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Growth, Anisotropy, and Residual Stresses in Arteries

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Abstract: A simple phenomenological theory of tissue growth is used in order to demonstrate that volumetric growth combined with material anisotropy can lead to accumulation of residual stresses in arteries. The theory is applied to growth of a cylindrical blood vessel with the anisotropy moduli derived from experiments. It is shown that bending resultants are developed in the ring cross-section of the artery. These resultants may cause the ring opening or closing after cutting the artery *in vitro* as it is observed in experiments. It is emphasized that the mode of the arterial ring opening is affected by the parameters of anisotropy.

keyword: Growth; residual stress; artery; continuum mechanics; nonlinear elasticity; anisotropy

1 1. Introduction

Rachev and Greenwald (2003) argue that the first notion of the artery ring opening after a radial cut can be traced to the PhD Thesis by Bergel of the University of London in 1960. Journal publications count the ring opening phenomenon since the reports by Chuong and Fung (1986) and Vaishnav and Vossoughi (1987). In any event, the opening (or closing) of the ring implies relieving the residual stresses, which exist in arteries. Chuong and Fung (1986) assumed that the radial cut *entirely* relieves the residual stresses. They introduced the concept of the *zero-stress state* (ZSS) with reference to an opened artery segment. This concept was lifted to generality by Rodriguez et al. (1994) who proposed a geometrical scheme of growth of living bodies shown in Fig.1. There are three states of the growing body in the figure: the original ZSS $B(t_0)$; the grown ZSS $B(t_1)$; and the observed grown state $B'(t_1)$ with residual stresses $\overset{\circ}{\mathbf{T}}$. The growth/deformation gradient is decomposed into pure growth $\mathbf{F}_g$ and elastic deformation $\mathbf{F}_e$ tensors accordingly. The elastic deformation is necessary in order to provide the compatibility of the grown *incompatible* body configuration: $\mathbf{F}_{eg} = \mathbf{F}_e\mathbf{F}_g$. During the last decade a number of growth theories based on the above multiplicative decomposition of the growth/deformation gradient was proposed: Taber and Eggers (1996); Taber and Humphrey (2001); Van Dyke and Hoger (2002), to list a few. These theories were used for explaining the artery cutting experiments *in vitro*. In spite of the interesting insights in mechanical behavior of arteries provided by the mentioned theories and experiments, the described approach is not entirely free of controversy. Particularly, both the idea of ZSS and related experiments suffer from the following two drawbacks.

(a) The multiplicative decomposition of the deformation gradient mentioned above is analogous to the multiplicative decomposition of the deformation gradient in the large strain plasticity: $\mathbf{F}_{ep} = \mathbf{F}_e\mathbf{F}_p$, where $\mathbf{F}_p$ corresponds to plastic deformation. This decomposition follows Lee's idea of introducing an intermediate stress-free configuration of the purely plastic deformation. Generally, the field of purely plastic deformation is not geometrically compatible, i.e., material particles undergoing purely plastic deformation separately can not be assembled into a continuous body and the stress-free configuration is possible only pointwisely. The same is true for deformation of pure growth. It is not compatible and, generally, configuration $B(t_1)$in Fig.1 is not observable in experiments because it requires an infinite number of infinitesimal cuts. We mention a few experimental attempts to find an observable ZSS. The first experiments by Vaishnav and Vossoughi (1987) and Chuong and Fung (1986) with the radial cut of an arterial ring were thought to be entirely stress-relieving and the opened artery ring was considered fully stress-free. However, Vossoughi et

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al. (1993) proceeded with the midline cut of a radially cut arterial ring. They found further changes in the shape of two circumferentially-cut halves of the segment. This meant that not all residual stresses were relieved after the single radial cut as was thought for some ten years. Since this work the artery-cutting experiments have flourished and the number of the pieces cut from the artery was more than 30 in some experiments (Matsumoto et al., 1995). This is not infinity yet. Recently, Greenwald et al. (1997) reported that "by stepwise removal of the inner or outer layers of the ... artery by machining frozen specimens, we have shown that the *true* stress-free state can only be reached by partial destruction of the vessel wall and that different layers of the wall may each have different zero-stress state". Despite the authors' optimism, the history teaches us that the "true stress-free state" may not be final. The fact that the ZSS is hardly attainable in experiments potentially prevents from the theory calibration, which is based on the multiplicative decomposition of the deformation gradient.

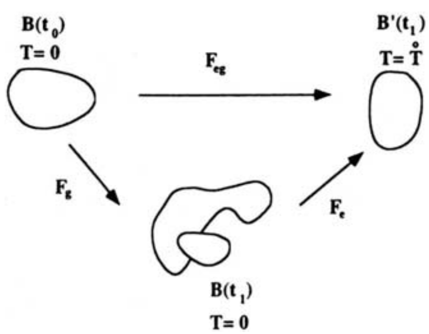


Figure 1 : Rodriguez-Hoger-McCulloch scheme of tissue growth.

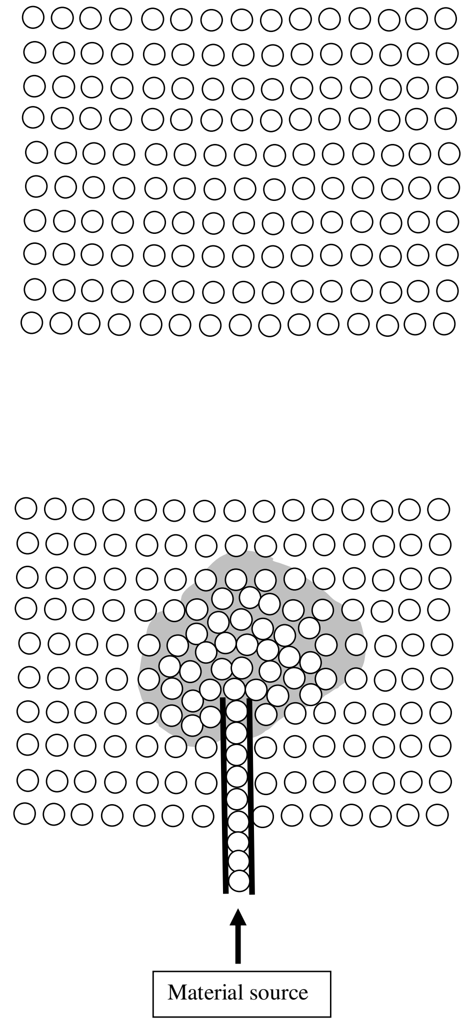


Figure 2 : Toy-tissue model: regular (top), point mass supply (bottom).

(b) Considerations above were devoted to the problems of reaching (or defining) the desirable zero-stress states of an arterial ring and calibrating growth theories based on the multiplicative decomposition of the deformation gradient. There is, however, another concern. Let us assume, for example, that after a series of very successful cuts we get a number of the arterial pieces, which are hopefully stress-free. Can we gain any information about the residual stresses in the artery from such an experiment? The answer is no. The problem is that every cut leads to a redistribution of residual stresses. This redistribution of stresses starts from the very first cut when a ring is taken from the artery. Indeed, such a ring is approximately under the plane strain state before the cut. However, after the cut the state of the ring is closer to the plane stress. Thus, extracting the ring from the artery we cause a new distribution of stresses in it. This same happens after all subsequent cuts. Actually, every subsequent cut relieves stresses in the previous configuration and not in the initial one. In order to illustrate this point we suggest the following experiment. Let two neighbor rings be cut from an artery in such place along the artery that it is reasonable to assume that the residual stresses are the same in both rings. These rings are further cut radially and circumferentially, i.e. along the midline. It is crucial, however, that the order of cutting is different for the different rings. One ring should be cut first radially and then circumferentially, while the other ring should be cut first circumferentially and then the two obtained thinner rings should be cut radially separately. Both initial rings are finally split into two open segments. This is done, however, in the different sequences of cuts. The final shape of the segments should be different in both cases because the order of cutting *does* matter for *large deformations* where the superposition principle is not valid. Indeed, the cutting is equivalent to applying tractions on the cut surfaces to make these surfaces traction-free. In the case of large deformations, the order of application of the external forces is crucial. The difference in the experimentally cut segments should be visible if the residual radial stresses are comparable in the magnitude with the residual circumferential stresses. The expected results of the described experiment could clearly illustrate insufficiency of the artery-cutting experiments for estimating the magnitudes of the residual stresses. It seems that the artery-cutting experiments can only be useful for the qualitative comparison purposes and not for the quantitative estimates of the residual stresses: the more artery is cut the less information can be gained about its residual stresses.

Guided by the difficulties of the traditional approach based on the ZSS concept, multiplicative decomposition of the deformation gradient, and the artery-cutting experiments, we look for alternative ways of attacking the problem of residual stresses. In the present work, we suggest a *qualitative* explanation of the residual stresses in arteries using a simple phenomenological framework for modeling tissue growth where material anisotropy is taken into account. This framework is not based on such concepts as ZSS and multiplicative decomposition of the deformation gradient and it is rooted in the work by Cowin and Hegedus (1976) on adaptive elasticity. We modify the Cowin-Hegedus theory by (a) including mass flux in the field equations; (b) introducing constitutive equations of 'thermoelastic' type; and (c) extending the theory to large strains. We apply this modified framework to the problem of growth of anisotropic cylindrical blood vessels. It is assumed that the material supply during growth is uniform and volumetric. Elastic properties of the anisotropic arterial wall are taken from the published experimental data.

2 Methods

2.1 Governing equations

A reasonable insight in the tissue growth mechanism can be gained by considering a very simple toy-tissue model (Fig.2). The regular initial tissue can be seen on the top of the figure. This is a collection of the regularly packed balls. The balls are interpreted as the tissue elementary components – cells, molecules of the extracellular matrix, and etc. The balls are arranged in a regular network for the sake of simplicity and clarity. They can be organized more chaotically – this will not affect the subsequent qualitative analysis. Let us assume that a new material, i.e. a number of new balls, is supplied as it is shown on the bottom of Fig.2. This supply is considered as a result of injection: the tube with the new material is a syringe. Usually, the new material is created in real tissues in a more complicated manner following a chain of the biochemical transformations. However, the finally produced new material still appears from the existing cells pointwisely. Thus, the injection of the balls is a quite reasonable model of tissue growth. Such model can be constructed physically, of course. It seems that the latter is not necessary and the toy-tissue model can be easily imagined. The result of such thought- experiment is shadowed in the figure and it can be described as follows:

(a) The number of the balls in the toy-tissue increases with the supply of the new ones.
(b) The new balls are concentrated at the edge of the tube and they do not spread uniformly over the tissue.
(c) The new balls cannot be accommodated at the point of their supply – the edge of the tube: they tend to spread over the area in the vicinity of the edge and the packing of the balls gets denser around the edge of the tube.
(d) The more balls are injected the less room remains for the new ones.
(e) The new balls press the old ones.
(f) The new balls tend to expand the area occupied by the tissue when the overall ball rearrangement reaches the tissue surface.

These six qualitative features of the toy-tissue microscopic behavior under the material supply can be translated into the language of the macroscopic theory accordingly:

(A) Mass of the tissue grows.
(B) Mass growth is not uniform – the mass density changes from one point to another.
(C) There is diffusion of mass.
(D) The diffusion is restricted by the existing tissue structure and its mass density: the denser is the tissue the less material it can accommodate.
(E) Growth is accompanied by stressing.
(F) The expansion of the tissue is volumetric – it is analogous to the thermal expansion of structural materials as steel, for example.

Three first features (A, B, and C) prompt the form of the mass balance equation. Indeed, the mass change means the failure of the mass conservation law, which covers most theories of Mechanics. The mass supply in the living tissues is possible through the biochemical interaction of the tissue with its environment. This means that the living tissue is an open system. The fact that non-uniform mass growth is related with the diffusion of mass is very important. It means that the mass balance law should include both the volumetric mass source and the surface mass flux. The latter is missed in most existing theories of growth. The absence of mass diffusion in the theory leads to a nonphysical conclusion that the density of the tissue will change only at the point of the material supply, i.e. pointwisely. In order to accommodate the non-uniform mass supply the mass diffusion should take place. Exceptions to this rule can occur, for example, when material supply is uniform and the tissue is not constrained geometrically.

The rest features (D, E, and F) motivate the constitutive law. They suggest that the stress-strain relations should be analogous to thermoelasticty where the role of the temperature is played by the mass density: the increase of the mass density results in the volume expansion of the tissue. On the other hand, the tissue should prevent the additional mass supply: the denser tissue the less is the new mass accommodation. Both these tendencies will be presented in the following equations of the growth theory. It is worth noting, however, that not all features of growth can be identified within the toy-tissue model. For example, the cells respond the applied mechanical stimuli biochemically – cellular mechanotransduction – by changing the program of the creation of new material. Thus, there is a competition between the stresses and supply of new material. This process is called adaptive growth. Under some circumstances, adaptive growth can be essential. We will not consider this issue in the present work restricting consideration by the purely genetic (programmed) growth.

Guided by the above reasoning the mass and momentum balance laws and initial/boundary conditions can be presented as follows accordingly (Volokh, 2003; 2004a; 2004b):

$$\frac{\partial \rho_R}{\partial t} = \mathrm{Div}\,\boldsymbol{\psi}_R + \xi_R, \tag{1}$$

$$\mathrm{Div}\,\mathbf{P} + \rho_R \mathbf{b} = \mathbf{0}, \tag{2}$$

$$\rho_R|_{t=0} = \rho_0^* \quad \text{in} \quad \Omega, \tag{3}$$

$$\begin{cases} \rho_R = \rho^* & \text{on} \quad \partial\Omega_\rho \\ \phi_R = \phi^* & \text{on} \quad \partial\Omega_\phi \end{cases}, \tag{4}$$

$$\begin{cases} \boldsymbol{\chi} = \boldsymbol{\chi}^* & \text{on} \quad \partial\Omega_{\boldsymbol{\chi}} \\ \mathbf{t} = \mathbf{t}^* & \text{on} \quad \partial\Omega_{\mathbf{t}} \end{cases}, \tag{5}$$

where subscript "R" designates the referential (Lagrangean) description of the body Ω; ρ_R is mass density; $\boldsymbol{\psi}_R$ is a vector of mass flux; ξ_R is volumetric mass supply; $\mathbf{P}$ is the 1$^{\text{st}}$ Piola-Kirchhoff stress tensor; $\mathbf{b}$ is the body force per unit mass; $\boldsymbol{\chi}(\boldsymbol{X})$ is a current placement of the particle $\mathbf{X}$; $\phi_R = \boldsymbol{\psi}_R \cdot \boldsymbol{N}$ is mass supply through the surface with the outward unit normal $\mathbf{N}$ in the reference configuration; $\mathbf{t}$ is a surface traction; and the quantities with the asterisk are given.

By introducing the deformation gradient $\boldsymbol{F} = \mathrm{Grad}\,\boldsymbol{\chi}(\boldsymbol{X})$ it is possible to write the following useful relationships:

$$\mathbf{t} = \mathbf{PN}\frac{dA_R}{dA}, \tag{6}$$

$$\boldsymbol{\sigma} = J^{-1}\mathbf{PF}^T. \tag{7}$$

Here A designates the surface area; $\boldsymbol{\sigma}$ is the "true" Cauchy stress tensor; $J = \det\mathbf{F}$.

Based on the thermoelastic analogy discussed above we introduce the constitutive equations in the following form

$$\mathbf{P} = \mathbf{F}\frac{\partial W}{\partial \mathbf{E}} - \mathbf{F}\boldsymbol{\eta}(\rho_R - \rho_0^*) \tag{8}$$

$$\boldsymbol{\psi}_R = \beta\,\mathrm{Grad}\,(\rho_R - \rho_0^*), \tag{9}$$

$$\xi_R = \omega + f[\mathbf{F},\mathbf{P}] - \gamma(\rho_R - \rho_0^*), \tag{10}$$

where $\mathbf{E} = (\mathbf{F}^T\mathbf{F} - \mathbf{1})/2$ is the Green strain tensor; and $\mathbf{1}$ is the identity tensor.

Material growth parameters are introduced in the constitutive equations as follows: $\beta > 0$ is a coefficient of mass conductivity of a tissue, which is analogous to the coefficient of thermal conductivity of a solid, and it counts for how much the mass supply changes for the spatially varying increment of mass density; $\omega > 0$ is the genetic mass supply, which is determined by the genetically controlled production of new cells and the extracellular matrix proteins by the existing cells; f is the epigenetic mass supply, which should depend on stress and/or strain measures (its correct expression is a key problem when tissue remodeling is considered); $\gamma > 0$ is a coefficient of tissue resistance, which defines the resistance of the tissue to accommodate new mass for the increasing mass density; $\boldsymbol{\eta}$ is a tensor of material growth parameters which is defined as follows in the case of material orthotropy

$$\begin{cases} \eta_{11} = c(c_1\alpha_1 + c_4\alpha_2 + c_6\alpha_3) \\ \eta_{22} = c(c_4\alpha_1 + c_2\alpha_2 + c_5\alpha_3) \\ \eta_{33} = c(c_6\alpha_1 + c_5\alpha_2 + c_3\alpha_3) \end{cases} \qquad \eta_{IJ} = 0, I \neq J \tag{11}$$

where the coefficients of growth expansion $\alpha_i > 0$ define how much the relative volume changes for the given increment of mass density. These coefficients are analogous to the coefficients of thermal expansion in the classical thermoelasticity. In order to provide the analogy with thermoelasticity at small strains the elastic parameters were chosen in accordance with the Fung pseudo-strain energy expression

$$\begin{aligned} W &= \frac{1}{2}ce^Q, \\ Q &= c_1E_{11}^2 + c_2E_{22}^2 + c_3E_{33}^2 + 2c_4E_{11}E_{22} \\ &\quad + 2c_5E_{33}E_{22} + 2c_6E_{11}E_{33} + c_7(E_{12}^2 + E_{21}^2) \\ &\quad + c_8(E_{23}^2 + E_{32}^2) + c_9(E_{13}^2 + E_{31}^2) \end{aligned} \tag{12}$$

where c is the only dimensional elastic parameter while c_i are dimensionless.

It is worth emphasizing that the volumetric mass supply should include the epigenetic term, f, on the right-hand side of Eq.(10) when the tissue adaptation is considered. Traditionally, growth is related to a merely adaptive process: "stress-modulated growth", "stress-dependent growth". The latter means that tissue growth is a result of the applied mechanical stimuli. Particularly, the increasing blood pressure (or/and the blood flow rate) is considered to stimulate artery growth. *There is no doubt that tissue adaptation takes place*. Such adaptation can be the main factor when tissue defects and healing are considered. Whether this adaptation is the *main* scenario of artery growth from embryo to maturity? There is no evidence of that. The argument that the blood flow is the *main* mechanical stimulus for tissue growth is questionable. Indeed, blood flow itself is a result of the heart and arteries' action. The increase of the rate and pressure of blood flow are themselves a result of the maturing heart and arteries. It is difficult to imagine that embryonic heart and arteries can produce the same blood flow as the mature ones. The influence of blood flow on the development of cardiovascular tissues is not one-directional: both blood and tissues affect each other. The latter means that the stimulus for tissue growth is not purely mechanical. Without some kind of genetic regulation, the organs were growing infinitely. We believe that genes significantly affect tissue growth from embryo to maturity: growth is not a purely adaptive process. In the subsequent consideration we ignore the tissue adaptation as subsidiary (f=0) and we consider free volumetric growth caused by the genetic factors primarily. This allows for uncoupling mass and momentum balance equations. Indeed, substituting Eqs.(9) and (10) in Eq.(1) and assuming β =constant we have

$$\frac{\partial \rho_R}{\partial t} = \beta \, \mathrm{Div}\,\mathrm{Grad}\,(\rho_R - \rho_0^*) - \gamma(\rho_R - \rho_0^*) + \omega. \tag{13}$$

This equation is completed with the initial and boundary conditions (3), (4). Substituting solution of Eq.(13) in Eqs.(8) and (2) and adding boundary conditions (5) it is possible to find the deformation and corresponding stress fields. The time-dependence of the mass density evolution is important when a transient process is considered. However, we will consider the steady (quasi-equilibrium) state when the time derivative can be ignored on the left-hand side of Eq.(13).

2.2 *Uniform free radial growth of an artery*

We consider artery growth as a radial growth of an infinite cylinder under the plane strain conditions where all variables depend on the radial coordinate only. In this case, the deformation gradient and the Green strain tensors take the following forms accordingly[4]

$$\mathbf{F} = \frac{\partial r}{\partial R}\mathbf{k}_r \otimes \mathbf{K}_R + \frac{r}{R}\mathbf{k}_\theta \otimes \mathbf{K}_\Theta + \mathbf{k}_z \otimes \mathbf{K}_Z, \tag{14}$$

$$\begin{aligned}\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{1}\right) = \frac{1}{2}\left(\frac{\partial r}{\partial R}\frac{\partial r}{\partial R} - 1\right)\mathbf{K}_R \otimes \mathbf{K}_R \\ + \frac{1}{2}\left(\frac{r^2}{R^2} - 1\right)\mathbf{K}_\Theta \otimes \mathbf{K}_\Theta\end{aligned} \tag{15}$$

Here $\{\mathbf{K}_R, \mathbf{K}_\Theta, \mathbf{K}_Z\}$ and $\{\mathbf{k}_r, \mathbf{k}_\theta, \mathbf{k}_z\}$ form the orthonormal bases in cylindrical coordinates at the reference and current configurations accordingly; r is the current placement of the particle, which occupied position R at the reference configuration.

In this case, we have

$$\begin{aligned}\frac{\partial W}{\partial \mathbf{E}} - \boldsymbol{\eta}\rho = \left(ce^Q\left(c_1E_{RR} + c_4E_{\Theta\Theta}\right) - \eta_{11}\rho\right)\ \mathbf{K}_R \otimes \mathbf{K}_R \\ + \left(ce^Q\left(c_2E_{\Theta\Theta} + c_4E_{RR}\right) - \eta_{22}\rho\right)\ \mathbf{K}_\Theta \otimes \mathbf{K}_\Theta \\ + \left(ce^Q\left(c_5E_{\Theta\Theta} + c_6E_{RR}\right) - \eta_{33}\rho\right)\ \mathbf{K}_Z \otimes \mathbf{K}_Z\end{aligned} \tag{16}$$

and

$$\begin{aligned}&\mathbf{F}\frac{\partial W}{\partial \mathbf{E}} - \boldsymbol{F}\boldsymbol{\eta}\rho \\ &= \frac{\partial r}{\partial R}\left(ce^Q\left(c_1E_{RR} + c_4E_{\Theta\Theta}\right) - \eta_{11}\rho\right)\ \mathbf{k}_r \otimes \mathbf{K}_R \\ &+ \frac{r}{R}\left(ce^Q\left(c_2E_{\Theta\Theta} + c_4E_{RR}\right) - \eta_{22}\rho\right)\ \mathbf{k}_\theta \otimes \mathbf{K}_\Theta \\ &+ \left(ce^Q\left(c_5E_{\Theta\Theta} + c_6E_{RR}\right) - \eta_{33}\rho\right)\ \mathbf{k}_z \otimes \mathbf{K}_Z\end{aligned} \tag{17}$$

where $\rho \equiv \rho_R - \rho_0^*$.

Assume also that new material is supplied uniformly. In this case,

$$\rho = \omega/\gamma = \text{constant} \tag{18}$$

is a solution of Eq.(13), where $\partial\rho_R/\partial t = 0$, and $\phi_R = \beta\mathbf{N}\cdot\mathrm{Grad}\,\rho = 0$ on the boundary.

Eqs. (11), (17), and (18) define the right-hand side of Eq.(8). The equilibrium equation (2) without the body forces takes the form (see Appendix)

$$\mathrm{Div}\,\mathbf{P} = \left(\frac{\partial P_{rR}}{\partial R} + \frac{P_{rR} - P_{\theta\Theta}}{R}\right)\mathbf{k}_r = \mathbf{0}, \tag{19}$$

[4] $\mathbf{K}_R = (\cos\Theta, \sin\Theta, 0)^T$; $\mathbf{K}_\Theta = (-\sin\Theta, \cos\Theta, 0)^T$; $\mathbf{K}_Z = (0,0,1)^T$ and $\mathbf{K}_M \otimes \mathbf{K}_N = \mathbf{K}_M\mathbf{K}_N^T$; $\mathbf{k}_r = (\cos\theta, \sin\theta, 0)^T$; $\mathbf{k}_\theta = (-\sin\theta, \cos\theta, 0)^T$; $\mathbf{k}_z = (0,0,1)^T$ and $\mathbf{k}_m \otimes \mathbf{k}_n = \mathbf{k}_m\mathbf{k}_n^T$

where all components have been defined already.

Summarizing the equations in the scalar form, we have the following system

$$\frac{\partial P_{rR}}{\partial R} + \frac{P_{rR} - P_{\theta\Theta}}{R} = 0, \tag{20}$$

$$\begin{cases} P_{rR} = c\dfrac{\partial r}{\partial R}\big(e^Q(c_1E_{RR} + c_4E_{\Theta\Theta}) \\ \qquad\qquad - (c_1 + c_4 + c_6)\alpha\omega/\gamma\big) \\ P_{\theta\Theta} = c\dfrac{r}{R}\big(e^Q(c_2E_{\Theta\Theta} + c_4E_{RR}) \\ \qquad\qquad - (c_2 + c_4 + c_5)\alpha\omega/\gamma\big) \end{cases}, \tag{21}$$

$$\alpha = \alpha_1 = \alpha_2 = \alpha_3$$

$$Q = c_1E_{RR}^2 + c_2E_{\Theta\Theta}^2 + 2c_4E_{RR}E_{\Theta\Theta}, \tag{22}$$

$$\begin{cases} E_{RR} = \dfrac{1}{2}\left(\dfrac{\partial r}{\partial R}\dfrac{\partial r}{\partial R} - 1\right) \\ E_{\Theta\Theta} = \dfrac{1}{2}\left(\dfrac{r^2}{R^2} - 1\right) \end{cases}, \tag{23}$$

$$\begin{cases} \sigma_{rr} = J^{-1}\dfrac{\partial r}{\partial R}P_{rR} \\ \sigma_{\theta\theta} = J^{-1}\dfrac{r}{R}P_{\theta\Theta} \end{cases}, \tag{24}$$

$$J = \frac{r}{R}\frac{\partial r}{\partial R}, \tag{25}$$

$$\begin{cases} \sigma_{rr}(R = 1) = 0 \\ \sigma_{rr}(R = 1.3) = 0 \end{cases}. \tag{26}$$

After substituting Eqs. (21)-(25) in the equilibrium equation (20) and boundary conditions (26), we have a two-point boundary value problem in terms of $r(R)$.

3 Results

The two-point BVP derived in the previous section was solved for a number of varying elastic and growth parameters. Solution was obtained by using the shooting method when the initial value problem (IVP) was solved iteratively until fitting the BVP solution. We used Mathematica IVP solver 'NDSolve'.

Radial displacements, radial and circumferential Cauchy stresses (Figs.3, 4) were computed for two sets of material parameters:

(1) Chuong and Fung (1986)

$$\begin{aligned} &c_1 = 0.0499; && c_2 = 1.0672; \\ &c_3 = 0.4775; && c_4 = 0.0042; \\ &c_5 = 0.0903; && c_6 = 0.0585; \\ &c_7 = c_8 = c_9 = 0, \end{aligned} \tag{27}$$

(2) Chuong and Fung (1984)

$$\begin{aligned} &c_1 = 1.744; && c_2 = 0.619; \\ &c_3 = 0.0405; && c_4 = 0.004; \\ &c_5 = 0.0667; && c_6 = 0.0019; \\ &c_7 = c_8 = c_9 = 0. \end{aligned} \tag{28}$$

Every set of material parameters was considered with four growth parameters varying by 4 orders of magnitude: $\alpha\omega/\gamma = 0.001; 0.01; 0.1; 1$.

Resulting displacements vary almost linearly along the radius (Figs.3a, 4a). Absolute values of the radial stresses increase towards the mid-surface of the wall (Figs.3b, 4b), while the absolute values of the circumferential stresses approach zero at the mid-surface and they vary almost linearly along the radius (Fig.3c, 4c). It should be noted that circumferential stresses are larger than the radial stresses by an order of magnitude in both cases of material parameters. It is interesting that the directions of the stresses are different for the two sets of material parameters. This is, particularly, critical for the circumferential stresses because it means that different bending resultants appear in the ring. If the ring is cut radially, then it opens as shown in Fig.5a (right) for the first set of elastic parameters. A ring with the second set of material parameters behaves differently: it closes (Fig.5b) after the cut, i.e. its edges overlap.

It is remarkable that the results of the computation are qualitatively similar for the essentially varying growth parameter even though the deformation is large – up to 50%.

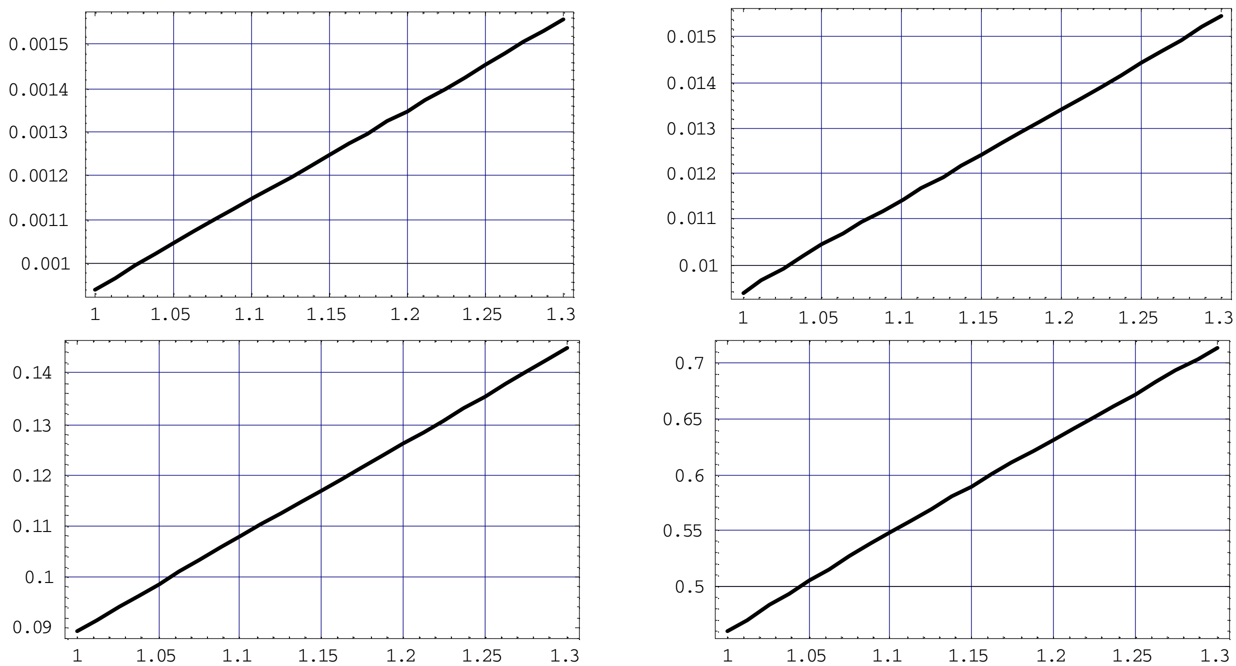
4 Discussion

A simple phenomenological theory of tissue growth has been used for explaining the phenomenon of the residual stresses in arteries qualitatively. Material anisotropy was included in the theoretical setting in accordance with the experimental data. The theory was applied to the problem of free and uniform radial growth of a cylindrical blood vessel. Displacement and stress fields were computed for the experimentally obtained values of the elasticity parameters. Computations give evidence of the appearance of the circumferential stresses resulting in the bending moments, which provide the compatibility of the grown arterial cross-section. The radial cut of the arterial ring will lead to the disappearance of the bending moments and opening or closing of the ring as it is observed in experiments.

It is important that the circumferential stresses, which are accumulated into the residual stresses during the long-term growth, appear due to anisotropy. These stresses would not appear in the 'isotropic artery' (Volokh, 2004b). The latter suggests the interpretation of the arterial anisotropy as a constraint imposed on the volumetric growth. It is interesting that this conclusion is novel as compared to the traditional point of view that the material inhomogeneity and differential growth are the main sources of the residual stresses (Fung, 1991; 1993). It is very likely that the material anisotropy is a complementary factor to the material inhomogeneity and differential growth in causing the residual stresses. In principle, the considered theory allows for including the material inhomogeneity and differential growth in analysis. Unfortunately, there is no clear enough experimental data to do so yet.

It is equally important that depending on the mutual ratio of the anisotropic elastic parameters various scenarios of the ring opening in the artery-cutting experiments are available. The ring can open up after cutting – resulting in positive opening angle; or the ring can close after the radial cut – resulting in negative opening angle. Both these scenarios are in excellent qualitative agreement with the experimental data (Fung, 1984; 1993; Rachev and Greenwald, 2003; Saini et al., 1995; Vaishnav and Vossoughi, 1987).

It should not be missed that also radial stresses appear in the considered arterial growth. The magnitude of these stresses is of lower order as compared to circumferential stresses. Nonetheless, the radial stresses can play a role in forming the global residual stresses. Particularly, the radial stresses are a good candidate for the explanation of Vossoughi et al. (1993) experiments. These authors cut the opened artery ring along the midline and found that the inside segment opened more while the outside segment closed more. Probably, this happened because the radial residual stresses had been relieved partially.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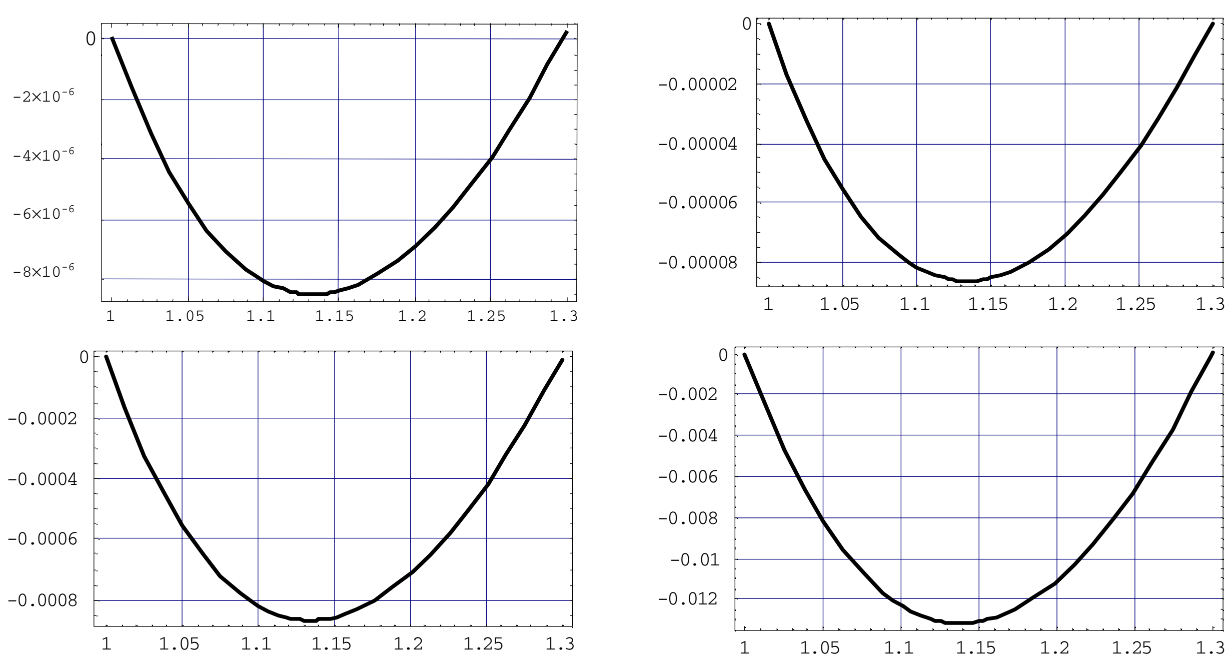
1 1.05 1.1 1.15 1.2 1.25 1.3

(a)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-4×10-6
-6×10-6
-8×10-6
1 1.05 1.1 1.15 1.2 1.25 1.3
0
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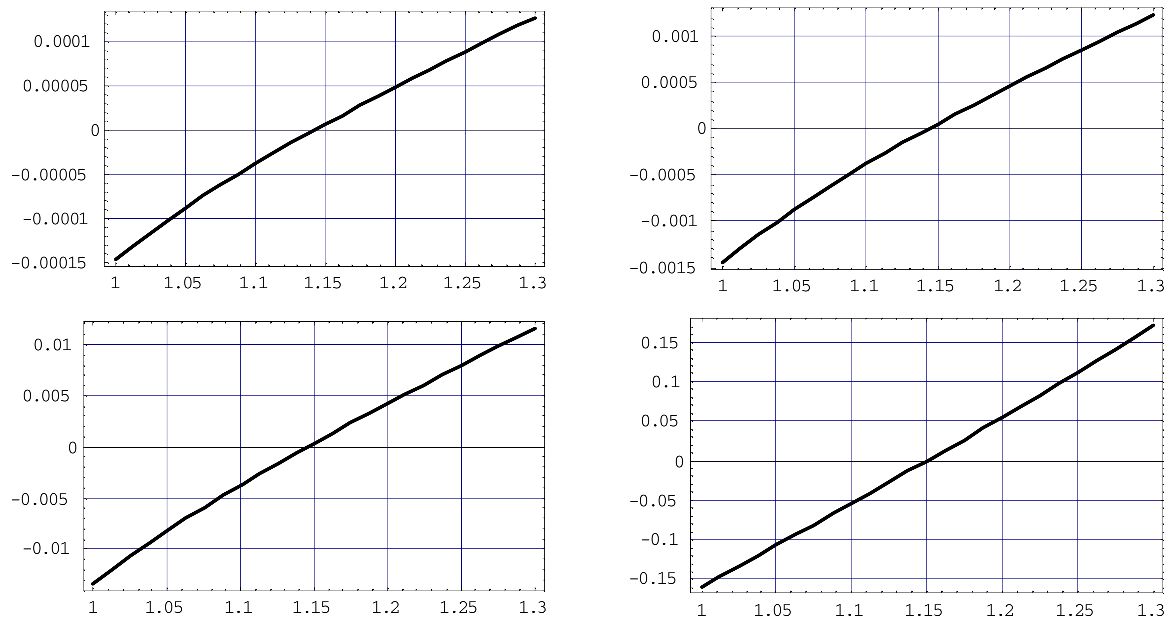
-0.00004
-0.00006
-0.00008
1 1.05 1.1 1.15 1.2 1.25 1.3
0
-0.0002
-0.0004
-0.0006
-0.0008
1 1.05 1.1 1.15 1.2 1.25 1.3
0
-0.002
-0.004
-0.006
-0.008
-0.01
-0.012
1 1.05 1.1 1.15 1.2 1.25 1.3

(b)

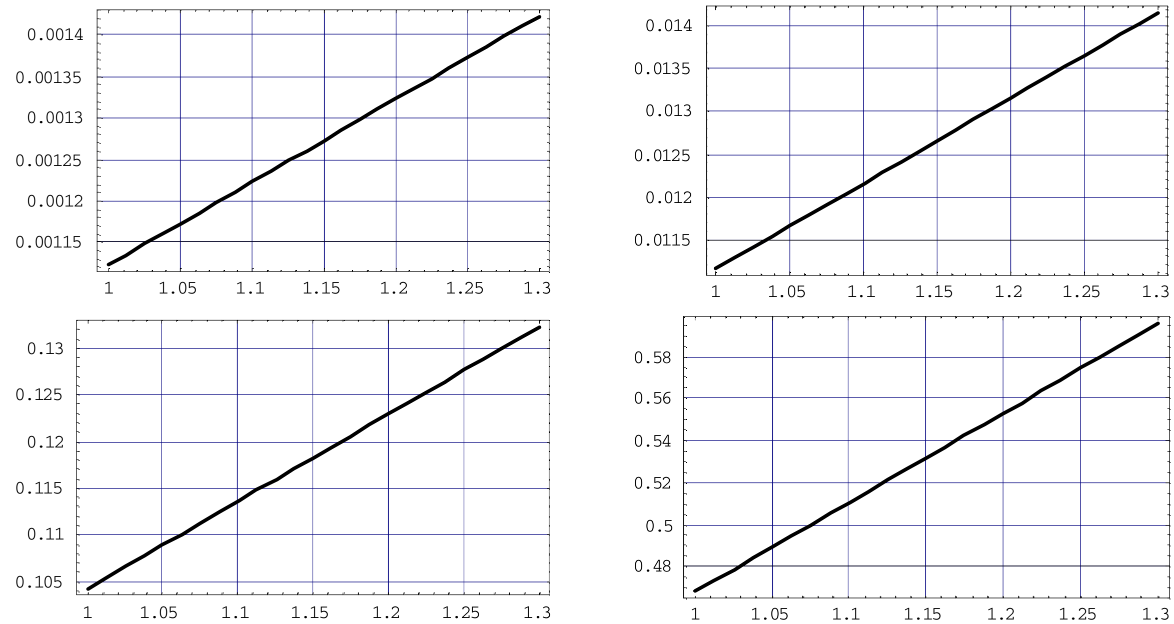


(c)

Figure 3 : (a) Normalized radial displacements $(r-R)/1$ (vertical axis) for the dimensionless growth parameter $\alpha\omega/\gamma$ equal to 0.001; 0.01; 0.1; 1.0 (from the top to the bottom accordingly) for free volumetric growth of the cylinder: the first set of material parameters Eq.(27). (b) Normalized radial stress σ_{rr}/c (vertical axis) for the dimensionless growth parameter $\alpha\omega/\gamma$ equal to 0.001; 0.01; 0.1; 1.0 (from the top to the bottom accordingly) for free volumetric growth of the cylinder: the first set of material parameters Eq.(27).(c) Normalized circumferential stress $\sigma_{\theta\theta}/c$ (vertical axis) for the dimensionless growth parameter $\alpha\omega/\gamma$ equal to 0.001; 0.01; 0.1; 1.0 (from the top to the bottom accordingly) for free volumetric growth of the cylinder: the first set of material parameters Eq.(27).

Comparing the theory considered in the present work to other theories of tissue growth we should emphasize that our purpose was to develop a *tractable* theory, i.e. a theory, which can guide an experiment and can be calibrated (or falsified!) by it. Indeed, all variables of our theory – placements and mass densities – are tractable and measurable. This is not the case of the theories based on the multiplicative decomposition of the deformation gradient. Both cofactors of this multiplicative decomposition as well as the concept of zero-stress state are *hidden*, i.e. unobservable and immeasurable, variables. Although we are far from stating that the theories including hidden variables are useless, some balance between this sort of theories and the theories without hidden variables is desirable. Another important feature of the theory presented in this work as compared to others is the use of a simple microscopic toy-tissue model. The fact that growth is accompanied by stressing immediately follows from this model. In our opinion, residual stresses exist in both hard and soft living tissues. However, the magnitude of the residual stresses in hard tissues is small as compared to their elastic parameters, while the magnitudes of the residual stresses and elastic parameters of soft tissues are comparable. The latter is the reason for the remarkable observations of the artery ring opening in experiments.

Finally, it is worth noting that the process of tissue growth is exceptionally complex and one can hardly expect creating a universal theory of growth covering the whole lifetime of a tissue. It seems that such ambitious project of 'great unification' is not a realistic one because not only the growth regime but also elastic properties of the tissue are exposed to essential alterations with time. Nonetheless, the attempts to find some rigid core of theoretical description of tissue growth may not be hopeless. Such theory could play a role analogous to the role of the linear elasticity theory for structural steels. The latter exhibit the whole range of rheological behavior in-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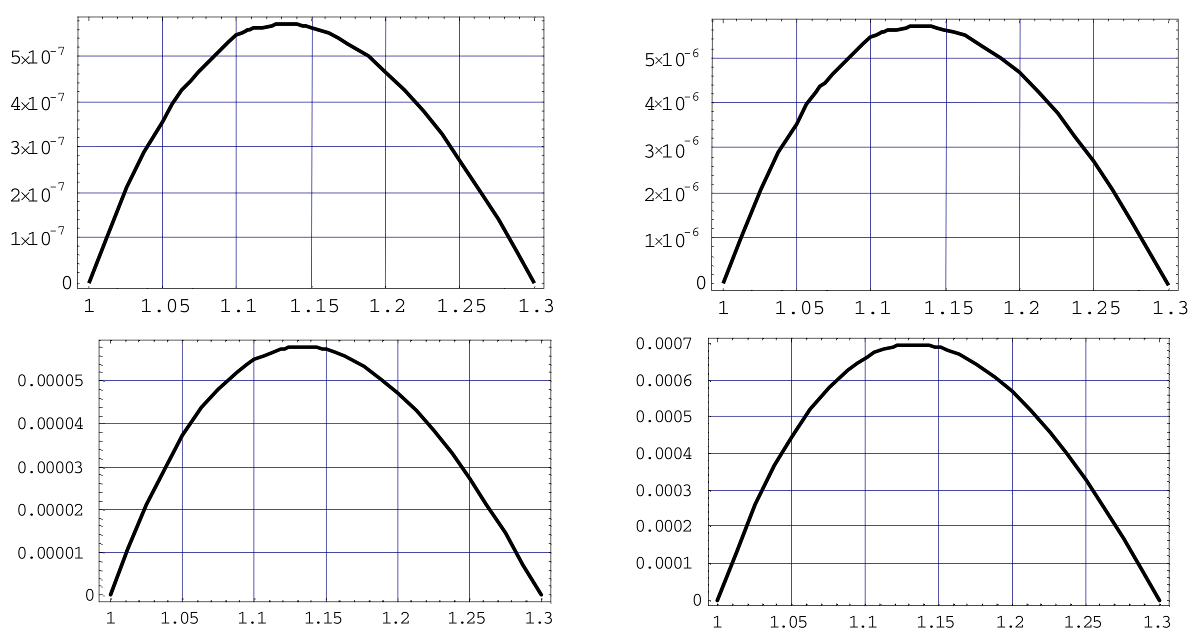
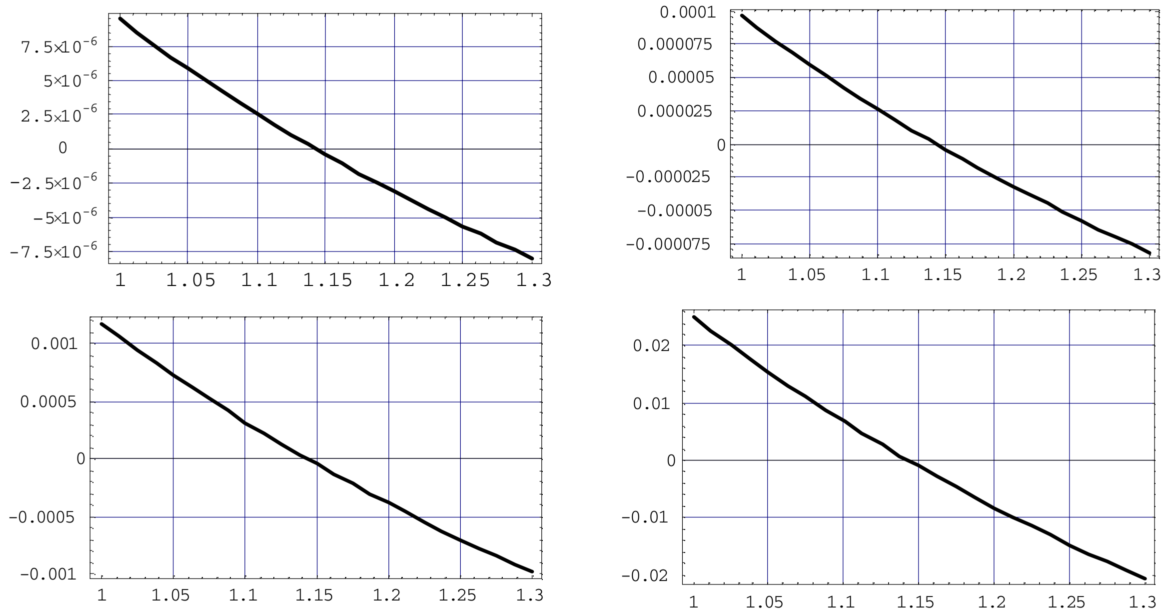
0.00135
0.0013
0.00125
0.0012
0.00115
1 1.05 1.1 1.15 1.2 1.25 1.3
0.014
0.0135
0.013
0.0125
0.012
0.0115
1 1.05 1.1 1.15 1.2 1.25 1.3
0.13
0.125
0.12
0.115
0.11
0.105
1 1.05 1.1 1.15 1.2 1.25 1.3
0.58
0.56
0.54
0.52
0.5
0.48
1 1.05 1.1 1.15 1.2 1.25 1.3

(a)


5×10⁻⁷
4×10⁻⁷
3×10⁻⁷
2×10⁻⁷
1×10⁻⁷
0
1 1.05 1.1 1.15 1.2 1.25 1.3
5×10⁻⁶
4×10⁻⁶
3×10⁻⁶
2×10⁻⁶
1×10⁻⁶
0
1 1.05 1.1 1.15 1.2 1.25 1.3
0.00005
0.00004
0.00003
0.00002
0.00001
0
1 1.05 1.1 1.15 1.2 1.25 1.3
0.0007
0.0006
0.0005
0.0004
0.0003
0.0002
0.0001
0
1 1.05 1.1 1.15 1.2 1.25 1.3

(b)



(c)

Figure 4 : (a) Normalized radial displacements $(r-R)/1$ (vertical axis) for the dimensionless growth parameter $\alpha\omega/\gamma$ equal to 0.001; 0.01; 0.1; 1.0 (from the top to the bottom accordingly) for free volumetric growth of the cylinder: the first set of material parameters Eq.(28).(b) Normalized radial stress σ_{rr}/c (vertical axis) for the dimensionless growth parameter $\alpha\omega/\gamma$ equal to 0.001; 0.01; 0.1; 1.0 (from the top to the bottom accordingly) for free volumetric growth of the cylinder: the first set of material parameters Eq.(28).(c) Normalized circumferential stress $\sigma_{\theta\theta}/c$ (vertical axis) for the dimensionless growth parameter $\alpha\omega/\gamma$ equal to 0.001; 0.01; 0.1; 1.0 (from the top to the bottom accordingly) for free volumetric growth of the cylinder: the first set of material parameters Eq.(28).

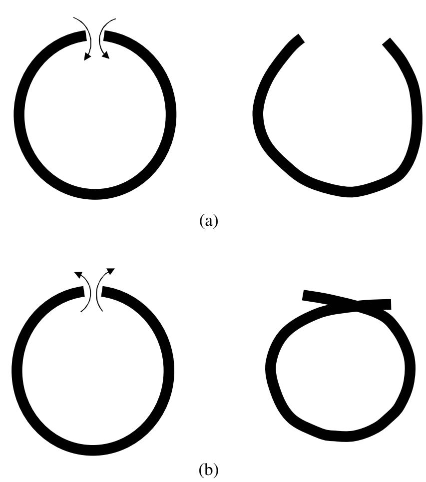


Figure 5 : (a) Bending moment (left) provides compatibility of the ring, which opens when cut (right). (b) Bending moment (left) provides compatibility of the ring, which closes when cut (right).

cluding plastic flow, hardening, viscosity, fatigue and etc. However, the linear elasticity is the core of the successful approach to describing mechanics of structural steels. Probably, something analogous can be also developed for describing mechanics of tissue growth.

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Appendix

Lagrangean scalar equilibrium equations in cylindrical coordinates are rarely discussed in the literature. The reason is that many soft materials are assumed incompressible what allows for using a simpler Eulerian description for obtaining some analytical solutions. This is not our case and we need the Lagrangean equilibrium equations in cylindrical coordinates. These equations can be derived from the *total covariant derivative* of the 1^{st} Piola-Kirchhoff stress tensor. Though this way may be elegant and short we prefer a more straightforward and lengthy way, which, however, does not require any knowledge of the general tensor calculus from the reader.

First, we write the divergence operator in the form

$$\mathrm{Div}\mathbf{P} = \frac{\partial \mathbf{P}}{\partial R}\mathbf{K}_R + \frac{\partial \mathbf{P}}{R\partial \Theta}\mathbf{K}_\Theta + \frac{\partial \mathbf{P}}{\partial Z}\mathbf{K}_Z. \tag{A29}$$

Now our plan is to compute the right-hand side of this equation term by term.

We start with

$$\frac{\partial \mathbf{P}}{\partial R}\mathbf{K}_R$$
$$= \left(\frac{\partial P_{rR}}{\partial R}\mathbf{k}_r\otimes\mathbf{K}_R + P_{rR}\frac{\partial \mathbf{k}_r}{\partial R}\otimes\mathbf{K}_R + P_{rR}\mathbf{k}_r\otimes\frac{\partial \mathbf{K}_R}{\partial R}\right)\mathbf{K}_R$$
$$+\left(\frac{\partial P_{r\Theta}}{\partial R}\mathbf{k}_r\otimes\mathbf{K}_\Theta + P_{r\Theta}\frac{\partial \mathbf{k}_r}{\partial R}\otimes\mathbf{K}_\Theta + P_{r\Theta}\mathbf{k}_r\otimes\frac{\partial \mathbf{K}_\Theta}{\partial R}\right)\mathbf{K}_R$$
$$+\left(\frac{\partial P_{rZ}}{\partial R}\mathbf{k}_r\otimes\mathbf{K}_Z + P_{rZ}\frac{\partial \mathbf{k}_r}{\partial R}\otimes\mathbf{K}_Z + P_{rZ}\mathbf{k}_r\otimes\frac{\partial \mathbf{K}_Z}{\partial R}\right)\mathbf{K}_R$$
$$+\left(\frac{\partial P_{\theta R}}{\partial R}\mathbf{k}_\theta\otimes\mathbf{K}_R + P_{\theta R}\frac{\partial \mathbf{k}_\theta}{\partial R}\otimes\mathbf{K}_R + P_{\theta R}\mathbf{k}_\theta\otimes\frac{\partial \mathbf{K}_R}{\partial R}\right)\mathbf{K}_R$$
$$+\left(\frac{\partial P_{\theta\Theta}}{\partial R}\mathbf{k}_\theta\otimes\mathbf{K}_\Theta + P_{\theta\Theta}\frac{\partial \mathbf{k}_\theta}{\partial R}\otimes\mathbf{K}_\Theta + P_{\theta\Theta}\mathbf{k}_\theta\otimes\frac{\partial \mathbf{K}_\Theta}{\partial R}\right)\mathbf{K}_R$$
$$+\left(\frac{\partial P_{\theta Z}}{\partial R}\mathbf{k}_\theta\otimes\mathbf{K}_Z + P_{\theta Z}\frac{\partial \mathbf{k}_\theta}{\partial R}\otimes\mathbf{K}_Z + P_{\theta Z}\mathbf{k}_\theta\otimes\frac{\partial \mathbf{K}_Z}{\partial R}\right)\mathbf{K}_R$$
$$+\left(\frac{\partial P_{zR}}{\partial R}\mathbf{k}_z\otimes\mathbf{K}_R + P_{zR}\frac{\partial \mathbf{k}_z}{\partial R}\otimes\mathbf{K}_R + P_{zR}\mathbf{k}_z\otimes\frac{\partial \mathbf{K}_R}{\partial R}\right)\mathbf{K}_R$$
$$+\left(\frac{\partial P_{z\Theta}}{\partial R}\mathbf{k}_z\otimes\mathbf{K}_\Theta + P_{z\Theta}\frac{\partial \mathbf{k}_z}{\partial R}\otimes\mathbf{K}_\Theta + P_{z\Theta}\mathbf{k}_z\otimes\frac{\partial \mathbf{K}_\Theta}{\partial R}\right)\mathbf{K}_R$$
$$+\left(\frac{\partial P_{zZ}}{\partial R}\mathbf{k}_z\otimes\mathbf{K}_Z + P_{zZ}\frac{\partial \mathbf{k}_z}{\partial R}\otimes\mathbf{K}_Z + P_{zZ}\mathbf{k}_z\otimes\frac{\partial \mathbf{K}_Z}{\partial R}\right)\mathbf{K}_R \tag{A30}$$

With account of orthonormality of the base vectors we have

$$\frac{\partial \mathbf{P}}{\partial R}\mathbf{K}_R = \frac{\partial P_{rR}}{\partial R}\mathbf{k}_r + P_{rR}\frac{\partial \mathbf{k}_r}{\partial R} + \frac{\partial P_{\theta R}}{\partial R}\mathbf{k}_\theta + P_{\theta R}\frac{\partial \mathbf{k}_\theta}{\partial R} + \frac{\partial P_{zR}}{\partial R}\mathbf{k}_z + P_{zR}\frac{\partial \mathbf{k}_z}{\partial R}. \tag{A31}$$

Differentiating the Eulerian basis, we get

$$\begin{aligned}
\frac{\partial \mathbf{k}_r}{\partial R} &= \frac{\partial \mathbf{k}_r}{\partial r}\frac{\partial r}{\partial R} + \frac{\partial \mathbf{k}_r}{\partial \theta}\frac{\partial \theta}{\partial R} + \frac{\partial \mathbf{k}_r}{\partial z}\frac{\partial z}{\partial R} = \frac{\partial \theta}{\partial R}\mathbf{k}_\theta\\
\frac{\partial \mathbf{k}_\theta}{\partial R} &= \frac{\partial \mathbf{k}_\theta}{\partial r}\frac{\partial r}{\partial R} + \frac{\partial \mathbf{k}_\theta}{\partial \theta}\frac{\partial \theta}{\partial R} + \frac{\partial \mathbf{k}_\theta}{\partial z}\frac{\partial z}{\partial R} = -\frac{\partial \theta}{\partial R}\mathbf{k}_r\\
\frac{\partial \mathbf{k}_z}{\partial R} &= \frac{\partial \mathbf{k}_z}{\partial r}\frac{\partial r}{\partial R} + \frac{\partial \mathbf{k}_z}{\partial \theta}\frac{\partial \theta}{\partial R} + \frac{\partial \mathbf{k}_z}{\partial z}\frac{\partial z}{\partial R} = \mathbf{0}.
\end{aligned} \tag{A32}$$

Now, substituting Eq.(A4) in Eq.(A3) we have

$$\frac{\partial \mathbf{P}}{\partial R}\mathbf{K}_R = \left(\frac{\partial P_{rR}}{\partial R} - P_{\theta R}\frac{\partial \theta}{\partial R}\right)\mathbf{k}_r + \left(P_{rR}\frac{\partial \theta}{\partial R} + \frac{\partial P_{\theta R}}{\partial R}\right)\mathbf{k}_\theta + \frac{\partial P_{zR}}{\partial R}\mathbf{k}_z. \tag{A33}$$

Analogously to (A2)-(A5) we calculate the last two terms on the right-hand side of Eq.(A1)

$$\frac{\partial \mathbf{P}}{R\partial \Theta}\mathbf{K}_\Theta$$
$$= \left(\frac{\partial P_{rR}}{R\partial \Theta}\mathbf{k}_r\otimes\mathbf{K}_R + P_{rR}\frac{\partial \mathbf{k}_r}{R\partial \Theta}\otimes\mathbf{K}_R + P_{rR}\mathbf{k}_r\otimes\frac{\partial \mathbf{K}_R}{R\partial \Theta}\right)\mathbf{K}_\Theta$$
$$+\left(\frac{\partial P_{r\Theta}}{R\partial \Theta}\mathbf{k}_r\otimes\mathbf{K}_\Theta + P_{r\Theta}\frac{\partial \mathbf{k}_r}{R\partial \Theta}\otimes\mathbf{K}_\Theta + P_{r\Theta}\mathbf{k}_r\otimes\frac{\partial \mathbf{K}_\Theta}{R\partial \Theta}\right)\mathbf{K}_\Theta$$
$$+\left(\frac{\partial P_{rZ}}{R\partial \Theta}\mathbf{k}_r\otimes\mathbf{K}_Z + P_{rZ}\frac{\partial \mathbf{k}_r}{R\partial \Theta}\otimes\mathbf{K}_Z + P_{rZ}\mathbf{k}_r\otimes\frac{\partial \mathbf{K}_Z}{R\partial \Theta}\right)\mathbf{K}_\Theta$$
$$+\left(\frac{\partial P_{\theta R}}{R\partial \Theta}\mathbf{k}_\theta\otimes\mathbf{K}_R + P_{\theta R}\frac{\partial \mathbf{k}_\theta}{R\partial \Theta}\otimes\mathbf{K}_R + P_{\theta R}\mathbf{k}_\theta\otimes\frac{\partial \mathbf{K}_R}{R\partial \Theta}\right)\mathbf{K}_\Theta$$
$$+\left(\frac{\partial P_{\theta\Theta}}{R\partial \Theta}\mathbf{k}_\theta\otimes\mathbf{K}_\Theta + P_{\theta\Theta}\frac{\partial \mathbf{k}_\theta}{R\partial \Theta}\otimes\mathbf{K}_\Theta + P_{\theta\Theta}\mathbf{k}_\theta\otimes\frac{\partial \mathbf{K}_\Theta}{R\partial \Theta}\right)\mathbf{K}_\Theta$$
$$+\left(\frac{\partial P_{\theta Z}}{R\partial \Theta}\mathbf{k}_\theta\otimes\mathbf{K}_Z + P_{\theta Z}\frac{\partial \mathbf{k}_\theta}{R\partial \Theta}\otimes\mathbf{K}_Z + P_{\theta Z}\mathbf{k}_\theta\otimes\frac{\partial \mathbf{K}_Z}{R\partial \Theta}\right)\mathbf{K}_\Theta$$
$$+\left(\frac{\partial P_{zR}}{R\partial \Theta}\mathbf{k}_z\otimes\mathbf{K}_R + P_{zR}\frac{\partial \mathbf{k}_z}{R\partial \Theta}\otimes\mathbf{K}_R + P_{zR}\mathbf{k}_z\otimes\frac{\partial \mathbf{K}_R}{R\partial \Theta}\right)\mathbf{K}_\Theta$$
$$+\left(\frac{\partial P_{z\Theta}}{R\partial \Theta}\mathbf{k}_z\otimes\mathbf{K}_\Theta + P_{z\Theta}\frac{\partial \mathbf{k}_z}{R\partial \Theta}\otimes\mathbf{K}_\Theta + P_{z\Theta}\mathbf{k}_z\otimes\frac{\partial \mathbf{K}_\Theta}{R\partial \Theta}\right)\mathbf{K}_\Theta$$
$$+\left(\frac{\partial P_{zZ}}{R\partial \Theta}\mathbf{k}_z\otimes\mathbf{K}_Z + P_{zZ}\frac{\partial \mathbf{k}_z}{R\partial \Theta}\otimes\mathbf{K}_Z + P_{zZ}\mathbf{k}_z\otimes\frac{\partial \mathbf{K}_Z}{R\partial \Theta}\right)\mathbf{K}_\Theta \tag{A34}$$

$$\begin{aligned}
\frac{\partial \mathbf{P}}{R\partial \Theta}\mathbf{K}_\Theta = {} & \frac{P_{rR}}{R}\mathbf{k}_r + \frac{\partial P_{r\Theta}}{R\partial \Theta}\mathbf{k}_r + P_{r\Theta}\frac{\partial \mathbf{k}_r}{R\partial \Theta} + \frac{P_{\theta R}}{R}\mathbf{k}_\theta\\
&+ \frac{\partial P_{\theta\Theta}}{R\partial \Theta}\mathbf{k}_\theta + P_{\theta\Theta}\frac{\partial \mathbf{k}_\theta}{R\partial \Theta} + \frac{P_{zR}}{R}\mathbf{k}_z\\
&+ \frac{\partial P_{z\Theta}}{R\partial \Theta}\mathbf{k}_z + P_{z\Theta}\frac{\partial \mathbf{k}_z}{R\partial \Theta}
\end{aligned} \tag{A35}$$

$$\begin{aligned}
\frac{\partial \mathbf{k}_r}{\partial \Theta} &= \frac{\partial \mathbf{k}_r}{\partial r}\frac{\partial r}{\partial \Theta} + \frac{\partial \mathbf{k}_r}{\partial \theta}\frac{\partial \theta}{\partial \Theta} + \frac{\partial \mathbf{k}_r}{\partial z}\frac{\partial z}{\partial \Theta} = \frac{\partial \theta}{\partial \Theta}\mathbf{k}_\theta\\
\frac{\partial \mathbf{k}_\theta}{\partial \Theta} &= \frac{\partial \mathbf{k}_\theta}{\partial r}\frac{\partial r}{\partial \Theta} + \frac{\partial \mathbf{k}_\theta}{\partial \theta}\frac{\partial \theta}{\partial \Theta} + \frac{\partial \mathbf{k}_\theta}{\partial z}\frac{\partial z}{\partial \Theta} = -\frac{\partial \theta}{\partial \Theta}\mathbf{k}_r.\\
\frac{\partial \mathbf{k}_z}{\partial \Theta} &= \frac{\partial \mathbf{k}_z}{\partial r}\frac{\partial r}{\partial \Theta} + \frac{\partial \mathbf{k}_z}{\partial \theta}\frac{\partial \theta}{\partial \Theta} + \frac{\partial \mathbf{k}_z}{\partial z}\frac{\partial z}{\partial \Theta} = \mathbf{0}
\end{aligned} \tag{A36}$$

$$\begin{aligned}
\frac{\partial \mathbf{P}}{R\partial \Theta}\mathbf{K}_\Theta = {} & \left(\frac{P_{rR}}{R} + \frac{\partial P_{r\Theta}}{R\partial \Theta} - \frac{P_{\theta\Theta}}{R}\frac{\partial \theta}{\partial \Theta}\right)\mathbf{k}_r\\
&+ \left(\frac{P_{r\Theta}}{R}\frac{\partial \theta}{\partial \Theta} + \frac{P_{\theta R}}{R} + \frac{\partial P_{\theta\Theta}}{R\partial \Theta}\right)\mathbf{k}_\theta\\
&+ \left(\frac{P_{zR}}{R} + \frac{\partial P_{z\Theta}}{R\partial \Theta}\right)\mathbf{k}_z
\end{aligned} \tag{A37}$$

$$
\begin{aligned}
&\frac{\partial \mathbf{P}}{\partial Z}\mathbf{K}_Z \\
&= \left(\frac{\partial P_{rR}}{\partial Z}\mathbf{k}_r \otimes \mathbf{K}_R + P_{rR}\frac{\partial \mathbf{k}_r}{\partial Z} \otimes \mathbf{K}_R + P_{rR}\mathbf{k}_r \otimes \frac{\partial \mathbf{K}_R}{\partial Z}\right)\mathbf{K}_Z \\
&+ \left(\frac{\partial P_{r\Theta}}{\partial Z}\mathbf{k}_r \otimes \mathbf{K}_\Theta + P_{r\Theta}\frac{\partial \mathbf{k}_r}{\partial Z} \otimes \mathbf{K}_\Theta + P_{r\Theta}\mathbf{k}_r \otimes \frac{\partial \mathbf{K}_\Theta}{\partial Z}\right)\mathbf{K}_Z \\
&+ \left(\frac{\partial P_{rZ}}{\partial Z}\mathbf{k}_r \otimes \mathbf{K}_Z + P_{rZ}\frac{\partial \mathbf{k}_r}{\partial Z} \otimes \mathbf{K}_Z + P_{rZ}\mathbf{k}_r \otimes \frac{\partial \mathbf{K}_Z}{\partial Z}\right)\mathbf{K}_Z \\
&+ \left(\frac{\partial P_{\theta R}}{\partial Z}\mathbf{k}_\theta \otimes \mathbf{K}_R + P_{\theta R}\frac{\partial \mathbf{k}_\theta}{\partial Z} \otimes \mathbf{K}_R + P_{\theta R}\mathbf{k}_\theta \otimes \frac{\partial \mathbf{K}_R}{\partial Z}\right)\mathbf{K}_Z \\
&+ \left(\frac{\partial P_{\theta\Theta}}{\partial Z}\mathbf{k}_\theta \otimes \mathbf{K}_\Theta + P_{\theta\Theta}\frac{\partial \mathbf{k}_\theta}{\partial Z} \otimes \mathbf{K}_\Theta + P_{\theta\Theta}\mathbf{k}_\theta \otimes \frac{\partial \mathbf{K}_\Theta}{\partial Z}\right)\mathbf{K}_Z \\
&+ \left(\frac{\partial P_{\theta Z}}{\partial Z}\mathbf{k}_\theta \otimes \mathbf{K}_Z + P_{\theta Z}\frac{\partial \mathbf{k}_\theta}{\partial Z} \otimes \mathbf{K}_Z + P_{\theta Z}\mathbf{k}_\theta \otimes \frac{\partial \mathbf{K}_Z}{\partial Z}\right)\mathbf{K}_Z \\
&+ \left(\frac{\partial P_{zR}}{\partial Z}\mathbf{k}_z \otimes \mathbf{K}_R + P_{zR}\frac{\partial \mathbf{k}_z}{\partial Z} \otimes \mathbf{K}_R + P_{zR}\mathbf{k}_z \otimes \frac{\partial \mathbf{K}_R}{\partial Z}\right)\mathbf{K}_Z \\
&+ \left(\frac{\partial P_{z\Theta}}{\partial Z}\mathbf{k}_z \otimes \mathbf{K}_\Theta + P_{z\Theta}\frac{\partial \mathbf{k}_z}{\partial Z} \otimes \mathbf{K}_\Theta + P_{z\Theta}\mathbf{k}_z \otimes \frac{\partial \mathbf{K}_\Theta}{\partial Z}\right)\mathbf{K}_Z \\
&+ \left(\frac{\partial P_{zZ}}{\partial Z}\mathbf{k}_z \otimes \mathbf{K}_Z + P_{zZ}\frac{\partial \mathbf{k}_z}{\partial Z} \otimes \mathbf{K}_Z + P_{zZ}\mathbf{k}_z \otimes \frac{\partial \mathbf{K}_Z}{\partial Z}\right)\mathbf{K}_Z
\end{aligned}
\tag{A38}
$$

$$
\begin{aligned}
\frac{\partial \mathbf{P}}{\partial Z}\mathbf{K}_Z &= \frac{\partial P_{rZ}}{\partial Z}\mathbf{k}_r + P_{rZ}\frac{\partial \mathbf{k}_r}{\partial Z} + \frac{\partial P_{\theta Z}}{\partial Z}\mathbf{k}_\theta \\
&\quad + P_{\theta Z}\frac{\partial \mathbf{k}_\theta}{\partial Z} + \frac{\partial P_{zZ}}{\partial Z}\mathbf{k}_z + P_{zZ}\frac{\partial \mathbf{k}_z}{\partial Z}.
\end{aligned}
\tag{A39}
$$

$$
\begin{aligned}
\frac{\partial \mathbf{k}_r}{\partial Z} &= \frac{\partial \mathbf{k}_r}{\partial r}\frac{\partial r}{\partial Z} + \frac{\partial \mathbf{k}_r}{\partial \theta}\frac{\partial \theta}{\partial Z} + \frac{\partial \mathbf{k}_r}{\partial z}\frac{\partial z}{\partial Z} = \frac{\partial \theta}{\partial Z}\mathbf{k}_\theta, \\
\frac{\partial \mathbf{k}_\theta}{\partial Z} &= \frac{\partial \mathbf{k}_\theta}{\partial r}\frac{\partial r}{\partial Z} + \frac{\partial \mathbf{k}_\theta}{\partial \theta}\frac{\partial \theta}{\partial Z} + \frac{\partial \mathbf{k}_\theta}{\partial z}\frac{\partial z}{\partial Z} = -\frac{\partial \theta}{\partial Z}\mathbf{k}_r, \\
\frac{\partial \mathbf{k}_z}{\partial Z} &= \frac{\partial \mathbf{k}_z}{\partial r}\frac{\partial r}{\partial Z} + \frac{\partial \mathbf{k}_z}{\partial \theta}\frac{\partial \theta}{\partial Z} + \frac{\partial \mathbf{k}_z}{\partial z}\frac{\partial z}{\partial Z} = \mathbf{0}.
\end{aligned}
\tag{A40}
$$

$$
\begin{aligned}
\frac{\partial \mathbf{P}}{\partial Z}\mathbf{K}_Z &= \left(\frac{\partial P_{rZ}}{\partial Z} - P_{\theta Z}\frac{\partial \theta}{\partial Z}\right)\mathbf{k}_r \\
&\quad + \left(\frac{\partial P_{\theta Z}}{\partial Z} + P_{rZ}\frac{\partial \theta}{\partial Z}\right)\mathbf{k}_\theta + \frac{\partial P_{zZ}}{\partial Z}\mathbf{k}_z.
\end{aligned}
\tag{A41}
$$

Finally, substituting Eqs.(A5), (A9), and (A13) in Eq.(A1) we have

$$
\begin{aligned}
\mathrm{Div}\mathbf{P} = &\Big(\frac{\partial P_{rR}}{\partial R} - P_{\theta R}\frac{\partial \theta}{\partial R} + \frac{P_{rR}}{R} + \frac{\partial P_{r\Theta}}{R\partial \Theta} \\
&\quad - \frac{P_{\theta\Theta}}{R}\frac{\partial \theta}{\partial \Theta} + \frac{\partial P_{rZ}}{\partial Z} - P_{\theta Z}\frac{\partial \theta}{\partial Z}\Big)\mathbf{k}_r \\
&+ \Big(P_{rR}\frac{\partial \theta}{\partial R} + \frac{\partial P_{\theta R}}{\partial R} + \frac{P_{r\Theta}}{R}\frac{\partial \theta}{\partial \Theta} + \frac{P_{\theta R}}{R} \\
&\quad + \frac{\partial P_{\theta\Theta}}{R\partial \Theta} + \frac{\partial P_{\theta Z}}{\partial Z} + P_{rZ}\frac{\partial \theta}{\partial Z}\Big)\mathbf{k}_\theta \\
&+ \left(\frac{\partial P_{zR}}{\partial R} + \frac{P_{zR}}{R} + \frac{\partial P_{z\Theta}}{R\partial \Theta} + \frac{\partial P_{zZ}}{\partial Z}\right)\mathbf{k}_z
\end{aligned}
\tag{A42}
$$