

## Mesh Quality Improvement for Unstructured Quadrilateral Multigrid Analysis

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### Summary

Due to more complex and severe design, more effective and faster finite element analyses are demanded. One of the most effective analysis ways is the combination of adaptive analysis and multigrid iterative solver, because an adaptive analysis requires several meshes with different node densities and multigrid solver utilizes such meshes to accelerate its computation. However, convergence of multigrid solver is largely affected by initial shape of each element. An effective mesh improvement method is proposed here. It is the combination of mesh coarsening and refinement. A good mesh can be obtained by the method to be applied to an initial mesh, and better convergence is obtained by the improved initial mesh.

### Introduction

FE analysis is one of the most powerful tools for simulation of physical phenomena. In the FE analysis it is necessary to divide an object into elements to predict the distribution of the amount of physical quantities, because sufficient accuracy with less calculation time is obtained. We developed an adaptive mesh generation technique for hexahedral element using local refinement [1]. In the adaptive refinement technique, two or more coarse meshes are generated with different node densities. However, it takes much time to calculate the finest mesh because of large degrees of freedoms. A combination of local hexahedral mesh refinement and a multigrid solver produce the efficient analysis method [2]. But distorted elements cause bad convergence or inconvergence of multigrid computation. Aspect ratio largely affects the convergence [3]. As aspect ratio increases, the convergence cannot be obtained in some cases.

In this paper, local mesh coarsening and refinement techniques to improve quadrilateral mesh quality and multigrid convergence in 2D are presented. Coarsening is removal of the existing distorted elements by modification of the mesh. Refinement is adding some elements to reduce connectivity restrictions on the mesh boundary. How techniques improve mesh quality and practical examples are presented. Effectiveness of the techniques for multigrid convergence is also presented and discussed.

### Local refinement and previous work

Local mesh refinement is a good method to control quality of the mesh. Zienkiewicz-Zhu estimator is used to find inaccurate elements of the mesh, and the elements are

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subdivided by local refinement patterns for a quadrilateral element in 2D as shown Fig. 1. The three patterns are convenient to use for multigrid method because all of nodes in a coarse mesh are located at the same position in a fine mesh.

Multigrid methods are a kind of iterative solver which is an acceleration technique to solve simultaneous linear equations. The fundamental idea is that combination of a fine mesh and a coarse mesh work for smoothing and correction of residual respectively. The employed multigrid technique is a geometric multigrid method. The method requires several hierarchy meshes. In this work the geometric multigrid method is employed.

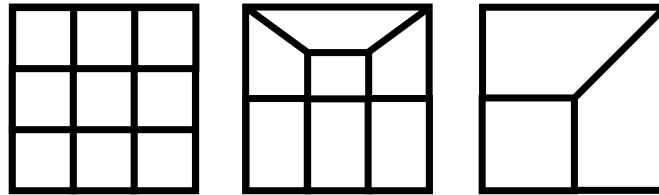


Figure 1: Local refinement patterns in 2D

The calculation of a cantilever problem [2] cannot be converged. Particularly a cantilever analysis with condition of high aspect ratio causes bad convergence of iterative solvers. Since it is necessary to understand the convergence, a cantilever problem in 2D is calculated using three adaptive meshes. The number of iterations increases as aspect ratio of elements increases. In case of aspect ratio over 7, multigrid computation is diverged. It is very important for less number of the iterations to keep initial shape of elements as possible as good.

### Mesh coarsening and refinement techniques

Basically an ideal quadrilateral element is square and one node is shared by four elements. According to this assumption, unnecessary nodes and elements are removed to be close to ideal element shape, and then local refinement is applied to the improved mesh. The proposed method consists of four processes, which are loop coarsening, boundary elements refinement, aspect ratio refinement and boundary smoothing. The effectiveness of the improvements is evaluated aspect ratio and rectangular degree. Aspect ratio is defined by the ratio of sum of quadrilateral opposite sides in Eq. 1.  $a$ ,  $b$ ,  $c$  and  $d$  represent length of sides of a quadrilateral element. Rectangular degree is defined by Eq. 2.  $A$ ,  $B$ ,  $C$  and  $D$  represent angles at a vertex of an element.

$$\text{Max in } \frac{a+c}{b+d} \text{ or } \frac{b+d}{a+c} \quad (1)$$

$$1 - \frac{1}{2\pi} \left( \left| A - \frac{\pi}{2} \right| + \left| B - \frac{\pi}{2} \right| + \left| C - \frac{\pi}{2} \right| + \left| D - \frac{\pi}{2} \right| \right) \quad (2)$$

**Loop coarsening.** The motivation of loop coarsening is to simplify STC chords and to reduce unnecessary elements. Furthermore an adaptive analysis doesn't require fine mesh as an initial mesh. A closed loop of STC chords is shown as a dark line in Fig. 2(a). This figure shows a mesh and corresponding STC chords. The elements with STC loop are deleted and the surrounding elements around the STC loop are connected to keep mesh connectivity. Couples of nodes with dotted lines as shown in Fig. 2(b) are combined. After the process, STC chords become more simply as shown Fig. 2(c) than the initial mesh. A self-intersected STC chord is a special case as shown in Fig. 3. A self-intersected STC chord can't be deleted as it stands. The element which has intersection of STC chord is collapsed as shown in Fig. 3(b). The self-intersected STC chord becomes closed STC loop, and it can be deleted as mentioned the above. The initial mesh becomes a mesh as shown in Fig. 3(c).

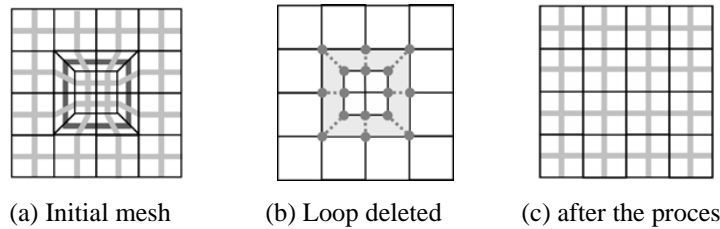


Figure 2: Loop coarsening of closed STC loop

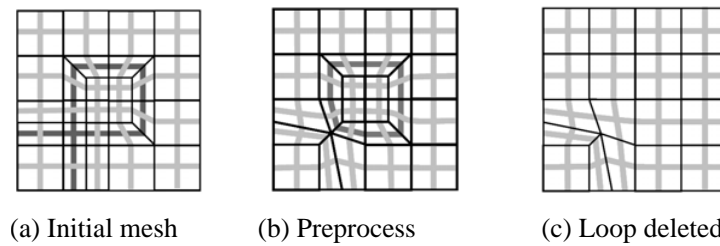


Figure 3: Loop coarsening of self-intersected loop

**Boundary elements refinement.** Sometimes elements on the boundary are generated in Fig. 4(a) and Fig. 5(a). The elements should be removed by reconstruction of the connectivity. Two types of elements concentration are considered. One is element concentration with three elements on straight boundary or almost straight boundary in Fig. 4(a). In this case, two new nodes on boundary are generated and the connectivity is reconstructed as shown in Fig. 4(b). Another is element concentration with four elements on straight boundary or almost straight boundary in Fig. 5(a). In this case, two new nodes on boundary and one new inside node are generated and the connectivity is reconstructed as shown in Fig. 5(b).

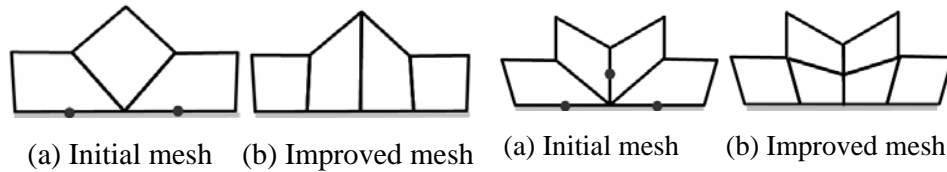


Figure 4: Three elements concentrated      Figure 5: Four elements concentrated

**Aspect ratio refinement.** This process is the most important process in order to keep good convergence of multigrid computation. An element with the highest aspect ratio is found as shown in Fig. 6(a). A quadrilateral mesh contains two STC chords as represented by dotted lines. The chord which intersects the longer side is chosen to subdivide elements. In this case vertical cord is chosen and the elements along the STC chord are subdivided into some or several elements. As a result the mesh is improved as shown in Fig. 6(b). If one STC chord contains the highest aspect ratio element and the lowest aspect ratio element simultaneously, there is no certain solution. To avoid a contradiction in the process, it is important to keep STC chords as simple as possible in the previous two processes.



(a) Initial mesh and STC chords of selected element      (b) Improved mesh

Figure 6: Aspect ratio refinement process

**Boundary smoothing.** Usually a mesh has no uniform node density for efficient and accurate analysis. If aspect ratio refinement is applied to such a mesh, the boundary of the mesh restricts the movements of inner nodes by smoothing. To obtain a good result of smoothing, boundary nodes should be smoothed with keeping original shape of the mesh by natural cubic spline interpolation. The two important points are how to move boundary nodes and which node is fixed. Corner nodes are fixed, and nodes on lines are smoothed and nodes on curves are also smoothed. After this process, Laplacian smoothing is applied to the improved mesh.

### Result and evaluation

Fig. 7, 8 and 9 show the example mesh of aspect ratio improvement. Fig. 7 shows initial model with high aspect ratio of 22.6. The mesh has 1397 nodes. Fig. 8 shows an improved mesh which has 2707 nodes and maximum aspect ratio is 5.3. Fig. 9 shows second improved mesh which has 3787 nodes and maximum aspect

ratio is 2.8. Adaptive local mesh refinement is applied to all of meshes according to the error estimation. Coarse grid correction is conducted in this example.

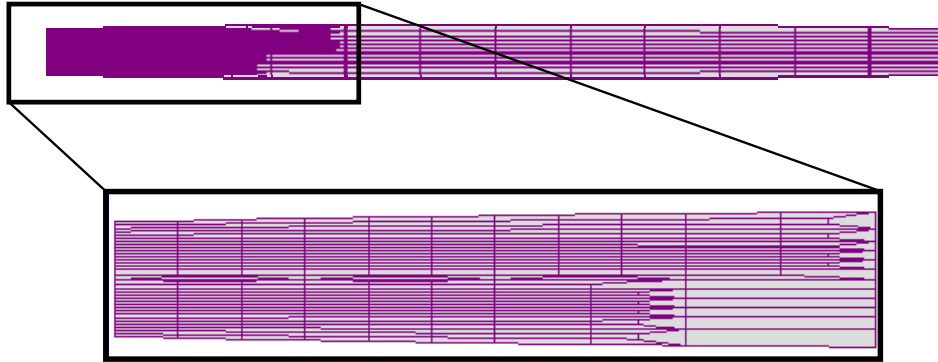


Figure 7: Initial refined mesh with mesh quality improvement: Max. aspect ratio 22.6 and average aspect ratio 18.4



Figure 8: Refined mesh with mesh quality improvement: Max. aspect ratio 5.3 and average aspect ratio 4.6



Figure 9: Refined mesh with mesh quality improvement: Max. aspect ratio 2.8 and average aspect ratio 2.4

Multigrid calculation is diverged in case of first mesh as shown in Fig. 7. In case of second mesh as shown in Fig.8, multigrid calculation is converged at 37 iterations. In case of last mesh as shown in Fig. 9, multigrid calculation is converged at 18 iterations. The number of degrees of freedoms increases, but the convergence is apparently improved by the proposed mesh modification.

Next result is a comparison of calculation time between multigrid and PICCG (diagonal scaling and incomplete Cholesky decomposition). This calculation is also a cantilever model. The mesh for PICCG is the same as the finest mesh of multigrid calculation. Multigrid calculation with modified meshes achieves faster time than PICCG solver.

### Conclusion

The improvement method for mesh quality is proposed. A better mesh can be obtained by the method from a view point of aspect ratio and rectangular degree. The improved mesh provides better convergence than the non-improved mesh, and

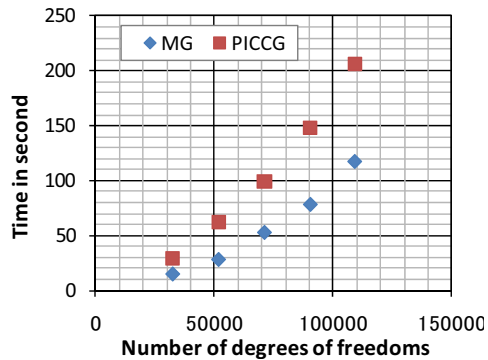


Figure 10: Comparison between PICCG and Multigrid solver  
faster calculation time than PICCG is achieved by multigrid calculation.

### References

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