The Progress of Energy Meshless Methods by Using Trial Functions as the Bases of Solution

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Abstract: For the linear differential operator equation equipped with boundary conditions we derive an energy identity. Then we propose an energy regularization technique to choose the energetic bases in the numerical solution of linear differential operator equation. In many meshless methods with some trial functions as the bases of numerical solution, there exist certain parameters in the numerical method. We derive a very simple energy gap functional and minimize it to determine the optimal parameters. The new methodology upon adopting optimal parameters by minimizing the energy gap functional can improve the accuracy of the meshless methods in the numerical solutions.

Keywords: Linear differential operator equation; energy regularization technique; energetic bases; energy gap functional; optimal parameters

1 Methodology

We consider

$$\mathcal{L}[u(\mathbf{x})] = f(\mathbf{x}), \mathbf{x} \in \Omega, \tag{1}$$

$$\mathcal{B}[u(\mathbf{x})] = g(\mathbf{x}), \mathbf{x} \in \Gamma := \partial\Omega, \tag{2}$$

where $\mathcal{L}$ is a linear differential operator, and $\mathcal{B}$ is a boundary operator to specify the boundary conditions.

The Galerkin method assumes that the trial solution $u(\mathbf{x})$ can be expressed by

$$u(\mathbf{x}) = \sum_{j=1}^{n} a_j \phi_j(\mathbf{x}), \tag{3}$$

where $\phi_j(\mathbf{x}), j = 1, \ldots, n$ are n given trial functions used as the bases, and then the expansion coefficients $a_j, j = 1, \ldots, n$ are determined from the weak-form formulation of Eqs. (1) and (2) by using $\phi_j(\mathbf{x})$ as test functions.

Multiplying Eq. (1) by $u(\mathbf{x})$, and integrating it over Ω:

$$\iint_{\Omega} u(\mathbf{x})\mathcal{L}[u(\mathbf{x})]\mathrm{ds} = \iint_{\Omega} f(\mathbf{x})u(\mathbf{x})ds. \tag{4}$$

Then by using the integration by parts, and the Gauss divergence theorem, etc., we can derive

$$\iint_{\Omega} I[u(\mathbf{x})]\mathrm{ds} = \int_{\Gamma} \mathcal{B}^*[u(\mathbf{x})]g(\mathbf{x})\mathrm{d}\ell + \iint_{\Omega} f(\mathbf{x})u(\mathbf{x})ds, \tag{5}$$

which is an energy identity, including internal energy, boundary work and external work. $\mathcal{B}^*$ is another boundary operator.

If u is a real solution of problem (1) and (2), it must satisfy the energy identity (5), and hence, the internal energy is balanced by the boundary work and external work. Due to numerical approximation and

error, the numerical solution of problem (1) and (2) usually does not satisfy the energy identity (5). Therefore, we can define the following energy gap functional:

$$\mathcal{G} := \int_\Gamma \mathcal{B}^*[u(\mathbf{x})]g(\mathbf{x})\mathrm{d}\ell + \iint_\Omega f(\mathbf{x})u(\mathbf{x})ds - \iint_\Omega I[u(\mathbf{x})]\mathrm{ds} \neq 0, \tag{6}$$

which means that the work done in the system is not fully transformed to the internal energy, and there is an energy gap $\mathcal{G} \neq 0$. The energy gap functional can help us to find the optimal parameters involved in the existent numerical method by minimizing $|\mathcal{G}|$:

$$\min_{\text{parameters}} |\mathcal{G}|. \tag{7}$$

It would be a very interesting topic for the numerical methods to be developed from this new idea.

The linear space $\mathbb{V}$ of all trial functions $\phi_j(\mathbf{x})$ is complete and satisfies:

$$\phi_j(\mathbf{x}) \in \mathbb{V},\ \phi_k(\mathbf{x}) \in \mathbb{V} \Rightarrow \phi_j(\mathbf{x}) + \phi_k(\mathbf{x}) \in \mathbb{V}, \tag{8}$$

$$\phi(\mathbf{x}) \in \mathbb{V} \Rightarrow \gamma\phi(\mathbf{x}) \in \mathbb{V}. \tag{9}$$

Unfortunately, $\mathbb{V}$ as a linear space is usually too large for $u(\mathbf{x}) \in \mathbb{V}$, such that the difficulty of Eq. (3) might appear.

We search the new bases by

$$E_j(\mathbf{x}) = \gamma_j\phi_j(\mathbf{x}) \in \mathbb{V},\ j \text{ not summed}, \tag{10}$$

where the weighting factor γ_j is to be determined, and at the same time, instead of Eq. (3), we assume that $u(\mathbf{x})$ can be expanded by

$$u(\mathbf{x}) = \sum_{j=1}^{n} b_j E_j(\mathbf{x}). \tag{11}$$

Inserting E_j for u into Eq. (5) we have

$$\iint_\Omega I\left[E_j(\mathbf{x})\right]\mathrm{ds} = \int_\Gamma \mathcal{B}^*[E_j(\mathbf{x})]g(\mathbf{x})\mathrm{d}\ell + \iint_\Omega f(\mathbf{x})E_j(\mathbf{x})ds, \tag{12}$$

and thus, by Eq. (10):

$$\iint_\Omega I\left[\gamma_j\phi_j(\mathbf{x})\right]\mathrm{ds} - \int_\Gamma \mathcal{B}^*[\gamma_j\phi_j(\mathbf{x})]g(\mathbf{x})\mathrm{d}\ell - \iint_\Omega \gamma_j f(\mathbf{x})\phi_j(\mathbf{x})ds = 0. \tag{13}$$

Because $\phi_j(\mathbf{x}), j = 1, \ldots, n$, $f(\mathbf{x})$ and $g(\mathbf{x})$ are given, we can solve $\gamma_j, j = 1, \ldots, n$ from the above equation. Consequently, $E_j(\mathbf{x})$ in Eq. (10) after inserting γ_j is an energetic basis, which shares a common property with $u(\mathbf{x})$, being both located on the same energy manifold $\mathbb{M}$, defined by Eq. (5):

$$u(\mathbf{x}) \in \mathbb{M},\ E_j(\mathbf{x}) \in \mathbb{M}. \tag{14}$$

The purpose of the present issue is to enhance the meshless method by considering the energy derived from Eqs. (1) and (2), such that we have a better meshless method: more stable, more efficient and more accurate, etc.

Basically, the new idea of energetic bases can be applied to the meshless methods which are based on the basis functions to expand the solutions, like as the boundary type method, the domain type method, as well as the Kansa type method.

2 Literature Survey

We have employed the new concept of energy gap to determine the optimal shape parameters in the MQ-RBF to solve the Cauchy problem of the Laplace Eq. [1]. On the other hand, we also used it to determine the optimal sources in the method of fundamental solutions (MFS) to solve the Cauchy problem of the Laplace Eq. [2].

We may expect that the energy plays a dominated role in the regularization of numerical solutions. The progress of this kind methods is revealed in [3-11], which include the boundary functions method, the Trefftz method, the MFS, and the MQ-RBF.

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