

## **Vibration of composite folded-plate structures using finite strips**

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### **Summary**

For the analysis of prismatic thin-walled structures, whether single or continuous spanned, the finite strip method is one of the most effective methods developed to date. Significant development of the method has been made, in particular, by adopting various analytical functions in the longitudinal direction to suit various support conditions, including the classical beam vibration functions and the spline functions. In contrast to analytically-defined functions, an alternative finite strip method is presented herein by exploring the use of computed beam vibration functions that takes into consideration explicitly the axial-bending coupling effect of unsymmetrical, cross-ply laminates as well as various combinations of boundary and internal support conditions in an easy and unified manner. These functions are computed a priori using conventional finite elements and defined in terms of the usual one-dimensional shape functions. Due to the limited scope of the paper, only one example is given herein to demonstrate the accuracy and effectiveness of the present method for analyzing composite folded plates.

**keywords:** finite strip; thin-walled structures; free vibration.

### **Introduction**

Finite strip analyses of prismatic thin-walled structures have been receiving a lot of research attention since its first publication in late 1960s by Cheung (1968). Whether the material is isotropic or composite, this type of structures with various boundary and internal support conditions has been studied intensively in the past few decades and summarized by Cheung and Tham (1998). For prismatic, single-span, thin-walled structures which are supported only at the two longitudinal ends, finite strips based on analytical beam vibration functions is particularly advantageous as only a few beam vibration modes are required to generate accurate response quantities. For thin-walled structures continuous over several spans, however, its applications are restricted to those made of isotropic materials and a limited number of support conditions. This limitation could be partly attributed to the fact that analytical beam vibration functions for composite laminated plates with general stacking sequences and various combinations of support conditions cannot be easily found and implemented. In this regard, an alternative finite strip is presented in this paper by exploring the use of computed beam vibration modes that takes into consideration various boundary and internal support conditions in the longitudinal direction of the structures as well as the axial-bending coupling effect of unsymmetrical laminates. Unlike analytical defined beam vibration functions, the present

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beam vibration functions are easily determined a priori by using conventional finite element approach with coupled beam elements. The method takes advantages of the versatility of the conventional finite elements in treating various support conditions in the longitudinal direction and the effectiveness of beam vibration modes in analyzing continuous-span structures. No complicated mathematical functions are involved in the formulation; only linear shape function and cubic beam shape functions are involved, which integration can be done explicitly without using any numerical integration. In this study, only thin-plate finite strip without transverse shear deformation is considered. Due to the limited scope of this paper, only a channel section comprising unsymmetrical, cross-ply laminates is given herein to demonstrate the accuracy and effectiveness of the method.

### Coupled Beam Vibration Modes

To illustrate the computation of coupled beam vibration modes, consider an unsymmetrical, cross-ply (0/90/0/90/. . . .) laminated finite strip of unit width and span  $L_y$  lying along the Y-axis (Figure 1). It is divided into a number of 2-node beam elements along its span, with 5 degrees of freedom per node including axial deformation, lateral and vertical bending deflections and the corresponding rotations. Cross-sectional distortion, twisting and warping effects of the strip are excluded.

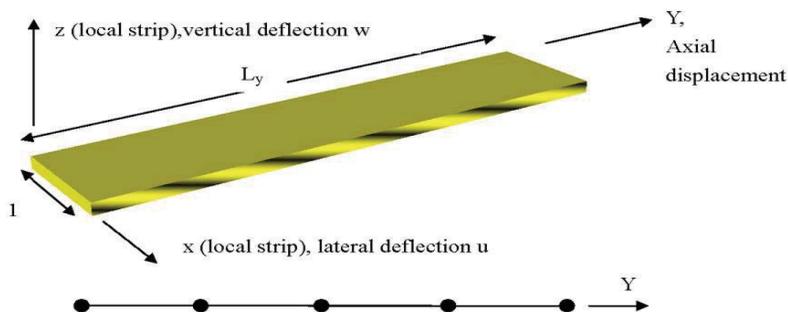


Figure 1: An unsymmetrically laminated finite strip with local strip axes  $(x, Y, z)$  is modeled as four coupled beam elements for computing the vibration modes. Y-axis of the strip corresponds to the longitudinal direction of the structure, while the x-z plane of the strip corresponds to the cross section (X-Z plane) of the structure. For a rigid X-Z plane diaphragm imposed in the structures at any position along the Y-axis, u and w are restrained in the corresponding node of the coupled beam model. For a fixed support at, say,  $Y=0$ , rotations  $du/dy$ ,  $dw/dy$  and axial displacement  $v$  are also restrained.

By ignoring shear deformation due to bending and transverse deformation, the displacement field of the beam strip with thickness  $t$  can be written in terms of three

independent displacement variables, that is:

$$\begin{aligned} U(x, y, z) &= u(x, y) - z \frac{\partial w(x, y)}{\partial x} \\ V(x, y, z) &= v(x, y) - z \frac{\partial w(x, y)}{\partial y} \\ W(x, y, z) &= w(x, y) \end{aligned} \quad (1)$$

Written in terms of the linear shape functions  $[L(y)]$  and cubic beam shape functions  $[N(y)]$ , the three displacement variables for element  $e$  become:

$$\begin{aligned} u^e(y) &= [N(y)]\{u\}^e \\ v^e(y) &= [L(y)]\{v\}^e \\ w^e(y) &= [N(y)]\{w\}^e \end{aligned} \quad (2)$$

where nodal displacements and rotations are represented by symbols in curly brackets  $\{ \}^e$ . The axial strain of the element can be expressed by:

$$\varepsilon_y = \frac{dv(y)}{dy} - z \frac{d^2w(y)}{dy^2} - x \frac{d^2u(y)}{dy^2} \quad (3)$$

Substituting the shape functions into (3) and following conventional finite element formulation, it can be shown that the stiffness and mass matrices for the unsymmetrical, cross-ply, laminated strip are given by:

$$[K]^e = \begin{bmatrix} K_a^v & K_c & 0 \\ K_c^T & K_b^w & 0 \\ 0 & 0 & K_b^u \end{bmatrix}; \quad [M]^e = \begin{bmatrix} M_a^v & 0 & 0 \\ 0 & M_b^w & 0 \\ 0 & 0 & M_b^u \end{bmatrix} \quad (4)$$

where  $K_a^v, K_b^w$  and  $K_b^u$  are the conventional axial and bending stiffness matrices, and  $K_c = \int_{-t/2}^{t/2} \int \frac{d[L(y)]^T}{dy} z D_{11} \frac{d[N(y)]}{dy} dz dy$  represents the axial-bending coupling matrix.

Only uni-directional, fiber-reinforced layers are considered in this paper. The constant  $D_{11}$  represent the material constant along direction of fibers after transforming to the strip axes. The mass matrices  $[M]^e$  represent the usual mass matrices for bar and beam elements respectively. Upon assembly and application of boundary conditions, the vibration modes can be obtained.

As there is no coupling between lateral bending and vertical bending-axial deformation, mode shapes for lateral bending can be obtained separately from the

vertical-axial ones. In general, the m-th mode shape for the lateral displacement can be written as:

$$Y_m^u(y) = \sum_{k=1}^{b_e} [N(y)] \{\alpha_m\}_k \quad (5)$$

Similarly, mode shapes for the vertical and axial displacements are given by:

$$Y_m^v(y) = \sum_{k=1}^{b_e} [L(y)] \{\beta_m\}_k$$

$$Y_m^w(y) = \sum_{k=1}^{b_e} [N(x)] \{\gamma_m\}_k \quad (6)$$

where  $\{\alpha_m\}_k$ ,  $\{\beta_m\}_k$ ,  $\{\gamma_m\}_k$  denotes the corresponding nodal displacements for element k of mode m.  $b_e$  is the total number of elements used. Axial and bending nodal displacements are extracted from the same mode whilst lateral nodal displacements are obtained separately.

Integration of these vibration modes can be done explicitly and, for illustration purpose, the integral of the products of two modes can be written as:

$$I_1 = \int_0^{L_y} Y_m^w Y_n^w dy = \sum_{k=1}^{b_e} \{\gamma_m\}_k^T \left( \int_0^{L_y} [N]^T [N] dy \right) \{\gamma_n\}_k$$

$$I_2 = \int_0^{L_y} \frac{dY_m^v}{dy} Y_n^w dy = \sum_{k=1}^{b_e} \{\beta_m\}_k^T \left( \int_0^{L_y} \frac{d[L]^T}{dy} [N] dy \right) \{\gamma_n\}_k \quad (7)$$

in which the first one represents the product of two vertical bending modes and second one represents the product of an axial strain mode with a vertical bending mode. Others integrals can be calculated in a similar manner. The explicit form of the integration of the beam shape functions and linear shape functions can be easily obtained by Matlab.

### Finite Strip formulation

Expanding the displacement field of a rectangular plate strip in terms of the coupled modes longitudinally whilst linear and cubic beam shape functions are

used across the strip, we have:

$$\begin{aligned}
 u(x, y) &= \sum_{n=1}^p [L(x)] Y_n^u(y) \{u\}_n \\
 v(x, y) &= \sum_{n=1}^p [L(x)] Y_n^v(y) \{v\}_n \\
 w(x, y, z) &= \sum_{n=1}^p [N(x)] Y_n^w(y) \{w\}_n
 \end{aligned} \tag{8}$$

where  $\{u\}_n$ ,  $\{v\}_n$ , and  $\{w\}_n$  denote the nodal line displacement parameters associated with the n-th vibration mode.  $p$  is the total number of vibration modes used. The nodal line displacement vector can be arranged as:

$$\{\delta\}_n = \{ \{u_1, u_2\}_n, \{v_1, v_2\}_n, \{w_1, \theta_1, w_2, \theta_2\}_n \}^T$$

in which  $\theta_i$  represents the rotation about y-axis of nodal line  $i$ .

Assuming small deformation, thin-plate theory, and following conventional virtual work formulation, explicit forms for the stiffness and mass matrices are obtained using the software Mathematica. No numerical integration is required in the calculation of these matrices. By summing up contributions from all strips and transforming to the global axes, the total stiffness matrix  $[K_p]$  and mass matrices  $[M_p]$  can be obtained. Frequencies and mode shapes are then solved using Matlab.

**Numerical Example – A 3-span continuous composite channel section**

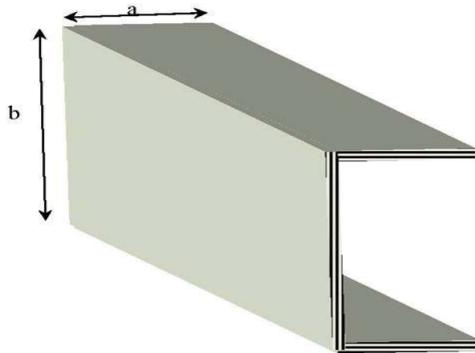


Figure 2: A composite channel section with depth  $b=2$  and equal flange width  $a=1$ .

A composite channel section (see Figure 2) is analyzed to demonstrate application of the present method to laminated, composite, thin-walled structures. Flanges

and web of the channel comprise the same 2-layer (0/90), cross-ply, laminated plates with material parameters  $E_1=25$ ;  $E_2=1$ ;  $G_{12}=0.5$ ;  $\mu_{12}=0.25$  and density=1.0. The section is of uniform thickness  $t=1$  and total span=10. Each flange is divided into 4 strips and the web is divided into 8 strips. The same channel section is supported by four internal rigid in-plane diaphragms at one-third and two-third of the span, in addition to one at either end of the structure. Axial displacement is also restrained at the left end ( $y=0$ ) of the structure. The lowest ten frequencies are obtained using five and eight numbers of coupled modes and 16 strips; by using only five coupled beam modes, excellent agreement with ABAQUS results can be seen from Table 1, with at most 5 percent difference.

Table 1: Comparison of natural frequencies ( $\times 10^{-2}$  Hz) for the continuous channel section

Mode	ABAQUS	FS (5 modes)	FS(8 modes)
1	1.89401	1.91202	1.90074
2	2.06832	2.18200	2.17343
3	2.39463	2.52855	2.51012
4	2.58992	2.61508	2.61074
5	2.71749	2.81050	2.78608
6	2.98037	3.07973	3.06854
7	4.29171	4.34465	4.30814
8	4.43717	4.51529	4.50972
9	4.52752	4.69491	4.66392
10	4.69537	4.79018	4.76309

### Conclusions

An alternative finite strip is presented using computed beam vibration functions instead of analytically-defined beam vibration functions. For the thin-walled structures continuous over several spans considered herein, the method gives accurate frequencies with rapid convergence; in all cases, only a few modes are required to give solutions with less than 5 percent difference for the first few frequencies. Other examples and further development of the method will be presented in the conference and published elsewhere.

### Acknowledgement

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### References

1. **Cheung Y.K.** (1968) Finite strip method in the analysis of elastic plates with two opposite ends simply supported. *Proceedings of Instn. Civ. Engrs.*, 1968, Vol. 40, 1-7.

2. **Cheung Y.K. and Tham L.G.** (1998) *Finite strip method*, Boca Raton, Fla: CRC Press.

