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On the solution of a coefficient inverse problem for the non-stationary kinetic equation

Mustafa Yildiz[1]

Summary

The solvability conditions of an inverse problem for the non-stationary kinetic equation is formulated and a new numerical method is developed to obtain the approximate solution of the problem. A comparison between the approximate solution and the exact solution of the problem is presented.

keywords: Kinetic Equation, Inverse Problem, Galerkin Method, Symbolic Computation

Introduction

Inverse problems appear in many important applications of physics, geophysics, technology and medicine. One of the characteristic features of these problems for differential equations is their being ill-posed in the sense of Hadamard. The general theory of ill-posed problems and their applications is developed by A. N. Tikhonov, V. K. Ivanov, M. M. Lavrent'ev and their students [8-10, 13-15]. Inverse problems for kinetic equations are important both from theoretical and practical points of view. Interesting results in this field are presented in Amirov [1-3], Anikonov, Kovtanyuk and Prokhorov [4], Anikonov and Amirov [5], Anikonov [6], Hamaker, Smith, Solmon and Wagner [7].

In this paper, the solvability conditions of an inverse problem for the non-stationary kinetic equation is formulated in the case where the values of the solution are known on the boundary of a domain. A new numerical method based on the Galerkin method is developed to obtain the approximate solution of the problem. A comparison between the computed approximate solution and the exact solution of the problem is presented.

The notations to be used in this paper are introduced below:

For a bounded domain G, $C^m(G)$ is the Banach space of functions that are m times continuously differentiable in G; $C^\infty(G)$ is the set of functions that belong to $C^m(G)$ for all $m \geq 0$; $C_0^\infty(G)$ is the set of finite functions in G that belong to $C^\infty(G)$; $L_2(G)$ is the space of measurable functions that are square integrable in G, $H^k(G)$ is the Sobolov space and $\overset{\circ}{H}{}^k(G)$ is the closure of $C_0^\infty(G)$ with respect to the norm of $H^k(G)$. These standard spaces are described in detail, for example, in Lions and Magenes [11] and Mikhailov [12].

[1]Zonguldak Karaelmas University, Faculty of Arts and Sciences, Department of Mathematics, 67100, ZONGULDAK / TÜRKİYE. mustafayildiz2002@hotmail.com

Statement of the Problem

In this work, the kinetic equation

$$\frac{\partial u}{\partial t}+\sum_{i=1}^{n}\left(v_i\frac{\partial u}{\partial x_i}+f_i\frac{\partial u}{\partial v_i}\right)-a(x,v,t)\,u=0 \tag{1}$$

is considered in the domain

$$\Omega=\left\{(x,v,t): x\in D\subset\mathbb{R}^n,\ v\in G\subset\mathbb{R}^n,\ n\geq 1,\ t\in(0,T)\right\},$$

where the boundaries ∂D, $\partial G\in C^2$, $a(x,v,t)$ is an unknown function and satisfies the equation

$$\left\langle a,\widehat{L}\eta\right\rangle=0,\ \widehat{L}=\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i\partial v_i} \tag{2}$$

for any $\eta\in H_{1,2}(\Omega)$ whose trace on $\partial\Omega$ is zero. $\langle .,.\rangle$ is a scalar product in $L_2(\Omega)$. $H_{1,2}(\Omega)$ is the set of all real-valued functions $u(x,v,t)\in L_2(\Omega)$ that have generalized derivatives u_{x_i}, u_{v_i}, $u_{x_iv_j}$, $u_{v_iv_j}$ $(i,j=1,2,\ldots,n)$, which belong to $L_2(\Omega)$.

Equation (1) is extensively used in plasma physics and astrophysics. In applications, $u(x,v,t)$ represents the number (or the mass) of particles in the unit volume element of the phase space in the neighbourhood of a point (x,v) at the moment t, $a(x,v,t)$ is the absorption term and $f=(f_1,\ldots,f_n)$ is the force acting on a particle.

Problem 1 *Determine the functions $u(x,v,t)$ and $a(x,v,t)$ defined in Ω from equation (1), provided that $u(x,v,t)>0$, the function $a(x,v,t)$ satisfies (2) and the trace of $u(x,v,t)$ is known on the boundary, i.e.,*

$$u|_{\partial\Omega}=u_0.$$

Remark 1 *It is easy to see that Problem 1 is non-linear because equation (1) contains a production of unknown functions $u(x,v,t)$ and $a(x,v,t)$.*

Remark 2 *In practise, the function $a(x,v,t)$ depends only on the argument x and t, i.e, the problem is overdetermined. In [3], a genereal scheme is presented to overcome this difficulty: It's assumed that the unknown coefficient in the problem depends not only on the variables x and t but also on the direction v in a specific way, that is, $\widehat{L}a=0$.*

Remark 3 *By introducing a new unknown function $\ln u=y$, Problem 1 can be reduced to the following problem:*

Problem 2 *Find a pair of functions* (y,a) *defined in* Ω *satisfying the equation*

$$Ly \equiv \frac{\partial y}{\partial t} + \sum_{i=1}^{n} \left(v_i \frac{\partial y}{\partial x_i} + f_i \frac{\partial y}{\partial v_i} \right) = a(x,v,t) \tag{3}$$

provided that $a(x,v,t)$ *satisfies* (2) *and* y *is known on* $\partial\Omega$*:* $y|_{\partial\Omega} = e^{u_0} = y_0$.

To formulate the solvability theorems for Problem 2, we need the following notation of $\Gamma(A)$:

We select a subset $\{w_1, w_2, ...\}$ of $\widetilde{C}_0^3 = \{\varphi : \varphi \in C^3(\Omega), \varphi = 0 \text{ on } \partial\Omega\}$ which is orthonormal and everywhere dense in $L_2(\Omega)$. Let P_n be the orthogonal projector of $L_2(\Omega)$ onto M_n, where M_n is the linear span of $\{w_1, w_2, ..., w_n\}$. $\Gamma(A)$ denotes the set of functions y with the following properties

i) $y \in L_2(\Omega)$, $Ay \in L_2(\Omega)$ in the generalized sense, where $Ay = \widehat{L}Ly$;

ii) There exists a sequence $\{y_k\} \subset \widetilde{C}_0^3$ such that $y_k \to y$ in $L_2(\Omega)$ and $\langle Ay_k, y_k\rangle \to \langle Ay, y\rangle$ as $k \to \infty$.

The condition that $Ay \in L_2(\Omega)$ in the generalized sense means that there exists a function $f \in L_2(\Omega)$ such that $\langle y, A^*\varphi\rangle = -\langle f, \varphi\rangle$ and $Ay = f$ for all $\varphi \in C_0^\infty(\Omega)$, where A^* is the differential operator conjugate to A in the sense of Lagrange.

Solvability of the Problem

Now we are in position to formulate the results which can be proved by a similar technique as that of Theorem 2.2.1 and Theorem 2.2.2 in [3] page 61.

Theorem 1 *Let* $f \in C^1(\Omega)$ *and assume that the following inequality holds for all* $\xi \in \mathbb{R}^n$ *:*

$$\sum_{i,j=1}^{n} \frac{\partial f_i}{\partial x_j} \xi^i \xi^j \geq \alpha_1 |\xi|^2, \tag{4}$$

where α_1 *is a positive number. Then Problem* 2 *has at most one solution* (y,a) *such that* $y \in \Gamma(A)$ *and* $a \in L_2(\Omega)$.

Problem 3 *Given the equation*

$$Ly = a + F$$

where the function a *satisfies* (2) *and* F *is a known function in* $H_2(\Omega)$*, find the pair of functions* (y,a) *under the condition that*

$$y|_{\partial\Omega} = 0.$$

Problem 2 can be reduced to Problem 3, a similar reduction is presented in [3] page 65 for an another kinetic equation.

Theorem 2 *Under the assumptions of Theorem 1, suppose that $F \in H_2(\Omega)$. Then there exists a solution (y,a) of Problem 3 such that $y \in \Gamma(A)$, $y \in H_1(\Omega)$, $a \in L_2(\Omega)$.*

Theorem 3 *Under the hypotheses of Theorem 1, assume that $u_0 \in H_2(\partial\Omega)$ and $u_0 \geq \alpha_0$, where α_0 is a positive number. Then there exists a solution (u,a) of Problem 1 such that $u \in H_1(\Omega)$, $a \in L_2(\Omega)$.*

Algorithm of Solving the Inverse Problem

An approximate solution to Problem 3 will be sought in the following form

$$y_N = \sum_{i=1}^{N} \alpha_{N_i} w_i.$$

For the solution algorithm, the domains $D = \{x \in \mathbb{R}^n : |x| < 1\}$, $G = \{v \in \mathbb{R}^n : |v| < 1\}$ are chosen. We consider the complete systems $\left\{x_1^{i_1} \ldots x_n^{i_n}\right\}_{i_1,\ldots,i_n=0}^{\infty}$, $\left\{v_1^{j_1} \ldots v_n^{j_n}\right\}_{j_1,\ldots,j_n=0}^{\infty}$, $\left\{1, t, t^2, \ldots\right\}$ in $L_2(D)$, $L_2(G)$ and $L_2(0,T)$ respectively. The approximate solution can be written in the following form:

$$y_N = \sum_{i_1,\ldots,i_n,j_1,\ldots,j_n,k=1}^{N} \alpha_{N_{i_1,\ldots,i_n,j_1,\ldots,j_n,k}} w_{i_1,\ldots,i_n,j_1,\ldots,j_n,k}\, \eta(x)\, \mu(v)\, \zeta(t) \tag{16}$$

where

$$w_{i_1,\ldots,i_n,j_1,\ldots,j_n,k} = \left\{x_1^{i_1} \ldots x_n^{i_n} v_1^{j_1} \ldots v_n^{j_n} t^k\right\}_{i_1,\ldots,i_n,j_1,\ldots,j_n,k=0}^{\infty}$$

$$\eta(x) = \begin{cases} 1 - |x|^2, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}, \quad \mu(v) = \begin{cases} 1 - |v|^2, & |v| < 1 \\ 0, & |v| \geq 1 \end{cases}, \quad \zeta(t) = \begin{cases} 1 - t^2, & |t| < 1 \\ 0, & |t| \geq 1 \end{cases}.$$

In expression (16), unknown coefficients $\alpha_{N_{i_1,\ldots,i_n,j_1,\ldots,j_n,k}}$, $i_1, \ldots, i_n$, $j_1, \ldots, j_n$, $k = 1, \cdots, N$ are determined from the following system of linear algebraic equations (SLAE):

$$\sum_{i_1,\ldots,i_n,j_1,\ldots,j_n,k=1}^{N} \left(A\left(\alpha_{N_{i_1,\ldots,i_n,j_1,\ldots,j_n,k}} w_{i_1,\ldots,i_n,j_1,\ldots,j_n,k}\right) \eta \mu \zeta, w_{i'_1,\ldots,i'_n,j'_1,\ldots,j'_n,k'}\right)_{L_2(\Omega)} \tag{17}$$
$$= \left(\mathscr{F}, w_{i'_1,\ldots,i'_n,j'_1,\ldots,j'_n,k'}\right)_{L_2(\Omega)}, \; i'_1, \ldots, i'_n, j'_1, \ldots, j'_n, \; k' = 1, \cdots, N.$$

Algorithm 1 $(LeftSLAE)$ *Left side of each equation in* (17) *is constructed.*

INPUT: N, i'_1, ..., i'_n, j'_1,..., j'_n, k', $w_{i'_1,\ldots,i'_n,j'_1,\ldots,j'_n,k'}$

OUTPUT: Left hand side of each equation in (17) : *LeftSum*

Set LeftSum=0;

For $i_1 = 1,...,N$ *do ... For* $i_n = 1,...,N$ *do*
For $j_1 = 1,...,N$ *do ... For* $j_n = 1,...,N$ *do For* $k = 1,...,N$ *do*
$LeftSum = LeftSum + \left(A\left(\alpha_{N_{i_1,...,i_n,j_1,...,j_n,k}} w_{i_1,...,i_n,j_1,...,j_n,k}\right)\eta\mu\zeta, w_{i'_1,...,i'_n,j'_1,...,j'_n,k'}\right)_{L_2(\Omega)}$
end k end j_n *...end* j_1 *end* i_n *...end* i_1
STOP (The procedure is complete.)

Algorithm 2 *This algorithm computes the approximate solution using Algorithm* 1.

INPUT: N, $F(x,v,t)$, $f(x,v,t)$
OUTPUT: Approximate solution u_N *and the coefficient* a
$SLAE = \{\}$, $y_N = 0$,
For $i'_1 = 1,...,N$ *do ... For* $i'_n = 1,...,N$ *do*
For $j'_1 = 1,...,N$ *do For* $j'_n = 1,...,N$ *do For* $k' = 1,...,N$ *do*
$SLAE = SLAE \cup \left\{LeftSLAE\left(i'_1,...,i'_n,j'_1,...,j'_n,k',N,\eta,\mu,\zeta,w_{i'_1,...,i'_n,j'_1,...,j'_n,k'}\right)\right\}$
$= \left(\mathscr{F}, w_{i'_1,...,i'_n,j'_1,...,j'_n,k'}\right)_{L_2(\Omega)}$
end k' *end* j'_n *...end* j'_1 *end* i'_n *...end* i'_1
Solve $\left(SLAE, \left\{\alpha_{N_{i_1,...,i_n,j_1,...,j_n,k}}\right\}\right)$
Principle Part
For $i_1 = 1,...,N$ *do ... For* $i_n = 1,...,N$ *do*
For $j_1 = 1,...,N$ *do ... For* $j_n = 1,...,N$ *do For* $k = 1,...,N$ *do*
$y_N = y_N + \left(\alpha_{N_{i_1,...,i_n,j_1,...,j_n,k}} w_{i_1,...,i_n,j_1,...,j_n,k}\right)\eta(x)\mu(v)\zeta(t)$
end k end j_n *...end* j_1 *end* i_n *...end* i_1
$u_N(x,v,t) = e^{y_N}$, $a(x,v,t) = L(y_N) - F(x,v,t)$
End of the Algorithm 2.

The algorithms have been implemented in the computer algebra system Maple and tested for several inverse problems. Two examples are presented below where U_N shows the computed solution at N and N is the order of sum in (16).

Example 1 *Let* $\Omega = \{(x,v,t)|\ x \in (-1,1),\ v \in (-1,1),\ t \in (-1,1)\}$, $F(x,v,t) = -2txv + 2txv^3 + 2tx^3v - 2tx^3v^3 - 3v^2x^2 + 3v^2x^2t^2 + 3x^2v^4 - 3x^2v^4t^2$ *and* $f_1(x,v,t) = 0$ *are given. Then, at* $N = 2$ *Algorithm* 1 *gives the result:* $U_2 = e^{(1-x^2)(1-v^2)\left(1-t^2\right)xv}$, $\lambda_2 = -2vx(1-x^2)(1-v^2)t + v(1-v^2)(1-t^2)(v(1-x^2) - 2vx^2) + 2txv - 2txv^3 - 2tx^3v + 2tx^3v^3 + 3v^2x^2 - 3v^2x^2t^2 - 3x^2v^4 + 3x^2v^4t^2$ *which is also the exact solution of the problem.*

Example 2 *In the domain* $\Omega=\{(x,v,t)|\ x\in(-1,1),\ v\in(1,2),\ t\in(-1,1)\}$, *according to the given functions* $F(x,v)=x^2(-4t+2t(v-2)^2)/v+(2tx^4)/v+x(v-2)^2(-2+2t^2-3tx+3v-3vt^2+txv-v^2+v^2t^2)+x^3v(6-6t^2)+vx(-6+6t^2)-2tx^2v+tx^4v-2x^3v^2+2x^3t^2v^2+2v^2x-2v^2xt^2$ *and* $f_1(x,v,t)=0$ *approximate solution of the problem at* $N=1$ *is* $U_1=e^{-\frac{1}{2}(1-x^2)(2-3v+v^2)\left(1-t^2\right)}$ *where the exact solution is* $u(x,v)=e^{\frac{1}{2v}\left(x^2+(2-v)^2-1\right)(1-x^2)(2-3v+v^2)\left(1-t^2\right)}$. *In Figure 1a and Figure 1b, a comparison between approximate solution (dotted, yellow graph) and exact solution* $u(x,v)$ *(solid, blue graph) of the problem is presented at* $N=1$ *and* $N=4$*, respectively. We didn't write the computed solution at* $N=4$ *explicitly because of the page limitation.* λ_2 *and* λ_4 *can be obtained from equation* $Ly=a+F$ *easily.*

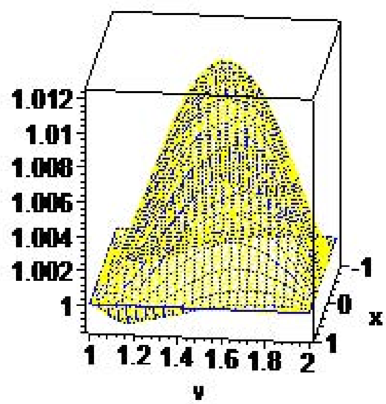

Figure 1a.

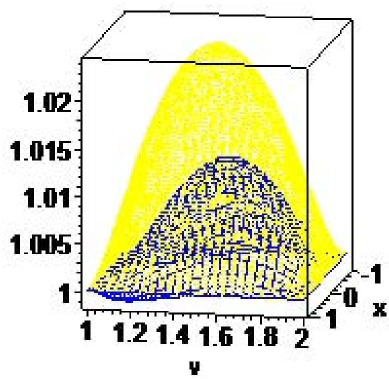

Figure 1b.

In example 1, computed solution at $N=2$ coincides with the exact solution of the problem and in example 2, as it can be seen from Figure 1b, approximate solution at $N=4$ is very closed to the exact solution. Consequently, the computational experiments show that the proposed algorithm gives efficient and reliable results.

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