 ICCES, vol.15, no.2, pp.57-64

A thermo-hydro-mechanical problem for an embedded disc inclusion

A.P.S. Selvadurai[1]

Summary

This paper examines the problem of the internal heating of a poroelastic halfspace by an embedded rigid disc inclusion. A computational approach is adopted for modelling the resulting coupled thermo-hydro mechanical processes associated with the heating by the disc inclusion.

Keywords: Thermo-hydromechanical phenomena, embedded rigid disc inclusion, coupled processes

Introduction

The three-dimensional formulation of the theory of consolidation was first proposed by Biot (1941). The theory was developed with classical Hooke's law as the constitutive relationship for the mechanical response of the porous skeleton and Darcy's law as the principle governing the flow of the fluid through the pore space. Within the context of the assumed forms of mechanical and transport behaviour, the theory developed by Biot is exact and the coupling is correctly accounted for through considerations of physics and mechanics of all the processes (Selvadurai, 1996; 2007). The theory of poroelasticity has found application in diverse areas ranging from geomechanics of resource exploration for oil and gas recovery to biomechanics. The need for extending Biot's classical theory to include other processes originated with the development of new methodologies of interest to environmental geomechanics, particularly those involving heating of fluid-saturated media. Here, the temperature is a dependent variable that has a coupling influence on both the mechanical and thermal phenomena. The motivation for considering thermo-poroelastic behaviour of geomaterials stems from the need for development of efficient and reliable techniques for deep geological disposal of heat-emitting nuclear fuel wastes (Laughton et al., 1986; Selvadurai and Nguyen, 1996).

The Governing Equations

We consider Thermo-Hydro-Mechanical (THM) processes in a fully saturated porous medium with an elastic fabric. The displacement vector defining the deformations of the porous skeleton is denoted by $\mathbf{u}(\mathbf{x},t)$, where $\mathbf{x}$ is a position vector and t is time. The pressure of the fluid occupying the pore space is denoted by $p(\mathbf{x},t)$. The

[1]Department of Civil Engineering and Applied Mechanics, McGill University, Montreal, QC, Canada H3A 2K6

temperature of all phases at a point in the saturated medium is denoted by $T(\mathbf{x},t)$. The effective stress in the porous skeleton is defined through

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}' + \alpha \mathbf{I}\, p \tag{1}$$

where $\boldsymbol{\sigma}$ is the total stress dyadic, $\boldsymbol{\sigma}'$ is the effective stress dyadic, $\mathbf{I}$ is the unit dyadic, α is the pore pressure parameter which takes into consideration the compressibilities of the solid and fluid phases and the porosity of the medium. As $\alpha \to 1$, (1) reduces to the classical effective stress equation proposed by Terzaghi (1925). For material isotropy of the deformable medium, the stress-strain relationship for the elastic porous solid, which also accounts for the thermal deformations of the solid skeleton, is given by the Duhamel-Neumann extension to Hooke's law; this can be written as

$$\boldsymbol{\sigma}' = 2G\boldsymbol{\varepsilon} + (\lambda \nabla.\mathbf{u} - \beta K_D T)\ \mathbf{I} \tag{2}$$

In (2), the strain dyadic $\boldsymbol{\varepsilon}$ is given by $\boldsymbol{\varepsilon} = (1/2)(\nabla\mathbf{u} + \mathbf{u}\nabla)$, where ∇ is the gradient operator, $\lambda = 2\nu G/(1-2\nu)$ is a Lamé constant, G and ν are, respectively, the skeletal linear elastic shear modulus and Poisson's ratio, $K_D(= 2G(1+\nu)/3(1-2\nu))$ is a bulk modulus and β is the coefficient of volume expansion of the porous skeleton. For the sign convention, we assume that compressive stresses are positive. Darcy's law describes the flow of the fluid through the pore space. In the definition of the flow, we need to take into consideration the relative motion between the fluid and the porous solid:

$$\mathbf{q}_r^f = \mathbf{v}_f - \mathbf{v}_s = -\frac{\mathbf{K}}{n\mu}\nabla p \tag{3}$$

where $\mathbf{q}_r^f$ is the relative measure of the flux associated with the liquid phase, , $\mathbf{K}$ is the permeability dyadic referred to the fully saturated condition, μ is the dynamic viscosity of the fluid. It can be shown that for non-negative dissipation of energy during fluid flow, $\mathbf{K} = \mathbf{K}^T$, where $\mathbf{K}^T$ is the transpose. In other words, the matrix of coefficients constructed with $\mathbf{K}$ is positive definite and symmetric.

The process of heat transfer in the saturated porous medium is assumed to be primarily due to heat conduction. Fourier's law of heat conduction in its general form can be written as

$$\mathbf{q} = -\boldsymbol{\kappa}\nabla T \tag{4}$$

where $\boldsymbol{\kappa}(= \boldsymbol{\kappa}^C)$ is the dyadic of thermal conductivity. Considering the balance equations applicable to linear momentum, we have, for quasi-static deformations of the partially saturated medium

$$\nabla.\boldsymbol{\sigma} + \mathbf{f} = \mathbf{0} \tag{5}$$

where $\mathbf{f}$ is a body force vector. The conservation of fluid mass takes into account storage originating from the consideration of water increase in a saturated medium, the storage originating from the compressibility of the solid matrix and storage/loss from a control volume due to mismatch of thermal expansion between the water and the porous solid. This gives

$$\rho_w \left[\frac{\partial}{\partial t}(\nabla.\mathbf{u}) + \frac{(1-n)}{\rho_s}\frac{\partial \rho_s}{\partial t} \right] = -\nabla.(\mathbf{q}_r^f - \mathbf{q}_l^v) \tag{6}$$

where n is the porosity and ρ_s is the mass density of the solid material constituting the porous solid. The conservation of heat energy gives

$$\nabla.\mathbf{q} + Q = \rho C \frac{\partial T}{\partial t} \tag{7}$$

where ρ and C are, respectively, the mass density and specific heat of the saturated porous solid which consists of the solid phase and the fluid phase. Combining these we obtain the following set of coupled partial differential equations for the dependent variables $\mathbf{u}$, p and T:

$$G\nabla.(\nabla\mathbf{u}) + (\lambda + G)\nabla(\nabla.\mathbf{u}) + \alpha\nabla p - \beta K_D \nabla T + \mathbf{f} = \mathbf{0} \tag{8}$$

$$\nabla.\left(\frac{\rho_w \mathbf{K}}{\mu}\left[\nabla p + \rho_w \mathbf{g}\right] \right) - C_w \frac{\partial p}{\partial t} + \rho_w \frac{\partial}{\partial t}(\nabla.\mathbf{u}) - \rho_w (n\beta_w + (1-n)\beta_s)\frac{\partial T}{\partial t} = 0 \tag{9}$$

$$\nabla.(\boldsymbol{\kappa}\nabla T) + Q = \rho C \frac{\partial T}{\partial t} \tag{10}$$

Computational Modelling

A 3D finite element code (Selvadurai and Nguyen, 1995) has been developed to solve, computationally, the system of coupled partial differential equations described by (8) to (10). When the problem governing THM processes is posed in terms of a system of governing partial differential equations along with a consistent set of boundary conditions and initial conditions, it is convenient to take advantage of the Galerkin-type weighted residual technique to formulate the computational scheme. Comprehensive expositions of the basic procedures are given in a number of advanced texts on finite element modelling and the reader is referred, in particular, to the texts by Lewis and Schrefler (1998) and the articles by Selvadurai (2007) and Selvadurai and Nguyen (1995). Considering standard finite element procedures, the domain is discretized into N^e elements. In the modelling of the geological medium with isoparametric elements, the displacements within the element are interpolated as functions of displacements of all nodes whereas the pore fluid and temperature are interpolated as functions of the values at the corner nodes. Details of the rationale for the procedure are well documented (Selvadurai and Nguyen, 1995). The Galerkin procedure, when applied to the partial differential equations (8) to (10), gives rise to matrix equations of the general form

$$\begin{aligned}
&\alpha_1 \begin{bmatrix} [\mathbf{K}] & \varsigma_2 [\mathbf{CP}] \\ \varsigma_2 [\mathbf{CP}]^T & -\alpha * (\Delta t) [\mathbf{KP}] - \varsigma_3 [\mathbf{CM}] \end{bmatrix} \begin{Bmatrix} \{\mathbf{d}\}^{(n+1)} \\ \{\mathbf{p}\}^{(n+1)} \end{Bmatrix} \\
&= \{\mathbf{f}\} + \begin{bmatrix} \left(\frac{\alpha*-1}{\alpha*}\right)[\mathbf{K}] & \varsigma_2 \left(\frac{\alpha*-1}{\alpha*}\right)[\mathbf{CP}] \\ \varsigma_2 [\mathbf{CP}]^T & (1-\alpha)(\Delta t)[\mathbf{KP}] - \varsigma_3 [\mathbf{CM}] \end{bmatrix} \begin{Bmatrix} \{\mathbf{d}\}^{(n)} \\ \{\mathbf{p}\}^{(n)} \end{Bmatrix} \\
&\begin{bmatrix} \frac{\varsigma_4}{\alpha*}[\mathbf{K}] & [\mathbf{0}] \\ [\mathbf{0}] & \varsigma_5 [\mathbf{CM}] \end{bmatrix} \begin{Bmatrix} (1-\alpha*)\{\mathbf{T}\}^{(n)} + \alpha * \{\mathbf{T}\}^{(n+1)} \\ \{\mathbf{T}\}^{(n+1)} - \{\mathbf{T}\}^{(n)} \end{Bmatrix}
\end{aligned} \tag{11}$$

$$\begin{aligned}
&[\,\alpha * [\mathbf{KH}] + \left(\tfrac{\varsigma_1}{\Delta t}\right)[\mathbf{CM}]\,]\{\mathbf{T}\}^{(n+1)} \\
&= \{\mathbf{FH}\} + \{\mathbf{FQ}\}[(1-\alpha*)\,[\mathbf{KH}] + \left(\tfrac{\varsigma_1}{\Delta t}\right)[\mathbf{CM}]\,]\{\mathbf{T}\}^{(n)}
\end{aligned} \tag{12}$$

where the unknowns are the nodal displacements $\{\mathbf{d}\}^{(n+1)}$, the nodal temperatures $\{\mathbf{T}\}^{(n+1)}$ and the nodal pore pressure values $\{\mathbf{p}\}^{(n+1)}$ at the current time step $\{\}^{(n+1)}$ and the superscripts $\{\}^{(n)}$ refer to the values of the corresponding variables at the preceding step. The parameters ς_i depend on the THM processes. Also $\{\mathbf{f}\}$ is the force vector, $\{\mathbf{FQ}\}$ and $\{\mathbf{FH}\}$ are heat flux vectors, Δt is a time increment and $\alpha*$ is a time integration parameter. This time integration parameter can vary between 0 and 1. It is found that by setting $\alpha* = 0.75$, a reliable stable solution is achieved. All other matrices $[\mathbf{K}]$, $[\mathbf{CP}]$, etc., are assembled from element matrices, which are dependent on thermal, mechanical and hydraulic properties of the individual elements and the interpolation functions used. The accuracy of the computational modelling has been verified by comparison with analytical solutions obtained for the one-dimensional TH, HM and THM behaviour of a finite column of a fluid saturated porous medium which is subjected, where appropriate, to surface tractions and /or to a temperature rise in the form of Heaviside step functions. In the computational scheme, the one-dimensional behaviour is simulated by considering a three-dimensional prismatic region, the surfaces of which are subjected to appropriate temperature, fluid pressure, displacement and traction boundary conditions, consistent with the requirements for one-dimensional behaviour. The computational estimates for the time-dependent responses of the dependent variables compare very accurately with results of analytical solutions.

The Heated Inclusion Problem

We apply the computational modelling to examine the THM behaviour of a fluid saturated medium of finite extent, which is internally heated by a rigid disc-shaped inclusion (Figure 1). The geometry of the heated area is a flat planar circular region, which is intended to model a deep repository of large area situated in a fluid saturated geological medium. The disc-shaped region is subjected to heating which has a time history of the form of a Heaviside step function and the excess pore fluid pressure generated due to the heating is allowed to dissipate within the medium itself. Since the problem is axisymmetric and since symmetry also exists about the plane of the heated inclusion, we can formulate the initial boundary value problem governing the resulting THM problem in relation to a cylindrical region where $r \in (0, \lambda a)$ and $z \in (0, \lambda a)$, where λ is an arbitrary parameter. Owing to the symmetries associated with the problem, the computational modelling can be restricted to a quarter-region of a halfspace, where appropriate symmetries are invoked to simulate the extended nature of the domain. The pore pressure boundary conditions at the heated inclusion region are specified as being of the homogeneous Dirichlet type. Figure 1 also illustrates the finite element discretization involving 20-node brick elements. The outer boundary of the quarter-halfspace domain is located at $20a$ where a is the radius of the heated inclusion region. Since the rigid disc inclusion is embedded in an infinite space region, by considering the symmetry conditions the boundary conditions can be identified on the planes of symmetry and along the axis of symmetry. In order to complete the formulation of the initial boundary value problem governing the THM process we need to specify initial conditions pertaining to the dependent variables and /or combinations of their spatial and time derivatives. In the problem dealing with the embedded rigid disc inclusion problem, we assume that the temperature and pore pressure fields relate to the excess values and that the poroelastic medium is at zero reference temperature and pressure.

The mesh discretization shown in Figure 1 provides a degree of mesh refinement to account for the steep gradients in the effective stress, pore pressure and temperature fields usually associated with the mixed boundary conditions prescribed on the plane of the disc inclusion. The computational modelling is performed using the developments presented previously. The thermo-poroelastic parameters used in the computations are as follows:

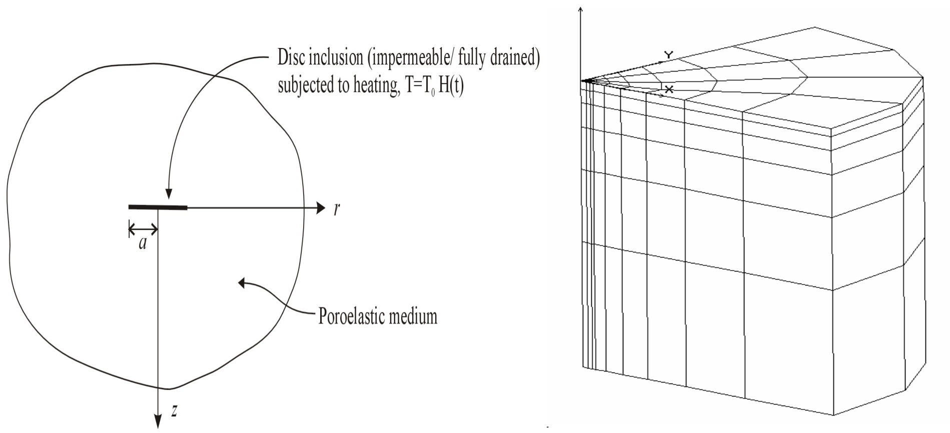

Figure 1: A disc inclusion problem for a poroelastic infinite space and the finite element discretization of the a sub-region

Mechanical: $E = 60 \times 10^9 Pa$; $\nu = 0.3$; $C_s = C_f = 0$

Thermal: $\beta' = \beta_s = 24.6 \times 10^{-6} \left({}^0C\right)^{-1}$; $\beta_f = 0.4 \times 10^{-3} \left({}^0C\right)^{-1}$; $\kappa = 4.0 W/m^0C$ $C_s = 718 J/kg/^0C$; $C_f = 4190 J/kg/^0C$

Physical: $\rho_s = 2700\, kg/m^3$; $\rho_f = 1000\, kg/m^3$; $\mu = 0.001\, kg/m/s$

Hydraulic: $K = 10^{-19} m^2$; $n = 0.01$

Time-dependent distributions for $\mathbf{u}$, $\boldsymbol{\sigma}'$, p and T within the poroelastic domain can be obtained from the computational analysis. Since the poroelastic medium is of infinite extent, the displacement field is of marginal interest. A result of some importance to the assessment of THM coupling relates to the time-dependent variations in the pore pressure field that result from the heating of the rigid disc inclusion and the thermo-mechanical deformations that materialize in the pore fluid as well as the porous skeleton of the poroelastic medium. In order to assess the greatest influence of the pore pressure generation effect, attention is restricted to the case of the impermeable rigid disc inclusion.

Figure 2 illustrates the time-dependent variation of pore fluid pressure at the central and edge locations of the poroelastic medium for four specific situations where both the far field boundary conditions and the location of the far field boundary are changed. These include, cases where either the boundaries are free or they are fixed and the extent of the cylindrical region is changed from 20a to 40a. Null boundary conditions are, of course, specified for the pore fluid pressure and the temperature at the outer surfaces of the region. It is found that for the specific problem examined, the far field boundary conditions and the extent of the domain have only a marginal effect on the time-dependent variation in the pore fluid pressure.

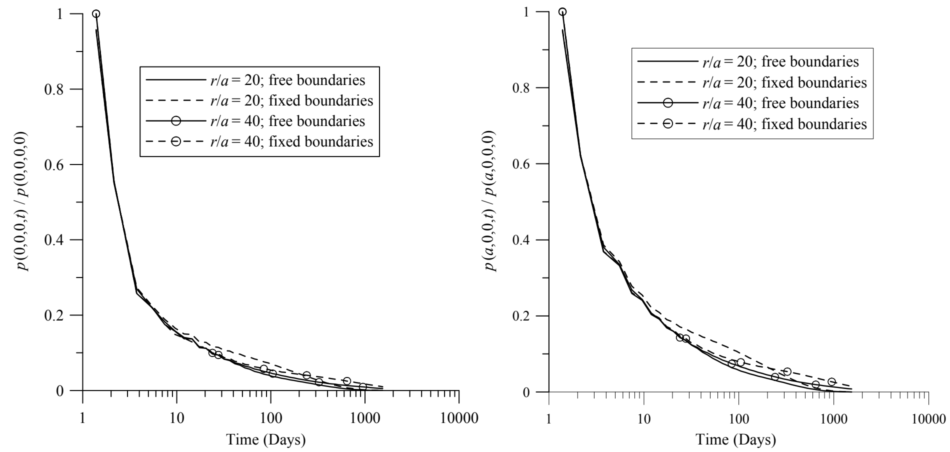

Figure 2: Time-dependent variation of pore fluid pressure at the center and at the edge of the rigid disc inclusion

Concluding remarks

The development of computational methodologies for the study of such THM problems is considered to be an important aspect that can enhance the application of such coupled theories to practical problems in environmental geomechanics. The reduction of the completely coupled system of equations into a weakly coupled form is an attempt to make the coupled problem more tractable. In certain classes of linear problems these weak links of coupling can be identified by through the physics of the problem. In this paper it is shown that in the context of the thermo-mechanical processes in fluid saturated porous geological materials with low permeability, the weak form of coupling results in a set of governing equations for which both the parameter identification and computational modelling are feasible exercises.

Acknowledgement

The work described in this paper was supported by the *Max Planck Forschungspreis in Engineering Sciences*, awarded to the author by the *Max Planck Gesellschaft*, Germany.

References

1. **Biot, M.A.** (1941): General theory of three-dimensional consolidation, *Journal of Applied Physics*, vol. 26, pp.155-164.

2. **Selvadurai, A.P.S. (Ed.)** (1996): *Mechanics of Poroelastic Media*, Kluwer Academic Publ., Dordrecht, The Netherlands.

3. **Selvadurai, A.P.S.** (2007): The analytical method in geomechanics, *Applied Mechanics Reviews*, vol. 60, pp. 87-106.
4. **Selvadurai, A.P.S.; Nguyen, T.S.** (1995): Computational modelling of isothermal consolidation of fractured media, *Computers and Geotechnics*, vol. 17, pp. 39-73.
5. **Selvadurai, A.P.S.; Nguyen, T.S**. (1996): Scoping analyses of the coupled thermal-hydrological-mechanical behaviour of the rock mass around a nuclear fuel waste repository, *Engineering Geology*, vol. 47, pp. 379-400.
6. **Laughton, A. S.; Roberts, L.; Wilkinson, D.; Gray, D.** (1986), *The disposal of long-lived and highly radioactive wastes*, In '*Proceedings of a Royal Society Discussion Meeting*', Royal Society, London.
7. **Lewis, R.W. and Schrefler, B.A.** (1998): *The Finite Element Method in the Static and Dynamic Deformation and Consolidation of Porous Media*, J. Wiley & Sons, New York, p. 492.
8. **Terzaghi, K.** (1925): *Erdbaumechanik auf bodenphysikalischer Grundlage*, Franz Deuticke, Leipzig.