

# An Analysis of Continued Plane Strain Compression by The Upper Bound Method

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## Summary

An efficient method for quick analysis and design of a class of metal forming processes is proposed. The method is based on the upper bound theorem and deals with continued deformation. Its main advantage is that numerical minimization involved in the upper bound method and solving differential equations for updating the configuration of deforming material are uncoupled. It significantly reduces computation time. Even though the accuracy of the method is unknown in advance, a procedure for verifying its accuracy a posteriori is proposed. The method is applied to plane strain compression of a viscous layer between two rough, parallel plates. In particular, the shape of the free surface in course of deformation is determined and is compared to the shape that might be obtained by means of a typical sequential limit analysis. It is shown that the difference between the shapes is negligible. A consequence of this finding is that the difference between the loads calculated by means of these two methods is also negligible.

**keywords:** Plane strain compression, upper bound method, new approach, viscoplasticity

## Introduction

The classical upper bound theorem is used for analysis of metal forming processes for a long time for example, Avitzur, (1980). The theorem is formulated for instantaneous flow, i.e. the configuration of the considered body must be prescribed. In order to get an approximate solution for continued deformation, a subsequence of upper bound solutions should be found in which the shape of the body is updated with the use of the velocity fields determined from the upper bound solutions. The method is called the sequential limit analysis for example, Yang(1993); Leu(2007); Leu(2008) among others. It is worthwhile to note here that the final solution cannot be considered as an upper bound solution because updated shapes are calculated by means of kinematically admissible velocity fields rather than with the use of the actual velocity field which is of course unknown. A remarkable example of the difference between an upper bound solution for continued deformation and a slip-line solution (i.e. a solution which exactly satisfies all the field equations and boundary conditions) is provided in Richmond(1969). It has been shown in this paper that the upper bound solution predicts a lower bound than the slip-line solution, except for the initial instant, indeed.

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The upper bound theorem is often used in conjunction with the finite element method, for example Lin and Wang(1997); Park, Kim and Bae(1997); Bramley(2001) among others. Analytical approaches have been adopted in Yeh and Yang(1996), Alexandrov and Richmond(2000), Alexandrov(2000), Alexandrova(2001), Wu and Yeh(2007) among others to increase the accuracy of kinematically admissible velocity fields in the vicinity of certain surfaces.

The presented paper concerns with an efficient approach to facilitate quicker analysis of continued deformation for a class of metal forming processes. The approach is based on the upper bound theorem and can be considered as a simplified version of the sequential limit analysis. A verification procedure proposed can be used to compare the solutions found by means of the new method and the sequential limit analysis. The approach is used to analyze compression of a viscous layer between two parallel, rough plates.

### **General approach**

The upper bound theorem is a convenient tool for finding approximate velocity fields in metal forming processes. In general, the velocity field found can be used to determine the field of stress Azarkhin and Richmond(1991). On the other hand, this method of finding approximate solutions is subject to severe restrictions Hill(1963). Therefore, efficient applications of the method can be developed by taking into account specific features of this or that process of deformation. This paper deals with a special type of plane strain metal forming processes where there are two planes of symmetry. A schematic diagram showing such a process is presented in Fig.1 where  $v_0$  is the speed of tool. Because there are two axes of symmetry, it is sufficient to consider the upper right-hand quadrant. The purpose of the approach proposed is to deal with continued deformation in un-steady processes. A difficulty here is that the upper bound theorem in the flow theory of plasticity is formulated for an instantaneous state of the velocity vector and, consequently, for the instantaneous shape of the body which must be known in advance. The key point of the present approach is that a significant portion of the external boundary is determined by the axes of symmetry and tool geometry after any amount of deformation. The only portion of the external boundary that should be found from the solution is  $AB$  (Fig.1).

Obviously, the shape of  $AB$  is given at the initial instant. The main *ad hoc* assumption of the approach proposed is that an approximate shape of  $AB$  after any amount of deformation is found by means of the equation of incompressibility without having a solution for the velocity. The functional following from the upper bound theorem can be minimized for this approximate shape. Of course, the solution obtained in such a manner cannot be considered as a strict upper bound solution. However, it is known that even crude approximations of the velocity field

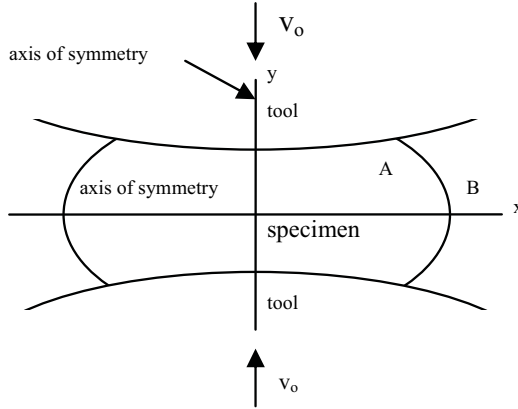


Figure 1: Schematic diagram showing a deformation process of the class considered.

can provide good upper bound solutions Hill, (1956). Therefore, it is first hypothesized that the replacement of the shape of  $AB$  that might be found by means of a series of upper bound solutions with the shape found by the aforementioned simplified method does not introduce a significant error. Then, a procedure to compare the accuracy of the approximation proposed to a standard upper bound approximation for continued deformation based on the kinematically admissible velocity field of the same complexity is introduced to verify the approach developed.

**Compression of a viscous layer between two parallel plates**

Geometry of the process at the initial instant and a Cartesian coordinate system  $x_1y_1$  are illustrated in Fig.2<sup>a</sup>.

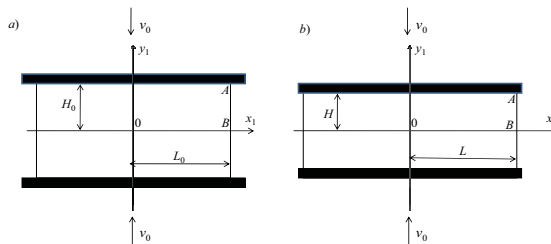


Figure 2: Compression of a layer between two parallel plates – notation.

Because of symmetry, it is sufficient to find the solution in the domain  $x_1 \geq 0$  and  $y_1 \geq 0$ . The initial thickness of the layer is  $2H_0$  and its initial width is  $2L_0$ . The thickness of the layer after any amount of deformation,  $2H$ , is given and is related to the speed of the plates,  $v_0$ , by the equation

$$\frac{dH}{dt} = -v_0 \tag{1}$$

where  $t$  is the time.

The constitutive equations of material consist of the yield condition in the form

$$\sigma_{eq} = K \zeta_{eq}^n \quad (2)$$

and its associated flow rule. The equivalent stress,  $\sigma_{eq}$ , and the equivalent strain rate,  $\xi_{eq}$ , involved in (2) are defined as

$$\sigma_{eq} = \sqrt{\frac{3}{2} \tau_{ij} \tau_{ij}}, \quad \xi_{eq} = \sqrt{\frac{2}{3} \xi_{ij} \xi_{ij}} \quad (3)$$

Here and in (2),  $\tau_{ij}$  are the deviatoric components of the stress tensor,  $\xi_{ij}$  are the components of the strain rate tensor,  $K$  is a rheological constant, and  $n$  is the strain rate sensitivity exponent. The upper bound theorem for the material model under consideration results in the following inequality Hill, (1956)

$$\iiint_V W^* dV - \iint_{\Sigma_t} T_i v_i^* d\Sigma - \iint_{\Sigma_{t+v}} T_i v_i^* d\Sigma \geq \iiint_V W dV - \iint_{\Sigma_t} T_i v_i d\Sigma - \iint_{\Sigma_{t+v}} T_i v_i d\Sigma \quad (4)$$

where  $V$  is the volume of the body,  $\Sigma_t$  and  $\Sigma_{t+v}$  are the parts of its surface,  $T_i$  are the tractions,  $v_i$  is the real velocity field and  $v_i^*$  is a kinematically admissible velocity field. Also, the tractions are prescribed on the part of the surface  $\Sigma_t$  and the mixed boundary conditions are given on the part of the surface  $\Sigma_{t+v}$ . The function  $W$  involved in (4) is defined by the following equation  $dW/d\xi_{eq} = \sigma_{eq}$ . Substituting (2) into this equation and integrating gives

$$W = \frac{K \xi_{eq}^{n+1}}{n+1} \quad (5)$$

Because of symmetry, the velocity boundary conditions are

$$v_y = 0 \quad (6)$$

at  $y_1 = 0$  and

$$v_x = 0 \quad (7)$$

at  $x_1 = 0$ . Here and in what follows  $v_x$  and  $v_y$  are the velocity components in  $x_1$  and  $y_1$  directions, respectively. The vertical component of the velocity vector is also prescribed at the plate surface

$$v_y = -v_o \quad (8)$$

at  $y_1 = H$ . The surface  $AB$  is traction free and the shear stress in the Cartesian coordinate system vanishes at  $y_1 = 0$  and  $x_1 = 0$ . The last boundary condition is a

friction law at  $y_1 = H$ . In the present paper, the maximum friction law is assumed. According to this law the regime of sliding may occur if and only if the friction stress is equal to the maximum possible shear stress supported by the material at a given magnitude of the equivalent strain rate. However, it has been demonstrated in Alexandrov, Danilov and Chikanova(2000) and Alexandrov and Alexandrova(2000) that the regime of sliding never occurs in viscous and viscoplastic materials under the maximum friction law. Therefore, the maximum friction law can be formulated in the form

$$v_x = 0 \tag{9}$$

at  $y_1 = H$ .

Under the aforementioned boundary conditions the functional (4) simplifies, with the use of (5), to Alexandrov(2000)

$$P \leq \frac{K}{v_o} \iiint_V \xi_{eq}^{*n+1} dV \tag{10}$$

This form of the functional has been adopted in Alexandrov(2000) and Tzou and Alexandrov(2006) to analyse the initial flow in compression of axisymmetric specimens. In contrast to the general case of the upper bound theorem for viscous materials, (10) gives an upper bound on the force  $P$  required to deform the specimen.

A conventional procedure of using (10) for continued deformation is to find parameters of a kinematically admissible velocity field chosen at the initial instant by minimizing (10) and, then, to calculate the shape of  $AB$  (Fig.2<sup>a</sup>) in a small time step prescribed. This subsequence of calculations can be repeated as many times as necessary to arrive at the final shape. This procedure is rather time consuming. According to the approach formulated in the previous section, the shape of  $AB$  after any amount of deformation is assumed to be known. In particular, it is natural in the problem under consideration to assume that  $AB$  is a straight line defined by the equation  $x_1 = L$  (Fig.2<sup>b</sup>). Then, it follows from the incompressibility equation that

$$H_o L_o = HL \tag{11}$$

Moreover, the functional (10) can be rewritten in the form

$$P_1 \leq \frac{K}{v_o} \int_o^H \int_o^L \xi_{eq}^{*n+1} dx_1 dy_1 \tag{12}$$

Where  $P_1$  is the force per unit length.

It is convenient to introduce the following dimensionless quantities

$$h = \frac{H}{H_o}, y = \frac{y_1}{H_o}, x = \frac{x_1}{L_o}, h_o = \frac{H_o}{H_o}, \zeta_{eq}^* = \frac{H_o}{v_o} \zeta_{eq}^*, \quad (13)$$

$$p = \frac{P_1}{L_o H_o} \left( \frac{K}{v_o} \right)^{-1} \left( \frac{v_o}{H_o} \right)^{-(n+1)}$$

Substituting (13) into (12) gives, with the use of (11),

$$P \leq \int_o^h \int_o^{h^{-1}} \zeta_{eq}^{*n+1} dx dy \quad (14)$$

Even though the material model is rate-dependent, it follows from (14) that the dimensionless force  $p$  is independent of the speed of loading. It significantly reduces the volume of calculations in the case of a parametric study of the process.

One of the simplest kinematically admissible velocity fields satisfying the incompressibility equation and the boundary conditions (6) to (9) is

$$\frac{v_y^*}{v_o} = - \left[ \alpha \sin \left( \frac{\pi y}{2 h} \right) + (\alpha - 1) \sin \left( \frac{3\pi y}{2 h} \right) \right], \quad (15)$$

$$\frac{v_x^*}{v_o} = \frac{\pi}{2 h h_o} \left[ \alpha \cos \left( \frac{\pi y}{2 h} \right) + 3(\alpha - 1) \cos \left( \frac{3\pi y}{2 h} \right) \right] x$$

where  $\alpha$  is a free parameter. Using (15) the components of the strain rate tensor can be calculated in the form

$$\xi_{xx}^* = -\xi_{yy}^* = \frac{\pi v_o}{2 h H_o} \left[ \alpha \cos \left( \frac{\pi y}{2 h} \right) + 3(\alpha - 1) \cos \left( \frac{3\pi y}{2 h} \right) \right], \quad (16)$$

$$\xi_{xy}^* = -\frac{\pi^2 v_o}{8 h^2 h_o H_o} \left[ \alpha \sin \left( \frac{\pi y}{2 h} \right) + 9(\alpha - 1) \sin \left( \frac{3\pi y}{2 h} \right) \right] x$$

The equivalent strain rate can be found by means of (3) and (16). Then, the integrand of (14) is determined with the use of (13). Obviously, the right hand side of (14) at a given value of  $h$  is a function of  $\alpha$ . In order to obtain the best solution based on the kinematically admissible velocity field chosen, it is necessary to find a minimum of this function with respect to  $\alpha$ . Thus the function  $\alpha(h)$  can be found. This function can be substituted into (15) to get the equations for determining the shape of  $AB$  resulting from the kinematically admissible velocity field.

### Numerical results

Numerical minimization of the right hand side of (14) has shown that  $\alpha$  is almost independent of  $h$ . For example,  $\alpha$  varies in the range  $0.959 < \alpha < 0.963$  when  $h$  decreases from 1 to 1/2 for  $n = 0.3$  and  $h_o = 0.2$ . Moreover, it has been found that an effect of  $n$  and  $h_o$  on the value of  $\alpha$  is also very small. Therefore, it is possible to take  $\alpha$  as a constant in calculations of the shape of  $AB$ . In particular, the relative difference defined as  $|\alpha - \alpha_o|/\alpha$  is less than 0.0075 in the ranges  $0.1 \leq n \leq 0.5$  and  $0.1 \leq h_o \leq 0.2$  for  $\alpha = \alpha_o = 0.961$ . Therefore,  $\alpha = \alpha_o$  is assumed in calculations of the shape of  $AB$ . By definition,  $dy_1/dt = v_y$  and  $dx_1/dt = v_x$ . In the case of the kinematically admissible velocity field these equations become:

$$dy_1/dt = v_y^*, \quad dx_1/dt = v_x^* \quad (17)$$

Substituting (15) into the first equation of this system and taking into account (1), it is possible to arrive with the use of (13) at

$$\frac{dy}{dh} = \left[ \alpha_o \sin \left( \frac{\pi y}{2h} \right) + (\alpha_o - 1) \sin \left( \frac{3\pi y}{2h} \right) \right] \quad (18)$$

The solution to this equation satisfying the initial condition  $y = Y$  at  $h = 1$  can be written in the form

$$Y = hu, \quad \ln h = \int_Y^u \frac{d\chi}{\alpha_o \sin \left( \frac{\pi \chi}{2} \right) + (\alpha_o - 1) \sin \left( \frac{3\pi \chi}{2} \right) - \chi} \quad (19)$$

Thus  $Y$  is a Lagrangian coordinate. In particular, it follows from (19) that  $u = Y$  at  $h = 1$ . Substituting (15) into the second equation of the system (17) and taking into account (1), it is possible to arrive with the use of (13) at

$$\frac{dx}{dh} = -\frac{\pi}{2h} \left[ \alpha_o \cos \left( \frac{\pi y}{2h} \right) + 3(\alpha_o - 1) \cos \left( \frac{3\pi y}{2h} \right) \right] x \quad (20)$$

It is convenient to rewrite this equation in terms of  $u$ . In particular,  $y/h$  and  $dh/h$  can be excluded by means of (19) to give

$$\frac{dx}{x} = -\frac{\pi}{2} \frac{[\alpha_o \cos \left( \frac{\pi u}{2} \right) + 3(\alpha_o - 1) \cos \left( \frac{3\pi u}{2} \right)]}{[\alpha_o \sin \left( \frac{\pi u}{2} \right) + 3(\alpha_o - 1) \sin \left( \frac{3\pi u}{2} \right) - u]} du \quad (21)$$

Since  $u = Y$  at  $h = 1$  and  $x = 1$  on line  $AB$  at the initial instant, the initial condition to equation (21) is  $x = 1$  at  $u = Y$ . Then, the solution to equation (21) can be written in the form:

$$\ln x = -\frac{\pi}{2} \int_Y^u \frac{[\alpha_o \cos \left( \frac{\pi \chi}{2} \right) + 3(\alpha_o - 1) \cos \left( \frac{3\pi \chi}{2} \right)]}{[\alpha_o \sin \left( \frac{\pi \chi}{2} \right) + 3(\alpha_o - 1) \sin \left( \frac{3\pi \chi}{2} \right) - \chi]} d\chi \quad (22)$$

The second of equations (19) determines the value of  $u$  corresponding to any stage of the process (any prescribed value of  $h$  in the range  $0 < h \leq 1$ ). This value depends on  $Y$ . Then, the first of equations (19) and equation (22) give the shape of  $AB$  at this stage of the process in parametric form with  $Y$  being a parameter whose range is  $0 < Y \leq 1$ . The point at the intersection of surface  $AB$  and the axis of symmetry  $Y = 0$  (or  $y = 0$ ) should be treated separately. In particular, equation (20) reduces to

$$\frac{dx}{dh} = -\frac{\pi}{2h}(4\alpha_o - 3)x \quad (23)$$

Using the initial condition  $x = 1$  at  $h = 1$  the solution to equation (23) can be written in the form

$$\ln x = -\frac{\pi}{2}(4\alpha_o - 3) \ln h \quad (24)$$

This solution determines the position of the point of  $AB$  at the axis of symmetry. The variation of the shape of the free surface with  $h$  is illustrated in Fig. 3 for  $n = 1$  and  $h_o = 0.2$ .

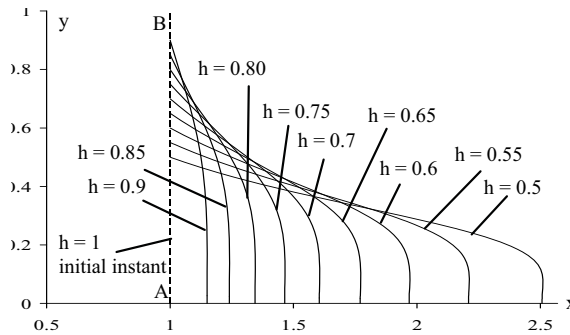


Figure 3: The variation of the shape of the free surface with the current thickness of the layer.

Thus it is necessary to distinguish several definitions for  $AB$  illustrated in Fig.4.

At the initial instant,  $AB$  is a straight line whose position is determined by the equation  $x=1$  (dashed lines in Fig. 3 and Fig. 4). When the simplified approach proposed in the present paper is adopted,  $AB$  is a straight line after any amount of deformation and its position is determined by the equations (11) and (13) in the form  $x = 1/h$ (solid straight line in Fig. 4). Curve 1 in Fig. 4 corresponds to the shape of  $AB$  calculated by means of minimization in (14) and subsequent solving equations (15). The evolution of this shape for the problem under consideration is shown in Fig. 3. Finally, curve 2 in Fig. 4 corresponds to the shape of  $AB$  that might be obtained by using sequential limit analysis. This method has been described, for example, in Yang(1993). The procedure adopted to get curve 1 is much simpler than the one that should be used to find curve 2. However, the latter



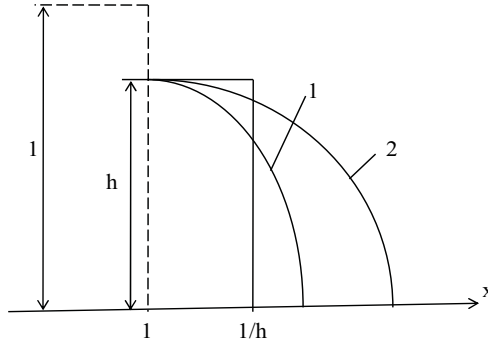


Figure 4: Illustration of various definitions for line  $AB$  (Fig.2) after an arbitrary amount of deformation.

is in general more accurate. Therefore, the question is how close curve 1 is to curve 2. To answer this question a verification procedure described in the next section is proposed.

### Verification of the approach

The velocity field (15) is kinematically admissible independently of the shape of  $AB$  (Fig. 1 and Fig. 4). Therefore, this field and the shapes found (Fig. 3) can be substituted into (10) and, after minimization, new values of  $\alpha$  and  $p$  can be determined. In particular, equation (14) should be replaced with

$$p \leq \int_0^h \int_0^{x_{AB}(y)} \zeta_{eq}^{*n+1} dx dy \quad (25)$$

Here the equation  $x = x_{AB}(y)$  determines the shape of  $AB$  at any stage of deformation (given value of  $h$ ) and is assumed to be known from the calculations completed in the previous section. In general, minimization in (25) can lead to values of  $\alpha$  different from those obtained in the previous section. The difference between values of  $\alpha$  found from (14) and (25) is a measure of the accuracy of determining the shape of  $AB$  with the use of the approach proposed as compared to sequential limit analysis. Numerical minimization in (25) has been carried out at  $h_0 = 0.2$ . The value of  $h$  has varied from 1 to  $1/2$ . The range of  $\alpha$ - values for several values of  $n$  and the arithmetic mean of these  $\alpha$ - values are shown in Table 1.

It is seen from this table that the value of  $\alpha$  is almost independent of  $h$  and  $n$ . Moreover, the difference between any  $\alpha$ - value in the table and  $\alpha_0$  is very small. Therefore, the effect of the simplified assumptions accepted in the approach proposed on the shape of  $AB$  is negligible. In other words, a standard approach to deal with continued deformation by means of the upper bound method for exam-

ple, Yang(1993); Leu(2007); Leu(2008) among others would give the practically same variation of the shape of  $AB$ . Of course, to get such a result, the kinematically admissible velocity field should be taken the same in each of the approaches.

Table 1: Variation of  $\alpha$  during the process for different values of  $n$  and at  $h_o = 0.2$ .

	Range of $\alpha$	Arithmetic mean
$n=0.1$	$0.949 < \alpha < 0.953$	0.95
$n=0.2$	$0.948 < \alpha < 0.957$	0.951
$n=0.3$	$0.948 < \alpha < 0.96$	0.952
$n=0.4$	$0.948 < \alpha < 0.962$	0.953
$n=0.5$	$0.948 < \alpha < 0.965$	0.954

Nevertheless, the value of  $p$  significantly depends on the shape of  $AB$ . In particular, the variation of  $p$ - values with  $h$  is depicted in Fig. 5 for several values of  $n$  and  $h_o = 0.2$ .

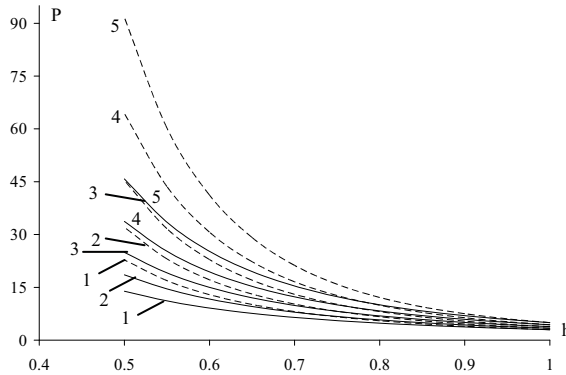


Figure 5: Variation of the dimensionless pressure with the current thickness of the layer at different values of  $n$  (curves 1 correspond to  $n = 0.1$ , curves 2 to  $n = 0.2$ , curves 3 to  $n = 0.3$ , curves 4 to  $n = 0.4$ , curves 5 to  $n = 0.5$ ).

The solid curves correspond to  $p$ - values found by means of (25) and the dashed curves to  $p$ - values found by means of (14). It is seen from this figure that equation (25) leads to much lower values of  $p$  than equation (14) as  $h$  decreases. However, it is not important for the approach proposed. For, once the value of  $\alpha$  and the function  $x = x_{AB}(y)$  at a given value of  $h$  have been found by means of the simplified approach, the value of  $p$  is determined from the right hand side of (25) by simple integration without any minimization.

### Conclusions

A new simplified approach based on the upper bound theorem has been proposed for quick analysis and design of continued deformation of a class of metal forming processes. The approach is restricted to configurations with one unknown

free surface. The shape of this surface after any amount of deformation is first assumed by means of the incompressibility equation. Based on this variation of the surface shape, free parameters of the kinematically admissible velocity field chosen are found by means of the upper bound theorem. Using the kinematically admissible velocity field the corresponding shape of the unknown surface in continued deformation can be found by solving two ordinary differential equations (for plane strain or axisymmetric processes). Finally, the load required to deform the specimen can be calculated from the upper bound theorem where integration should be performed over the configuration found on the previous step. Even though, no firm recommendation can be made on the applicability of the approach to each specific problem, the verification procedure has been proposed. Using this procedure it is possible to reveal an effect of the *ad hoc* assumptions on the final result. In the case when the approach is applicable, its accuracy should be the practically same as the accuracy of standard upper bound solutions for continued deformation based on the same class of kinematically admissible velocity fields. In particular, finite element versions of the upper bound theorem for instantaneous flow, for example Lin and Wang(1997); Park, Kim and Bae(1997); Bramley(2001) among others, can be combined with the approach proposed.

Even though the method has been used in conjunction with a history-independent model, it is reasonable to hypothesize that history-dependent models can be included in the formulation in the same manner as the change in configuration has been taken into account. Of course, this assumption should be verified separately. It is also expected that the method can be extended to a wider class of processes, such as upsetting of hollow cylinders. In this case the minimization of a functional for instantaneous flow, such as (14), should be coupled with solving the differential equations for the change of the shape of free surfaces, such as (15). However, the resulting system is still much simpler than in the case of sequential limit analysis.

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