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Solvability of a Plane Integral Geometry Problem and a Solution Algorithm

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Summary

In this work we deal with solvability and aproximation to the solution of the two dimensional integral geometry problem for a family of regular curves of given curvature. Solvability of the problem is proved by using the Galerkin method and an algorithm is developed to compute the approximate solution of the problem.

keywords: Integral Geometry Problem, Inverse Problem, Galerkin Method, Symbolic Algorithm.

Introduction

Solvability of a two dimensional integral geometry problem (IGP) is considered by reducing it to the equivalent inverse problem for the general transport equation and an efficient aproximation method proposed for the solution of IGP. To demonstrate the computational feasibility of the given approximation method, some computational experiments are performed and the results are presented at the end of the paper. Investigating the solvability of problems of integral geometry by reducing them to equivalent inverse problems for differential equation was first carried out in Lavrent'ev and Anikonov [5]. Similar reduction is demonstrated for IGP formulated below. It is assumed that in a domain D, a family of regular curves is given by curvature such that curvature of the curve passing from each point $x \in D$, in any direction $v = (\cos\varphi, \sin\varphi)$ is $K(x,\varphi) = f_2(x)\cos\varphi - f_1(x)\sin\varphi$ and there exists a curve passing from every $x \in D$ in the arbitrary direction v, with endpoints on the boundary of D. Suppose lengths of these curves in D are upper-bounded by the same constant. Let us denote the family of these curves by $\{\Gamma\}$.

IGP. *Find a function $\lambda(x)$ in a domain D from the integrals of λ along the curves of a given family of curves $\{\Gamma\}$.*

Suppose that $\lambda(x) \in C(\mathbb{R}^2)$ vanishes outside D. Introduce an auxiliary function

$$u(x,\varphi) = \int_{\gamma(x,\varphi)} \lambda ds, \tag{1}$$

where $\gamma(x,\varphi)$ is the curve passes through x in the direction v and has the curvature $K(x,\varphi)$. Differentiating (1) in the direction v at x,

$$Lu \equiv u_{x_1}\cos\varphi + u_{x_2}\sin\varphi + K(x,\varphi)u_\varphi = \lambda(x), \tag{2}$$

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is obtained (see Amirov [2], p.11). By the hypothesis of IGP, $u(x,\varphi)$ is 2π-periodic in φ and is known for $x \in \Gamma_1 = \partial D \times (0,2\pi)$, i.e.

$$u|_{\Gamma_1} = u_0(x,\varphi), \quad u(x,\varphi) = u(x,\varphi+2\pi). \tag{3}$$

Problem 1. *Find a pair of functions* (u,λ) *from the equation* (2) *provided that the function* $K(x,\varphi)$ *is known and the solution* $u(x,\varphi)$ *satisfies conditions* (3).

Given the function $K(x,\varphi)$, a set of curves $\{\Gamma\}$ such that $K(x,\varphi)$ is the curvature at $x \in D$ of the curve that passes through $x \in D$ in the direction $(\cos\varphi, \sin\varphi)$ can be constructed. Integrating both sides of the equality (2) along the curve $\gamma(x,\varphi)$ and observing (3), IGP is obtained. Thus, it is proved that in the corresponding spaces (see Theorem 1 below), IGP is equivalent to Problem 1, where $K(x,\varphi)$ is a given sufficiently smooth function (see [2]).

One of the applications of integral geometry problem is computerized tomography. Namely, problems of integral geometry provide the mathematical background of the computerized tomography. The goal of the tomography is to recover the interior structure of a nontransparent object using external measurements. The object under investigation is exposed to radiation at different angles, and the radiation parameters are measured at the points of observations. The basic problem in computerized tomography is the reconstruction of a function from its line or plane integrals and there are many applications related with computerized tomography; geophysics, X-ray tomography, diagnostic radiology, astronomy, seismology, radar and many other fields (see [3], [4], [7]).

Definitions and Solvability of the Problem

Based on the works by Amirov [1, 2], let us introduce some definitions and notations, which will be used throughout the paper. Let $C^3_\pi(\Omega)$ denote the set of real-valued functions $u(x,\varphi)$ that are 2π-periodic in φ and three times continuously differentiable on Ω with respect to all arguments, where $\Omega = \{(x,\varphi) : x \in D \subset \mathbb{R}^2,\ \varphi \in (0,2\pi),\ \partial D \in C^3\}$. Let introduce the scalar product $(u,z)_{1,c} = (u,z)_1 + \int_\Omega \left(u_{x_1\varphi} z_{x_1\varphi} + u_{x_2\varphi} z_{x_2\varphi}\right) d\Omega$ in $C^3_\pi(\Omega)$, where $(u,z)_1 = \int_\Omega \left(uz + u_{x_1} z_{x_1} + u_{x_2} z_{x_2} + u_\varphi z_\varphi\right) d\Omega$ and $d\Omega = dx d\varphi$.

Set $\|u\|_1 = [(u,u)_1]^{1/2}$ and $\|u\|_{1,c} = [(u,u)_{1,c}]^{1/2}$. Let $H^\pi_{1,c}(\Omega)$, $H^\pi_m(\Omega)$, and $H^\pi_{1,2}(\Omega)$ be the completions of $C^3_\pi(\Omega)$ with respect to the norms $\|\cdot\|_{1,c}$, $\|\cdot\|_{H^m}$ $(m = 1,2)$ and $\|\cdot\|_{1,2}$ respectively, where $\|u\|_{1,2} = [(u,u)_{1,2}]^{1/2}$ and $(u,u)_{1,2} = (u,u)_{1,c} + \int_\Omega u^2_{\varphi\varphi} d\Omega$.

Let $C^3_{\pi 0} = \{\psi : \psi|_{\partial D} = 0,\ \psi \in C^3_\pi\}$ and select a set $\{\omega_1, \omega_2, \omega_3, \ldots\} \subset C^3_{\pi 0}$ which is complete and othonormal in $L_2(\Omega)$. It may be assumed that the linear span of this set is everywhere dense in $\mathring{H}^\pi_{1,2}(\Omega)$, where $\mathring{H}^\pi_{1,2}(\Omega)$ is the completion of $C^3_{\pi 0}$ with respect to the norm $\|\cdot\|_{1,2}$. We denote the orthogonal projector of $L_2(\Omega)$

onto $\mathscr{M}_n$ by $\mathscr{P}_n$, where $\mathscr{M}_n$ is the linear span of $\{w_1, w_2, ...w_n\}$. Furthermore, let $Au \equiv \widetilde{L}Lu$, where

$$\begin{aligned} \widetilde{L}u &= \frac{\partial^2}{\partial l \partial \varphi} u = \frac{\partial}{\partial l} u_\varphi, \\ \frac{\partial}{\partial l} &= (\cos\varphi)\left(\frac{\partial}{\partial x_2} - f_1 \frac{\partial}{\partial \varphi}\right) - (\sin\varphi)\left(\frac{\partial}{\partial x_1} - f_2 \frac{\partial}{\partial \varphi}\right). \end{aligned}$$

$\Gamma''(A)$ is the set of all functions $u(x,\varphi) \in L_2(\Omega)$ such that for any $u \in \Gamma''(A)$ there exists $y \in L_2(\Omega)$ such that $(u, A^*\eta)_{L_2(\Omega)} = (y, \eta)_{L_2(\Omega)}$, holds for every $\eta \in C^3_{\pi 0}(\Omega)$ where, A^* is the differential expression conjugate to A in the sense of Lagrange; $y \stackrel{def}{=} Au$ and $(u,v)_{L_2(\Omega)}$ is a scalar product of functions u and v in $L_2(\Omega)$.

Take a subset $\Gamma(A) \subset \Gamma''(A)$ such that for any $u \in \Gamma(A)$ there exists a sequence $\{u_k\} \subset C^3_{\pi 0}(\Omega)$ such that $u_k \to u$ weakly in $L_2(\Omega)$ and $(Au_k, u_k)_{L_2(\Omega)} \to (Au, u)_{L_2(\Omega)}$ as $k \to \infty$. $\Gamma'(A)$ is the closure of $C^3_{\pi 0}(\Omega)$ with respect the norm $\|u\|_{\Gamma(A)} = \|u\|_{L_2(\Omega)} + \|Au\|_{L_2(\Omega)}$. It is clear that $\Gamma'(A) \subset \Gamma(A) \subset \Gamma''(A)$ and $\Gamma''(A) \cap \mathring{H}^\pi_{1,c} \subset \Gamma(A) \subset L_2(\Omega)$.

Since the unknown function λ depends only on x, Problem 1 is overdetermined and this problem can be replaced by the following determined problem;

Problem 2. *Find a pair of functions (u,λ) defined in Ω that satisfies*

$$\begin{aligned} Lu &= \lambda(x,\varphi), &(4)\\ u|_{\Gamma_1} &= u_0, \quad u(x,\varphi) = u(x,\varphi+2\pi) &(5)\\ \tilde{L}\lambda &= 0 &(6) \end{aligned}$$

Here the equation (6) is satisfied in generalized functions sense. If $u_0 \in C^3(\Gamma_1)$ and $\partial D \in C^3$ then, Problem 2 can be reduced to the following problem (see Amirov [2], p.20);

Problem 3. *Find a pair of functions (u,λ) defined in Ω that satisfies*

$$\begin{aligned} Lu &= \lambda + F, &(7)\\ u|_{\Gamma_1} &= 0, \quad u(x,\varphi) = u(x,\varphi+2\pi), &(8)\\ \widetilde{L}\lambda &= 0. &(9) \end{aligned}$$

Theorem 1. *If $f_1(x), f_2(x) \in C^2(\bar{D}), \forall x \in \bar{D}$ $f_{1x_1} + f_{2x_2} > 0$ and $F \in H^\pi_2(\Omega)$ then Problem 3 has a unique solution (u,λ), that satisfies the conditions $u \in \Gamma(A) \cap H^\pi_1(\Omega), \lambda \in L_2(\Omega)$, and the inequality $\|u\|_{H^\pi_1(\Omega)} + \|\lambda\|_{L_2(\Omega)} \leq C(\|F\|_{L_2(\Omega)} + \|F_\varphi\|_{L_2(\Omega)})$ holds, where $C > 0$ depends on D and $\bar{D}$ is the closure of D.*

Proof. To prove the uniqueness part of the theorem, we show that the corresponding homogeneous linear problem has only trivial solution satisfying the conditions of the theorem. If we apply $\tilde{L}$ to the equation (7) and take into account (9), we get $Au = 0$. Since $u \in \Gamma(A)$, there exists a sequence $\{u_k\} \subset C^3_{\pi 0}$ such that $u_k \to u$ weakly in $L_2(\Omega)$ and $(Au_k, u_k)_{L_2(\Omega)} \to 0$ as $k \to \infty$. It is easy to show that

$$-Au_k u_k = \frac{\partial}{\partial \varphi}(Lu_k)\frac{\partial u_k}{\partial l} - \frac{\partial}{\partial l}\left(\frac{\partial}{\partial \varphi}(Lu_k)u_k\right), \tag{10}$$

and

$$-2\,(Au_k, u_k)_{L_2(\Omega)} = \int\limits_{\Omega} [(u_{kx_1} + u_{k\varphi} f_2)^2 + (u_{kx_2} + u_{k\varphi} f_1)^2 + u^2_{k\varphi}(f_{1x_1} + f_{2x_2})]d\Omega, \tag{11}$$

since the divergent terms are all zero. It is clear that, if $f_{1x_1} + f_{2x_2} > 0$, then the quadratic form $J(\nabla u_k)$ in $(u_{kx_1} + u_{k\varphi} f_2)$, $(u_{kx_2} + u_{k\varphi} f_1)$, $u_{k\varphi}$ is positive definite, where

$$J(\nabla u_k) = (u_{kx_1} + u_{k\varphi} f_2)^2 + (u_{kx_2} + u_{k\varphi} f_1)^2 + u^2_{k\varphi}(f_{1x_1} + f_{2x_2}).$$

Since the domain D is bounded and $u_k = 0$ on Γ_1, we have $\|u_k\|^2 \leq C\int\limits_{\Omega} J(\nabla u_k)d\Omega$, where $C > 0$ depends on D. From (11) and the definition of $\Gamma(A)$ we get

$$\|u\|^2 = \lim_{k\to\infty} \|u_k\|^2 \leq C \lim_{k\to\infty} \int\limits_{\Omega} J(\nabla u_k)d\Omega = -2C \lim_{k\to\infty} (Au_k, u_k)_{L_2(\Omega)} = 0. \tag{12}$$

Inequality (12) implies $\|u\| = 0$, i.e., $u = 0$ and equation (7) implies $\lambda = 0$. This completes the proof of uniqueness part.

Secondly we will prove existence of the solution of the problem in the same set : $\Gamma(A)$. From (7)-(9), we obtain the following problem: Find u defined in Ω that satisfies

$$Au = \tilde{L}F = \mathscr{F}, \tag{13}$$

$$u|_{\Gamma_1} = 0, \quad u(x,\varphi) = u(x, \varphi + 2\pi). \tag{14}$$

An approximate solution of order N of the problem (13)-(14)

$$u_N = \sum_{i=1}^{N} \alpha_{Ni} w_i(x,\varphi); \quad \alpha_N = (\alpha_{N_1}, \alpha_{N_2}, ..., \alpha_{N_N}),$$

is defined as a solution to the following problem: Find the vector α_N from the system of linear algebraic equations

$$\int\limits_{\Omega} \tilde{L}(Lu_N - F)w_j d\Omega = 0, \quad j = 1, 2, ..., N, \quad d\Omega = dxd\varphi. \tag{15}$$

We shall prove that there exists a unique solution α_N of the system (15) for any $F \in H_2^{\pi}(\Omega)$ under the hypotheses of the theorem. To this end, it suffices to prove that the homogeneous version of system (15) has only trivial solution. Assume the contrary. Let the homogeneous version of system (15) have a nonzero solution $\bar{\alpha}_N = (\bar{\alpha}_{N_1}, \bar{\alpha}_{N_2}, ..., \bar{\alpha}_{N_N})$. In the system (15) with $F = 0$, substituting $\bar{\alpha}_N$ for α_N, multiplying the ith equation by $-2\bar{\alpha}_{N_i}$ and summing with respect to i from 1 to N, we obtain

$$-2\int_{\Omega} \tilde{L}L\bar{u}_N\bar{u}_N d\Omega = 0 \tag{16}$$

where $\bar{u}_N = \sum_{i=1}^{N} \bar{\alpha}_{Ni}w_i$. By (11) and (16),

$$\int_{\Omega} [(\bar{u}_{Nx_1} + \bar{u}_{N\varphi}f_2)^2 + (\bar{u}_{Nx_2} + \bar{u}_{N\varphi}f_1)^2 + \bar{u}_{N\varphi}^2(f_{1x_1} + f_{2x_2})]d\Omega = 0.$$

Therefore, since $J(\nabla u_k)$ is positive definite and $\bar{u}_N = 0$ on Γ_1, we have $\bar{u}_N = 0$ in D, and since $\{w_i\}$ is linearly independent we get $\bar{\alpha}_{N_i} = 0$. This contradicts the condition $\bar{\alpha}_{N_i} \neq 0$. Thus, system (15) has a unique solution α_N for any $F \in H_2^{\pi}(\Omega)$. Now we estimate the solution u_N of system (15) in terms of F. For this purpose, we multiply the ith equation of (15) by $-2\alpha_{N_i}$ and sum the obtained relations with respect to i from 1 to N, we have

$$-2\int_{\Omega} u_N\tilde{L}(Lu_N)d\Omega = -2\int_{\Omega} u_N\tilde{L}F d\Omega. \tag{17}$$

Since $u_N|_{\Gamma_1} = 0$, the right hand side of (17) is as follows:

$$-2\left|\int_{\Omega} u_N\tilde{L}F d\Omega\right| \leq \frac{2}{\delta_0}\int_{\Omega} |F_{\varphi}|^2 d\Omega + \frac{\delta_0}{2}\int_{\Omega} |\nabla_{x,l}u_N|^2 d\Omega.$$

Then, using (10), (11), (17) and Schwartz inequality for sufficiently large $\bar{\delta}_0 > 0$, we get

$$\int_{\Omega} J(\nabla u_N)d\Omega \leq \frac{2}{\bar{\delta}_0}\int_{\Omega} F_{\varphi}^2 d\Omega + \frac{\bar{\delta}_0}{2}\int_{\Omega} |\nabla_x u_N|^2 d\Omega$$

or

$$\int_{\Omega} |Lu_N|^2 d\Omega \leq C, \quad \int_{\Omega} |\nabla_x u_N|^2 d\Omega \leq C \Rightarrow \int_{\Omega} u_N^2 d\Omega \leq C.$$

This implies that the sets of functions $\{u_N\}$ and $\{Lu_N\}$ are bounded in $L_2(\Omega)$ and $\|u_N\|_{\mathring{H}_1^{\pi}(\Omega)} \leq C\|F_{\varphi}\|_{L_2(\Omega)}$. Since $L_2(\Omega)$ is a Hilbert space, the sets$\{u_N\}$ and $\{Lu_N\}$

are weakly compact in $L_2(\Omega)$. Therefore, there exist subsequences (we again denote them by $\{u_N\}$ and $\{Lu_N\}$) such that $u_N \to u$, $Lu_N \to \lambda$ $(N \to \infty)$ weakly converge and this implies the inequality $\|u\|_{\mathring{H}_1^\pi(\Omega)} \le C\left\|F_\varphi\right\|_{L_2(\Omega)}$. If we take into consideration the fact that the operator L is weakly closed, we see that $\lambda = Lu$. Since $u_N|_{\Gamma_1} = 0$, and $u_N \to u$ we have $u|_{\Gamma_1} = 0$. Multiplying the equations (15) by $\beta \in \mathbb{R}^1$, and taking into account the condition $u_N|_{\Gamma_1} = 0$, and calculating the action of $\tilde{L}w_j$ on β, for $N \ge j$, we obtain $\int\limits_\Omega (Lu_N - F)\tilde{L}(\beta w_j)\, d\Omega = 0$. Since the linear span of $\{w_j\}$ is dense on the space $\mathring{H}^\pi_{1,2}(\Omega)$, we get

$$\int\limits_\Omega (Lu - F)\tilde{L}\eta\, d\Omega = 0, \tag{18}$$

for every $\eta \in \mathring{H}^\pi_{1,2}(\Omega)$. If we put $\lambda = Lu - F$, we obtain $\|\lambda\|_{L_2(\Omega)} \le C\|u\|_{H_1^\pi(\Omega)} + \|F\|_{L_2(\Omega)}$. From (18), we get $\tilde{L}\lambda = 0$ in the sense of generalized functions.

To complete the proof of the theorem, we have to show $(Au_N, u_N)_{L_2(\Omega)} \to (Au, u)_{L_2(\Omega)}$ as $N \to \infty$. From the equation (17), we obtain $\mathscr{P}_N Au_N = \mathscr{P}_N \mathscr{F}$. Since the system $\{w_1, w_2, ..., w_N\}$ is orthogonal in $L_2(\Omega)$, $\mathscr{P}_N\mathscr{F}$ converges strongly to $\mathscr{F}$ in the sense of L_2 as $N \to \infty$, in other words we get $\mathscr{P}_N Au_N \to \mathscr{F} = Au$ as $N \to \infty$. Then, since we have $u_N \to u$ weakly and $\mathscr{P}_N Au_N \to Au$ in L_2 as $N \to \infty$, we obtain $(\mathscr{P}_N Au_N, u_N)_{L_2(\Omega)} \to (Au, u)_{L_2(\Omega)}$ as $N \to \infty$. By the definitions of $\mathscr{P}_N$ and u_N (since $\mathscr{P}_N$ is self adjoint in $L_2(\Omega)$) we obtain

$$(\mathscr{P}_N Au_N, u_N)_{L_2(\Omega)} = (Au_N, \mathscr{P}^*_N u_N)_{L_2(\Omega)} = (Au_N, \mathscr{P}_N u_N)_{L_2(\Omega)} = (Au_N, u_N)_{L_2(\Omega)}.$$

Hence $(Au_N, u_N)_{L_2(\Omega)} \to (Au, u)_{L_2(\Omega)}$ as $N \to \infty$ which completes the proof of the theorem. □

Solution Algorithm and Some Numerical Results

To be able to define the solution algorithm, the domain $\Omega = D \times (0, 2\pi)$ (where $D = (-1,1) \times (-1,1)$) is selected. Approximate solution u_N is written in the following form:

$$u_N = \sum_{i,j,k=0}^{N} \left(\alpha_{i,j,k} v_{i,j,k} + \beta_{i,j,k} w_{i,j,k}\right) \eta\left(x\right) \tag{19}$$

where $\eta\left(x\right) = \begin{cases} \left(1 - x_1^2\right)\left(1 - x_2^2\right), & x \in D \\ 0, & x \notin D \end{cases}$.

$\left\{v_{i,j,k}\right\}_{i,j,k=0}^N$ and $\left\{w_{i,j,k}\right\}_{i,j,k=0}^N$ are complete systems in $L_2\left(\Omega\right)$ where $v_{i,j,k} = x_1^i x_2^j \sin(k\varphi)$ and $w_{i,j,k} = x_1^i x_2^j \cos(k\varphi)$. In expression (19), unknown coefficients

$\alpha_{i,j,k}$, and $\beta_{i,j,k}$, $i,j,k=0,...,N$ are determined from the following system of linear algebraic equations:

$$\sum_{i,j,k=0}^{N}\left(A\left(\alpha_{i,j,k}v_{i,j,k}+\beta_{i,j,k}w_{i,j,k}\right)\eta, v_{i',j',k'}\eta\right)_{L_2(\Omega)} = \left(F, v_{i',j',k'}\eta\right)_{L_2(\Omega)}, \tag{20.a}$$

$$\sum_{i,j,k=0}^{N}\left(A\left(\alpha_{i,j,k}v_{i,j,k}+\beta_{i,j,k}w_{i,j,k}\right)\eta, w_{i',j',k'}\eta\right)_{L_2(\Omega)} = \left(F, w_{i',j',k'}\eta\right)_{L_2(\Omega)} \tag{20.b}$$

where $i',j',k'=0,...,N$.

Algorithm 1. *INPUT : N, $F(x_1,x_2,\varphi)$, $f_1(x_1,x_2)$, $f_2(x_1,x_2)$.*
OUTPUT : u_N and λ_N approximations to u and λ.

Procedure SysA(i',j',k') (*Construct the left side of* $(20.a)$)
$Left := 0$, for $i=0,...,N$ do, for $j=0,...,N$ do, for $k=0,...,N$ do
begin $Left := Left + \left(A\left(\alpha_{i,j,k}v_{i,j,k}+\beta_{i,j,k}w_{i,j,k}\right)\eta,\, v_{i',j',k'}\eta\right)_{L_2(\Omega)}$ end;
Procedure SysB(i',j',k') (*Construct the left side of* $(20.b)$)
$Left := 0$, for $i=0,...,N$ do, for $j=0,...,N$ do, for $k=0,...,N$ do
begin $Left := Left + \left(A\left(\alpha_{i,j,k}v_{i,j,k}+\beta_{i,j,k}w_{i,j,k}\right)\eta,\, w_{i',j',k'}\eta\right)_{L_2(\Omega)}$ end;
Procedure *SYS* (*Construct the systems* $(20.a)-(20.b)$)
$Set := \{\}$, $\mathscr{F} := \tilde{L}F$
for $i=0,...,N$ do, for $j=0,...,N$ do, for $k=0,...,N$ do
begin
$Set := Set \cup \Big\{ SysA\,(i',j',k') = \left(\mathscr{F},\, v_{i',j',k'}\eta\right)_{L_2(\Omega)},$
$SysB\,(i',j',k') = \left(\mathscr{F},\, w_{i',j',k'}\eta\right)_{L_2(\Omega)} \Big\}$ end;
Solve $\left(SYS, \{\alpha_{i,j,k}\}, \{\beta_{i,j,k}\}\right)$ (*Solve the systems* $(20.a)-(20.b)$)
for $i=0,...,N$ do, for $j=0,...,N$ do, for $k=0,...,N$ do
begin $u_N = u_N + \left(\alpha_{i,j,k}v_{i,j,k}+\beta_{i,j,k}w_{i,j,k}\right)\eta$ end;
$\lambda_N = L(u_N) - F$
end. (End of the algorithm)

Two examples, whose approximate solutions obtained by Algorithm 1, are presented below, where U_N and λ_N are the approximations at N and N is the order of sum in (19).

Example 1. *Let* $\Omega=(-1,1)\times(-1,1)\times(0,2\pi)$, $F(x_1,x_2,\varphi)=\left(x_1^3x_2-x_1x_2^3\right)\sin 2\varphi\frac{1}{2}+\left(x_1^4+x_2^4-\left(x_1^2x_2^2+5\right)\left(x_1^2+x_2^2\right)+8x_1^2x_2^2+2\right)\cos 2\varphi$, $f_1(x_1,x_2)=x_1$ *and* $f_2(x_1,x_2)=$

x_2 are given. Then, the approximation at $N=1, U_1 = \left(\left(1-x_1^2\right)\left(1-x_2^2\right)(x_1\cos\varphi - x_2\sin\varphi)\right)$, $\lambda_1 = \frac{1}{2}\left(x_2^4 - x_1^4 + \left(x_1^2 - x_2^2\right)(x_1^2x_2^2-1)\right)$ which is also the exact solution of the problem.

Example 2. *In the domain $\Omega = (-1,1)\times(-1,1)\times(0,2\pi)$, according to the given functions, $F(x_1,x_2,\varphi) = \frac{1}{2}\left(e^{x_1}\left(x_2^2-3\right)\left(x_1x_2 - x_1^3x_2\right) + e^{x_2}\left(x_1^2-3\right)\left(x_2^3x_1 - x_1x_2\right)\right)\sin(2\varphi)$ $+\frac{1}{2}\left(e^{x_1}\left(x_1^4 - x_1^3 - 4x_1^2 + x_1 + 1\right)\left(1-x_2^2\right) + e^{x_2}\left(x_2^4 - x_2^3 - 4x_2^2 + x_2 + 1\right)\left(1-x_1^2\right)\right)\cos(2\varphi)$, $f_1(x_1,x_2) = x_1$ and $f_2(x_1,x_2) = x_2$, comparison of U_1 and U_3 with exact u is represented in Figure 1 (a)-(b) respectively (λ_1 and λ_3 can be obtained easily from (7)), where the exact solution is; $u(x_1,x_2,\varphi) = \left(1-x_1^2\right)\left(1-x_2^2\right)\left(x_1e^{x_1}\cos\varphi - x_2e^{x_2}\sin\varphi\right)$, $\lambda(x_1,x_2,\varphi) = \frac{1}{2}\left(\left(1-x_2^2\right)\left(1+x_1-2x_1^2-x_1^3-x_1^4\right)e^{x_1} - \left(1-x_1^2\right)\left(1+x_2-2x_2^2-x_2^3-x_2^4\right)\right)e^{x_2}$.*

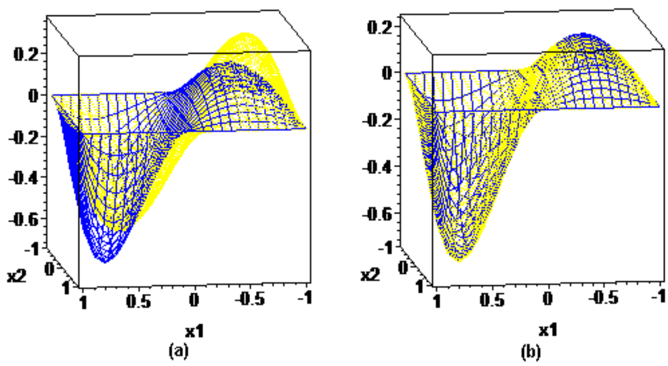

Figure 1: Comparison of approximate (dotted, yellow graph) and exact solution (solid, blue graph) for Example 2 at $\varphi = \pi$ (a) $N=1$, (b) $N=3$.

The computations are performed using MAPLE program on a PC with Intel Core 2 T7200 2.00 GHz CPU, 1 Gb memory, running under Windows Vista. In Example 1, approximation at $N=1$ coincides with the exact solution and in Example 2, as it can be seen from Figure 1b, approximation at $N=3$ is very closed to the exact solution. Consequently, the obtained results show very high accuracy and the proposed method which is given by Algorithm 1, is an efficient method in solving the IGP.

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