Visualizing the Rotation of a Rigid Body

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Summary

The torque-free rotation of a rigid body has long been a topic of interest to physicists, mathematicians and engineers, including Euler in the 18th century, and continuing to the present day. Numerical solutions to Euler's equations are easy to achieve in an era of cheap computation, but classic solutions to the problem have used principles of angular momentum and energy to develop geometric insights to the problem. Static diagrams can help display these insights, but modern computer graphics hardware and software can truly bring these solutions to life via user-controlled animation.

Besides making visible such abstractions as the angular momentum and angular velocity vectors, animated computer graphics can show Poinsot's construction, Binet's ellipsoid, and the space and body cones made by the angular velocity vector. Commonly available hardware, including hand-held devices such as smart phones, can create these animations in real time, allowing users to set initial conditions, vary mass parameters, and even change the point of view mid-animation.

Binet's ellipsoid in particular benefits from advanced computer graphics. Recall that the angular momentum vector, although constant in inertial coordinates, can vary in body coordinates. Binet noted that in a body-fixed frame, the angular momentum is constrained to lie on a sphere (due to its constant magnitude) and on an ellipsoid (due to conservation of kinetic energy). The angular momentum vector must therefore lie on the intersection of these surfaces. Rendering these three-dimensional surfaces as translucent objects makes their intersection easier to understand than traditional line drawings would, especially when the viewer can vary the viewpoint and watch the angular velocity vector progress along its path as the object rotates.

Poinsot's construction is another classic solution, based on a body-fixed inertia ellipsoid. Poinsot showed that the tip of the angular velocity vector moves on an invariable plane, with the inertia ellipsoid rolling without slipping on the invariable plane. The path of the tip of the angular momentum vector is known as the herpolhode, when plotted on the invariable plane, or the polhode, when plotted on the inertia ellipsoid.

This presentation aims to give students and researchers a better understanding of the rotation of a rigid body by studying the equations of motion from a geometric approach, and then showing interactive animations that illustrate the solutions. I hope you will find this combination of physics, mathematics, and computer graphics an engaging way of approaching this classic problem.