).

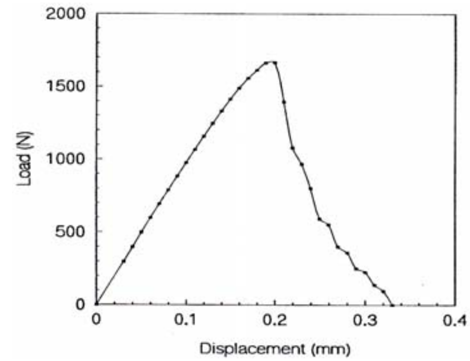

Figure 5: Predicted load-displacement diagram using modified lattice model for a beam tested by Nallathambi et al. [1984], (600×76×80 mm, a = 15.2 mm, P_{exp} = 1700 N).

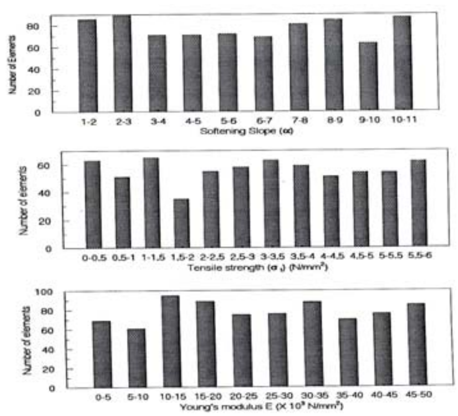

Figure 6: Uniform distribution of the properties (E, σ_t, α) over the elements for a beam tested by Nallathambi et al. [1984] (600×76×80 mm).

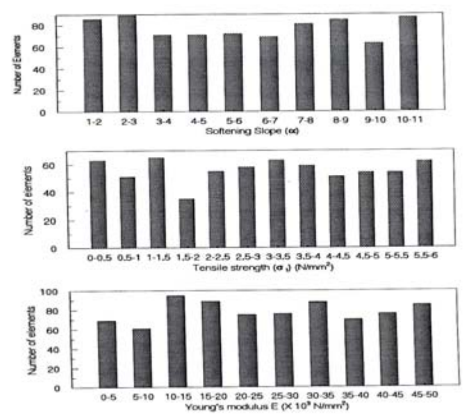

Figure 7: Normal distribution of the properties (E, σ_t, α) over the elements for a beam tested by Nallathambi et al. [1984] (600×76×80 mm).

beams were analyzed by giving constant values of the properties E, σ_t and α of the truss members representing mortar. Next the values of the properties E, σ_t and α of the truss members are varied according to uniform random distribution and normal random distribution as shown in Figs. 6 and 7. The figures include the beam details tested experimentally. A typical load-displacement diagram obtained using the three different random distributions are shown in Fig. 8.

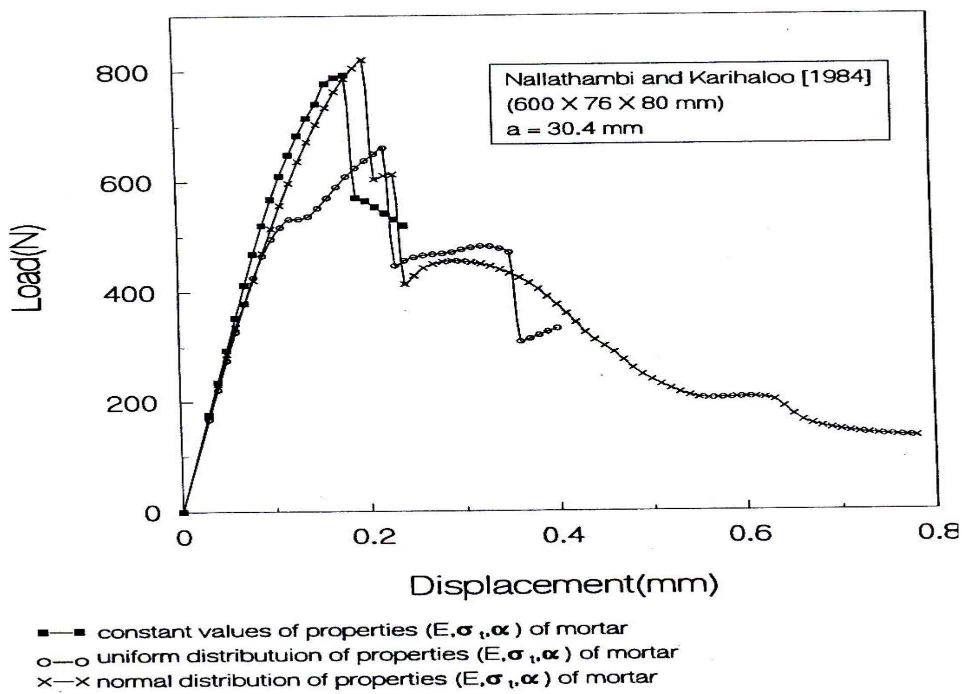

Figure 8: Comparative load-displacement diagrams of a beam tested by Nallathambi et al. [1984] (600×76×80 mm, a = 30.4 mm), for three different distributions of properties (E, σ_t, α) of mortar.

Sensitivity study from the modified lattice model

The variation of the maximum load of a beam with the variation of Young's modulus E, tensile strength σ_t and softening slope parameter α of the mortar matrix has been studied. All the lattice elements representing the mortar are given a set of values to the initial Young's modulus E, tensile strength σ_t and softening slope parameter α. The aggregates and the homogeneous concrete are given different set of values. The effect of the variation of the tensile strength σ_t of the mortar on the maximum load of the beam is studied by varying this parameter and by keeping the other values of properties constant. It is observed that an increase in the tensile strength σ_t of the mortar causes an increase in the maximum load of the beam. Table 1 gives the variation of the maximum load with the changing tensile strength σ_t for a few beams. The values of E and α of the mortar are also mentioned in the Table 1. It can be seen from the fig.1 that as the tensile strength σ_t of a lattice element representing mortar is increased, keeping E and α constant, the area under the stress-strain curve of the element increases. The area under the stress-strain curve of the element represents the fracture energy G_f, which is a measure of the energy required to completely fracture the element. Hence an increase in the tensile strength σ_t causes an increase in the energy required to completely fracture the element. This results in an increase in the maximum load of the beam. It can be seen from Table 1 that the maximum load of a beam is very sensitive to the variation in the tensile strength σ_t of the mortar. The results using the present model on the beam tested (600 x 76 x 80 mm, a = 15.2 mm) indicate, an increase of 167% in the tensile strength σ_t of the mortar, causes an increase of 119% in the maximum load of the beam. Similarly an increase of 31.2% in the maximum load of the beam is observed for an increase of 40% in the tensile strength of the mortar for the beam 600x76x80mm, a = 30.4 mm. Thus, it can be said that the maximum load of a beam is very sensitive to the variation of tensile strength σ_t of the mortar.

Table 1: Sensitivity of maximum load of a beam to the variation in tensile strength of mortar

Details of the beam	Tensile strength N/mm^2	P_{max} N
	3.0	737.1
Nallathambi and Karihaloo	5.0	1120.0
(600×76×80) mm	6.0	1257.3
a = 15.2 mm , P_{exp} = 1700 N	7.0	1489.9
E = 25000.0 N/mm2 , α = 3.0	8.0	1618.2
Nallathambi and Karihaloo		
(600×76×80) mm	5.0	674.0
a = 30.4 mm , P_{exp} = 900 N	6.0	792.1
E = 25000.0 N/mm2, α = 3.0	7.0	884.3
Gjorv et al.	5.0	95.8
(550×50×50) mm	6.0	111.5
a = 25.0 mm , P_{exp} = 150 N	7.0	127.1
E = 25000.0 N/mm2, α = 5.0	8.0	142.3

The variation of maximum load of a beam as a result of variation of initial Young's modulus of the mortar is studied. In this process, the other two properties of mortar, namely, tensile strength σ_t and softening slope parameter α are kept constant. The properties of aggregates and the surrounding homogeneous zone are also kept constant. It is observed that as the initial Young's modulus E of mortar is increased keeping σ_t and α constant, the maximum load of the beam reduces. Table 2 gives the variation of the maximum load with the changing initial Young's modulus for some of the beams analyzed. The values of the other two parameters of the mortar are also mentioned in the table. The observed decrease in the maximum load with increasing initial Young's modulus E of an element is explained based on energy concepts. As the initial Young's modulus E of an element is increased keeping σ_t and α constant, the area under the stress-strain curve of the element decreases as can be seen from fig.1. Thus, as the fracture energy G_f decreases the load bearing capacity decreases. Hence the maximum load of the beam decreases with an increase in initial Young's modulus E, keeping σ_t and α constant for mortar. Table 2 indicates that the maximum load of a beam is less sensitive to the variation in the initial Young's modulus E of the mortar. The results for the beam shown in Table 2 (600×76×80 mm, a = 30.4 mm) indicate that an increase of 120% in the initial Young's modulus E, causes a mere decrease of 6% in the maximum load of the beam when σ_t = 4.0 N/mm^2, α = 7.0. Similarly, for the same beam with different value of softening slope parameter α = 5.0, it is seen that an increase of 80% in the initial Young's modulus E, causes a mere decrease of 4.8% in the maximum load of the beam. Thus, it can be said that the maximum load of a beam is very less sensitive to the variation of initial Young's modulus E of the mortar.

Table 2: Sensitivity of maximum load of a beam to the variation in Young's modulus of mortar

Details of the beam	Young's modulus N/mm^2	$P_{\max}$ N
Nallathambi and Karihaloo	25000	727.0
(600×76×80) mm	35000	704.0
a = 30.4 mm , P_{exp} = 900 N	45000	690.4
σ_t = 4.0 N/mm2 ,α = 7.0	55000	679.6
Nallathambi and Karihaloo	25000	674.0
(600×76×80) mm	35000	653.0
a = 30.4 mm , P_{exp} = 900 N	45000	641.6
σ_t = 4.0 N/mm2 , α = 5.0		
Gjorv et al.	25000	95.8
(550×50×50) mm	35000	95.5
a = 25.0 mm , P_{exp} = 150 N	45000	95.6
σ_t = 4.0 N/mm2 , α = 5.0	55000	96.4
	65000	95.8

The effect of the variation of the softening slope parameter α of the mortar on the maximum load of a beam is studied in a similar manner. Here, the values of softening parameter α, of the lattice element representing mortar were varied keeping E and σ_t constant. The properties of the aggregates and the homogeneous concrete zone for a beam are kept constant during this sensitivity study. The increase in the softening parameter α of an element causes an increase in the fracture energy G_f of the element as shown in Fig 1.Thus, more energy is needed to completely fracture this element. This causes an increase in the maximum load of the beam. The results in the Table 3 indicate that the maximum load of a beam is sensitive to the variation in the softening parameter α of the mortar. It is found from the results for the beam in Table 3 (600x76x80 mm, a = 15.2 mm) that, an increase of 100% in the softening slope parameter α causes an increase of 12.1% in the maximum load of the beam. Similar results on the beam 600x76x80 mm, a = 30.4 mm indicate that an increase of 120% in the softening slope parameter α, causes an increase of 16.4% in the maximum load of the beam. Thus, α does not seem to significantly influence the value of the maximum load of the beam.

Conclusions from the present study

The modified lattice model presented here is more realistic as compared to other existing model due to modeling of heterogeneity of concrete around the crack tip which is subjected to the fracture process, developing micro-cracks and process zone. Elsewhere, the beam is normally not subjected to the fracture process. The load-displacement curves obtained from the present model are found to match the experimentally obtained load-displacement curves. It is seen that by varying the values of properties of mortar in a region according to normal random distribution, better results can be obtained for load-displacement diagram. From the sensitivity study, it is observed that the maximum load of a beam is most sensitive to the tensile strength of mortar, less sensitive to the softening slope parameters of mortar and least sensitive to the Young's modulus of mortar. Hence, it can be concluded that using a mortar of higher tensile strength the load carrying capacity of a beam can be increased.

Table 3: Sensitivity of maximum load of a beam to the variation in softening slope parameter α of mortar

Details of the beam	Young's modulus N/mm^2	$P_{\max}$ N
Nallathambi and Karihaloo	3.0	1120.0
(600×76×80) mm	4.0	1172.2
a = 15.2 mm , P_{exp} = 1700 N	5.0	1218.1
σ_t = 4.0N/mm2 , E = 25000 N/mm2	6.0	1255.6
Nallathambi and Karihaloo	5.0	674.0
(600×76×80) mm	7.0	727.0
a = 30.4 mm , P_{exp} = 900 N	9.0	762.9
σ_t = 4.0N/mm2 , E = 25000 N/mm2	11.0	784.3
Gjorv et al.	5.0	95.8
(550×50×50) mm	7.0	98.6
a = 25.0 mm , P_{exp} = 150 N	9.0	103.6
σ_t = 4.0 N/mm2 , E = 25000 N/mm2	11.0	109.2

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