 SID, vol.1, no.2, pp.107-120, 2005

Reliable Fracture Analysis of OF 2-D Crack Problems Using NI-MVCCI Technique

G.S. Palani [1], Nagesh R. Iyer[1] and B. Dattaguru [2]

Abstract: *A posteriori* error estimation and adaptive refinement technique for 2-D/3-D crack problems is the state-of-the-art. In this paper a new *a posteriori* error estimator based on strain energy release rate (SERR) or stress intensity factor (SIF) at the crack tip region has been proposed and used along with the stress based error estimator for reliable fracture analysis of 2-D crack problems. The proposed *a posteriori* error estimator is called the K-S error estimator. Further, h-adaptive mesh refinement strategy which can be used with K-S error estimator has been proposed for fracture analysis of 2-D crack problems. The performance of the proposed *a posteriori* error estimator and the h-adaptive refinement strategy have been demonstrated by employing 4-noded, 8-noded and 9-noded plane stress finite elements. The proposed error estimator together with the h-adaptive refinement strategy will facilitate automation of fracture analysis process to provide reliable solutions.

keyword: Finite element method; Fracture analysis; Error estimation; Adaptive refinements.

1 Introduction

The finite element method (FEM) offers solution to almost all structural analysis problems once a suitable formulation and computational model are adopted. FEM has been used ever since 1950 as an analytical tool for solving many of the problems related to solid and structural mechanics. The power and versatility of FEM are generally exploited by developing suitable software packages or by using commercially available software. In FEM, the primary goal is to determine how a structure and its components will respond to a given set of environmental conditions such as loads, boundaries, discontinuities, etc. The results of finite element analysis (FEA) can be used to understand the behavior of structure and can also be used to improve and optimize the structural design. This is based on the assumption that the structure is correctly modelled, the environmental conditions are properly defined and FEM software itself performs correctly. The quality of finite element (FE) formulations and FE mesh/idealization employed in FEA will have a direct effect on the solution time, cost, accuracy and reliability of the results. Further, FEA of any practical structure involves large amount of data to be prepared to represent the physical structure. This gives scope for errors in input and inefficient modelling. Therefore, efficient/appropriate use of any software based on FEM, commercial or otherwise, requires basic knowledge of the method and also about the software. As a result, the use of software has been mainly restricted to professionals, trained analysts and those who are knowledgeable in this field.

One of the major sources of error in FEA is from the discretization of the structure into relatively simple elements to represent the complex structure and its behaviour. Reliability of solutions obtained by using FEA depends on modelling of the structure, element formulations and the errors associated with the solution process. Despite the significant advances made on discretization methods, the selection of FE model for a particular problem is largely based on the intuition and experience gained from solving similar problems. The discretization error is the result of modelling a continuum with a computational model that has a finite number of degrees of freedom (DOF). The governing equation that is the equilibrium conditions can be satisfied only in the weak sense and at global level. Likewise, the imposed natural boundary conditions are not fulfilled in an exact manner. The stress solution is discontinuous across element boundaries. The obtained stresses at nodes are of lower quality than within the domain. To remove the element of uncertainty involved in modelling, research has been directed towards evolving methodologies for *a posteriori* error estimation and adaptive mesh refinements that can be used

[1] Structural Engineering Research Centre (SERC), CSIR Campus, Chennai, 600 113, India.
[2] Indian Institute of Science (IISc), Bangalore, 560 012, India

with FEM. These attempts are intended to provide automatic means to improve mesh design and consequently to offer reliable solution with least computational cost.

Attention has been focused since mid 1970s on developing procedures for error estimation and adaptive mesh refinements. An extensive literature survey on techniques for error estimation and adaptive refinements in FEA was presented by Iyer (1993), Palani et al. (1998) and Zienkiewicz and Taylor (2000). The superconvergent patch recovery (SPR) technique, proposed by Zienkiewicz and Zhu (1992), is effective and recovers superconvergent derivatives. It is also shown that if the recovery technique is superconvergent, that is, if the recovered derivatives are superconvergent, Zienkiewicz and Zhu error estimator will always be asymptotically exact in energy norm. The procedure is a least square fit of finite element solutions over a local patch of elements at pre-selected points, where the rate of convergence is higher than the global rate or at least more accurate. An enhancement of SPR technique, called superconvergent patch recovery with equilibrium (SPRE) [Wiberg and Abdulwahab (1993)] is achieved by adding residual of the equilibrium equation for the improved solution to the smoothing procedure. A similar post-processing technique that accounts for violations of the governing equations was developed by Blacker and Belytschko (1994). Numerical experiments revealed [Wiberg, Abdulwahab and Ziukas (1994)] that while recovering high quality derivatives at inner nodes in both SPR and SPRE techniques the quality of recovered solution relatively deteriorate at boundaries. This deterioration was remedied by switching to a new method that was devised to account for the imposed boundary conditions. This enhanced method proposed by Wiberg, Abdulwahab and Ziukas (1994) is called the superconvergent patch recovery technique incorporating equilibrium and boundary conditions (SPREB). SPREB is designed in such a way that the prescribed boundary conditions are satisfied in the sense of least squares. The technique is tested for linear problems and the quality of the recovered solutions has been increased considerably. Wiberg, Li and Abdulwahab (1996) and Wiberg and Li (1999) used the same idea for adaptive FEA of linear, plasticity and non-linear dynamic problems. Schlenpen and Ramm (2000) presented a local and global error estimation procedure for linear dynamic problems.

It is observed from the literature that work on development of error estimation and adaptive refinement procedures have been largely concerned with linear static and dynamic analysis. While considerable successes have been achieved on error estimation and adaptivity for linear static and dynamic problems, development concerning adaptive FEM for nonlinear, plasticity, coupled and fracture problems is far from complete and is very desirable. It is observed from the literature that the application of *a posteriori* error estimation and adaptive refinement techniques for fracture analysis of 2-D/3-D crack problems is the state-of-the-art [Meshii and Watanabe (2003) and Giner, Fhenmayor and Tarancon (2004)]. For the adaptive FEA of fracture analysis problems, most of the proposed procedures [Koenke, Harte, Kratzig, and Rosenstein (1998) and Min, Bass and Spradley (1994)] make use of the conventional stress based *a posteriori* error estimator [Zienkiewicz and Zhu (1992)]. In the recently proposed [Meshii and Watanabe (2003)] error estimator for fracture problems, a procedure evaluating an error index for SIF was presented. This procedure involves use of displacement extrapolation technique for SIF computation. Therefore, this error index has the limitation that the analytical function of the crack tip displacement is known, which may not be available for many of the practical problems. A critical review of the numerical methods for fracture analysis was presented by Aliabadi and Rooke (1991) and the recent developments in fatigue crack growth modelling was presented by Atluri (2005). Palani, Iyer and Dattaguru (2004) proposed a generalized technique, called numerically integrated modified virtual crack closure integral technique, for accurate computation of SIF. It is well known that the solutions obtained by using FEM are approximate. Despite the fact that NI-MVCCI technique is used for computation of SIF, the errors due to discretization and the choice of crack tip element size have significant influence on the reliability of SIF values. In order to refine the mesh adaptively for accurate evaluation of SIF, it is essential to estimate the errors involved in the computation of SIF and also in stresses. In the present study an *a posteriori* error estimator based on SERR/SIF at the crack tip region has been developed and used along with the stress based error estimator [Zienkiewicz and Zhu (1992)]. The proposed *a posteriori* error estimator is called the K-S error estimator. Further, h-adaptive mesh refinement strategy has been proposed for fracture analysis of 2-D crack problems. This will facilitate automation of fracture analysis process to provide reliable solutions.

2 Formulation of K-S error estimator

In the proposed *a posteriori* K-S error estimator for fracture analysis problems, a combination of SIF and stress based approach is adopted for computing the element and domain errors. For the elements meeting at the crack tip, which are the ones contributing to SERR, SIF based approach is proposed for computing K-error estimator. As the rest of the elements in the domain are still in the linear and static state, stress based error estimator [Zienkiewicz and Zhu (1992)] is used for computing the domain as well as element errors. SIF based error estimator and stress based error estimator together forms the K-S error estimator. It may be noted that K-error is used in conjunction with S-error during the solution. The procedure to compute the error estimators is presented in the following:

2.1 Proposed SIF based (K) error estimator

Consider a typical FE mesh at the crack tip as shown in Figure 1. SERR can be evaluated by multiplying the stress distribution along OA (ahead of crack tip) with the corresponding displacement distribution along OB (behind crack tip) and integrating this product over Δa. For evaluation of SERR for mode I crack (G_I) the stress distribution on the crack extension line OA can be expressed in terms of the nodal forces $F_{y,j}$, $F_{y,(j+1)}$, etc. acting at the nodes j, (j+1), etc. respectively. COD distribution along OB can be expressed in terms of the nodal values at j, (j-1), (j-1)', etc. SERR for mode I crack is derived by evaluating the energy required to close the crack over a length 'Δa' in terms of these nodal forces and displacements.

The corresponding SERR for mode I and II (G_I and G_{II}) can be expressed as

$$G_I = \underset{\Delta a \to 0}{Lt} \frac{1}{2\Delta a} \int_{\Delta a} \sigma_{yy}(\xi) U_y(\xi') dx \tag{1}$$

$$G_{II} = \underset{\Delta a \to 0}{Lt} \frac{1}{2\Delta a} \int_{\Delta a} \sigma_{xy}(\xi) U_x(\xi') dx \tag{2}$$

The integrals associated with the constants required for representing the stress distribution and the integrals in eqns (1) and (2) can be evaluated by Gauss numerical integration technique with the appropriate rule [Palani, Iyer and Dattaguru (2004)].

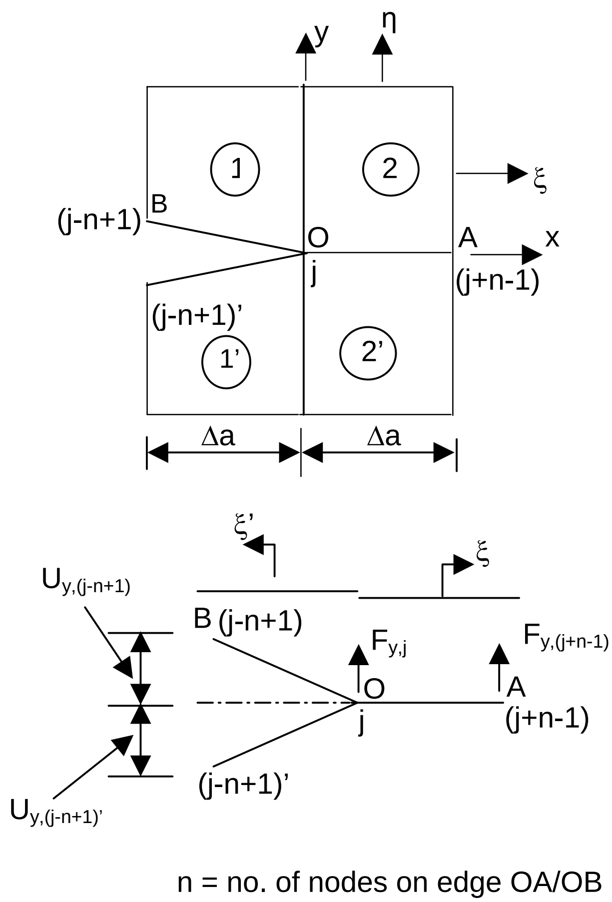


Figure 1 : Typical FE Mesh of Crack Tip Region

If the exact value of SERR for mode I crack (G_I) is known, the error in G_I computed using the results of FEA can be expressed as

$$G_{I,error} = G_{I,exact} - -G_I \tag{3}$$

The value of G_I can be updated with these errors $G_{I,error}$ to compute the improved value of G_I as

$$G_{I,impr} = G_I + G_{I,error} \tag{4}$$

By assuming plane stress or strain conditions, SIF values for mode I (K_I) for the current analysis and the true value of SIF for mode I ($K_{I,impr}$) can be computed using the values of G_I and $G_{I,impr}$ respectively. The error in SIF for mode I can then be expressed as

$$K_{I,error} = K_{I,impr} - -K_I \tag{5}$$

It may be noted SERR and SIF values in eqns (3) to (5) are scalar quantities. The predicted error indicator or relative percentage error in SIF can be expressed as

$$\eta_{K_I}(\%) = \frac{K_{I,error}}{K_{I,impr}} \times 100 \quad (6)$$

If the exact (analytical) values are available, the actual error indicator with respect to these values can be evaluated using

$$\eta_{aK_I}(\%) = \frac{K_{I,exact} - K_I}{K_{I,exact}} \times 100 \quad (7)$$

The reliability of the error estimator is measured by effectivity index [Zienkiewicz and Taylor (2000)], which is defined by the ratio of the predicted error indicator and the actual error indicator. The effectivity index for K-error estimator can be expressed as

$$\theta_{K_I} = \frac{\eta_{K_I}}{\eta_{aK_I}} \quad (8)$$

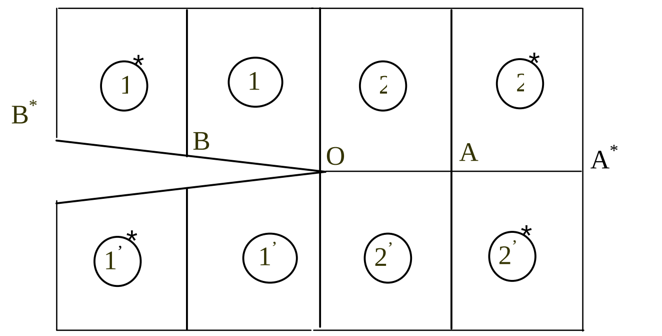
Similar expressions for mode II crack can be obtained by replacing G_I and K_I by G_{II} and K_{II}, respectively in eqns (3) to (8). Since the exact values of G_I are not known, the exact values ($G_{I,exact}$) are replaced by "improved" or "smoothed" values, which are supposed to be better than the computed values. The procedure for computing the "improved" values of G_I and G_{II} is explained below.

It is proposed to evaluate the improved values of SERR by assuming a polynomial of one order higher than that required for the finite elements employed at the crack tip for representing the stress distribution along OA as well as the corresponding displacement distribution along OB. In order to achieve this, referring to Fig. 2(a), one additional element (#2*) ahead of the crack tip (OAA*) and an additional element (#1*) behind crack tip (OBB*) is used for representing the stress and the displacement distribution respectively. The displacement variation along OBB* for computing improved SERR can be expressed as function of ξ' as

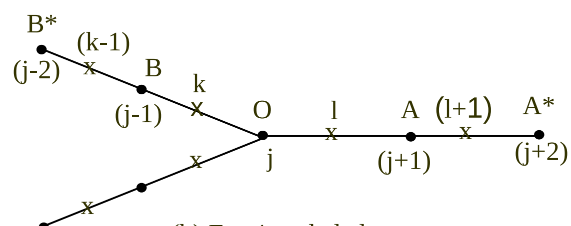
$$U_y(\xi')* = a_0 + a_1\xi' + ... + a_{(n-1)}\xi'^{(n-1)} n = 3,4,... \quad (9)$$

where $U_y(\xi')*$ is a polynomial of order (n-1). It may be noted that the value of n is assumed to be one order higher, for example n=3 for 4-noded elements and n=5 for 8-noded and 9-noded elements. Referring to Fig. 2(b) and (c), the constants a_0, a_1,....,$a_{(n-1)}$ can be evaluated by matching the displacement conditions at node B* and at the intermediate points k, (k-1), etc. in element numbers 1 and 1*. The values at intermediate points k, (k-1), etc. can be evaluated by interpolation using the respective element shape functions as

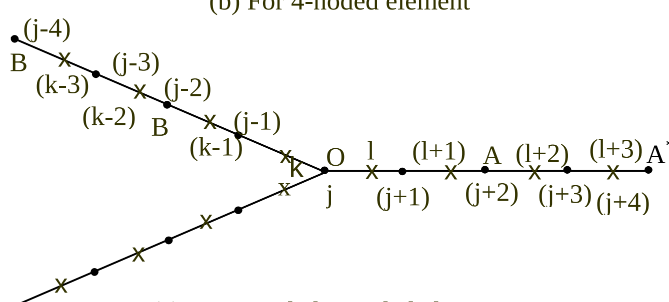
$$U_{y,k} = \sum N_i U_{y,i} \quad (10)$$



(a) FE mesh at crack tip



(b) For 4-noded element



(c) For 8-noded/9-noded element

Figure 2 : Details of FE Mesh at Crack Tip Region for Improved SERR Evaluation

The stress distribution along OA for computing improved SERR can be expressed as a function of ξ

$$\sigma_{yy}(\xi)^* = b_0 + b_1\xi + ...b_{(n-1)}\xi^{(n-1)} n = 3,4,... \quad (11)$$

where $\sigma_{yy}(\xi)^*$ is a polynomial of order (n-1). The value of n is assumed to be one order higher than the element shape functions similar to the case of displacements. Referring to Fig. 2, the constants b_0, b_1,....,$b_{(n-1)}$can be computed by matching the nodal forces at node O and the forces evaluated at the intermediate points l, (l+1), etc. with the derived consistent load vector from FE analysis. The values at intermediate points l, (l+1), etc. can be evaluated by interpolation using the respective element shape functions as

$$F_{y,k} = \sum N_i F_{y,i} \tag{12}$$

By replacing $\sigma_{yy}(\xi)$ and $U_y(\xi')$ in eqn (1) with $\sigma_{yy}(\xi)^*$ (eqn (9)) and $U_y(\xi')^*$ (eqn (11)), improved SERR for mode I cracks, $G_{I,impr}$ can be expressed as

$$G_{I,impr} = \underset{\Delta a \to 0}{Lt} \frac{1}{2\Delta a} \int_{\Delta a} \sigma_{yy}(\xi)^* U_y(\xi')^* dx \tag{13}$$

Similarly for mode II cracks,

$$G_{II,impr} = \underset{\Delta a \to 0}{Lt} \frac{1}{2\Delta a} \int_{\Delta a} \sigma_{xy}(\xi)^* U_x(\xi')^* dx \tag{14}$$

The integrals associated with the constants required for representing the stress distribution and the integrals in eqns (13) and (14) can be evaluated by Gauss numerical integration technique with the appropriate rule [Palani, Iyer and Dattaguru (2004)].

2.2 Stress based (S) error estimator [Zienkiewicz and Zhu (1987)]

The solution of a linear elastic problem consists of displacements and stresses. Considering the equilibrium equation for linear static problems based on principle of virtual work, the strain-displacement and stress-strain relations can be expressed as,

$$\varepsilon = B\delta \quad \text{and } \sigma = DB\delta \tag{15}$$

The displacements $\delta$and the stresses $\sigma$are approximate solutions and differ from the exact values (computed at Gauss points) as given below:

$$e = \delta^* - \delta \quad \text{for displacements} \tag{16a}$$

$$\text{and } es = \sigma^* - \sigma \quad \text{for stresses} \tag{16b}$$

The difference between the corresponding values given in the above equations form the pointwise errors in displacements and stresses respectively. However, since pointwise errors are difficult to compute, integral measures are conveniently adopted. Among these measures, the "energy norm" is the most commonly used. This can be expressed as,

$$\|e\| = \left[\int_{\Omega} e^T \, \Gamma \, e \, d\Omega \right]^{1/2} \tag{17}$$

where Γ is the self-adjoint operator in the linear equation and Ω is the domain. The predicted error indicator or relative percentage error can be defined as,

$$\eta_s\,(\%) = \frac{\|e\|}{\|u\|} \text{ x } 100 \tag{18}$$

where $\|u\|$ is the exact energy norm.

The following steps can be used to estimate the *a posteriori* errors in linear static FE solutions:

1. compute Gauss point stresses (σ)
2. compute smoothed/improved stresses, that is, σ to nodal points (σ_{nd})
3. compute projected stresses at the plate Gauss points (σ^*_{pnd}) using the appropriate element shape functions
4. compute the error estimate in energy norm in the domain using,

$$\|e\| = \left[\sum^{N} \int_{A} (\sigma^*_{nd} - \sigma) D^{-1} (\sigma^*_{nd} - \sigma)\, dA \right]^{1/2} \tag{19}$$

5. compute error indicator (η_s) for the domain and elements using eqn (18).

Since the exact values are not known, these are replaced by "improved" or "smoothed" values, which are better than the computed values. The "improved" values are obtained by using stress smoothing techniques such as global smoothing technique [Zienkiewicz and Zhu (1987)] and Zienkiewicz and Zhu (1989)], modified global smoothing technique [Iyer (1993)], nodal averaging approach [Byrd (1988)] and SPR technique [Zienkiewicz and Zhu (1992)]. Among these, the preferred technique used to determine the "improved" solution is the superconvergent patch recovery technique as described below:

3 SPR technique

In this approach a single and continuous polynomial expansion of the function describing the derivatives is used on an element patch surrounding the nodes at which recovery is required [Zienkiewicz and Zhu (1992)]. This expansion can be made to fit locally the superconvergent points in a least square manner or simply be an L_2 projection of the consistent FE derivatives. It is assumed that the nodal stress values σ^* belong to a polynomial expansion $\sigma_p{}^*$ of the same complete order p as that present in the shape function N and which is valid over an element patch surrounding the particular assembly node considered. Such a patch represents a union of elements containing this vertex node. This polynomial expansion will be used for each component of σ_p^* or the derivatives as expressed below,

$$\sigma_p^* = \mathbf{Pa} \tag{20}$$

where **P** contains the appropriate polynomial terms and is a set of unknown parameters. More details on the polynomial terms and 2-D element patches are available in Ref. 5. The equations for each vertex node is assembled and solved to determine the parameters 'a'. The recovered nodal values are computed by appropriately inserting the coordinates into the expression of $\sigma_p{}^*$. Zienkiewicz and Zhu (1992) proved that this approach generally leads to superconvergent recovery of nodal derivatives (or stresses).

4 h-Adaptive mesh refinement strategy

Adaptive FEA aims at achieving more accurate and reliable solutions with least computational effort by employing efficient modelling techniques. The domain error indicator is compared with the specified error indicator and the adaptive FEA iterative cycles are continued till the specified accuracy is achieved. Error indicators are computed for all elements in a domain to decide about further refinement of the element to improve the accuracy of solution. Five adaptive refinement strategies generally used [Zienkiewicz and Taylor (2000)] are (i) refinement of elements (h-refinements), (ii) increase in order of polynomial (p-refinements), (iii) relocation of nodes (R-refinements), (iv) simultaneous refinement of elements and increase in order of polynomial (h-p refinements) and (v) combination h-p-R refinements in sequence. Among these strategies h-refinements is popularly used as it ensures the convergence of solutions and is easy to implement in existing FEA software.

The present study aims at developing efficient methodologies for achieving an acceptable FE mesh in single level of adaptive mesh refinements for fracture analysis of 2-D crack problems. The domain error indicators, η_{KI} and η_s, as given by eqns (6) and (18) are compared with the specified error indicators $\overline{\eta}_{KI}$ and $\overline{\eta}_s$ respectively to check whether the specified accuracy has been achieved. It is proposed to compute the element error indicators for those elements meeting at the crack tip based on K-error estimator and for the rest of the elements in the domain based on S-error estimator. An adaptive mesh refinement strategy involving graded mesh guided by a new h-distribution has been used for obtaining the adaptive FE mesh in single level of refinements.

For the elements meeting at the crack tip, the element refinement parameter can be obtained by equally distributing $G_{I,error}$ among these elements.

$$K_{I,error} \text{ for an element, } (K_{I,error})_i = \frac{K_{I,error}}{N_k} \tag{21}$$

$$\text{element refinement parameter, } \xi_{ki} = \frac{(K_{I,error})_i}{\overline{\eta}_{kI}\frac{K_{I,impr}}{N_k}} \tag{22}$$

where N_k is the number of elements meeting at the crack tip, $K_{I,error}$ is error in SIF (eqn (5)), $K_{I,impr}$ is computed using $G_{I,impr}$ value as given in eqn (4) and $\overline{\eta}_{KI}$ is the user specified acceptable percentage error for SIF value.

The element error indicator and refinement parameter [Zienkiewicz and Taylor (2000)] based on stress based error estimator can be computed based on the principle of equal distribution of the errors among the elements in the domain as given by,

$$\text{element error indicator, } \eta_i = \frac{\|e\|_i}{\left(\frac{\|u\|^2+\|e\|^2}{N}\right)^{1/2}} \tag{23}$$

$$\text{element refinement parameter, } \xi_i = \frac{\|e\|_i}{\overline{\eta}_s\left(\frac{\|u\|^2+\|e\|^2}{N}\right)^{1/2}} \tag{24}$$

where N is the number of elements in the mesh excluding those elements considered for evaluating K-error estimator, $\|e\|$ is the energy norm for the error (eqn (19)) in the domain, $\|u\|$ is the energy norm computed using FE solution, $\|e\|_i$ is the energy norm for the error in element i and $\overline{\eta}_s$ is the user specified acceptable percentage error.

It is proposed to use the values of ξ_{ki} for the elements meeting at the crack tip (eqn (22)) and the values of ξ_i for the rest of the elements in the domain (eqn (24)) to decide on the refinement level required for the next level of adaptive mesh. If each of this parameter is greater than unity, then further refinement is carried out, if the parameters are equal to unity no further refinement is carried out and if the parameters are less than one then derefinement may be attempted. Based on the refinement parameter values ($\xi_{ki}$and ξ_i), it is proposed carry out the refinements simultaneously for the elements at the crack tip as well as the rest of the elements in the domain. The current level of FE mesh will be replaced with a refined unstructured graded mesh. The error indicator for the domain will be used to check whether the specified level of accuracy has been reached.

5 Numerical studies

In order to verify and validate the K-S error estimator and the h-adaptive mesh refinement strategy proposed above, fracture analysis of 2-D cracked plates (modes I and II) has been conducted by employing 4-noded, 8-noded and 9-noded plane stress finite elements. Static analysis of the plates has been conducted by using FEM and the stress based errors (S-errors) have been estimated as explained above. NI-MVCCI technique has been employed for computing SERR and SIF and SIF based errors (K-errors). Gauss integration technique with appropriate rules has been employed for evaluating the integrals associated with NI-MVCCI technique. Plane strain conditions have been assumed at the crack tip to compute SIF by using SERR values obtained using NI-MVCCI technique. Example problems on mode I and mode II cracks have been chosen to validate and study the performance of the K-S error estimator. In order to compare and perform the convergence studies of the error estimator and adaptive refinement strategies, both uniform and adaptive mesh refinements have been carried out. In all the example problems, four levels of uniform refinements for 4-noded elements and two levels of uniform refinements for 8-noded and 9-noded elements have been used to demonstrate the monotonic convergence of the error estimator. Single level of adaptive refinements have been carried out to generate unstructured graded mesh for 4-noded, 8-noded and 9-noded elements based on the element refinement parameters computed by assuming an accuracy of 5% ($\overline{\eta}_s$) for S-errors [Zienkiewicz and Taylor (2000)] and 1% ($\overline{\eta}_K$) for K-errors [Rooke and Cartwright (1976)], which is generally recommended for engineering applications.

Example-1: Rectangular Plate with Center Crack under Uniaxial Tension

A rectangular plate with center crack subjected to uniaxial tensile loading (mode I) as shown in Fig. 3(a) has been analysed to compute SERR and SIF at the crack tip. One quarter of the plate with symmetric boundary conditions has been idealized using 4-noded, 8-noded and 9-noded finite elements. The basic mesh employed for each of these elements is shown in Fig. 4. The meshes for the final level of uniform mesh refinement are shown in Fig. 5 and the graded adaptive meshes are shown in Fig. 6. The plot of the deformed shape superposed with σ_{xx} stress contour is shown in Fig. 7 for basic mesh, level IV of uniform mesh and adaptive mesh for 4-noded element. The convergence of K-errors obtained by using uniform and adaptive meshes is shown in Fig. 8. Table 1 presents SERR and SIF values and K-S domain error estimates obtained in the present study using 4-noded, 8-noded and 9-noded elements along with the finite plate solution for SIF available in the literature [Rooke and Cartwright (1976)].

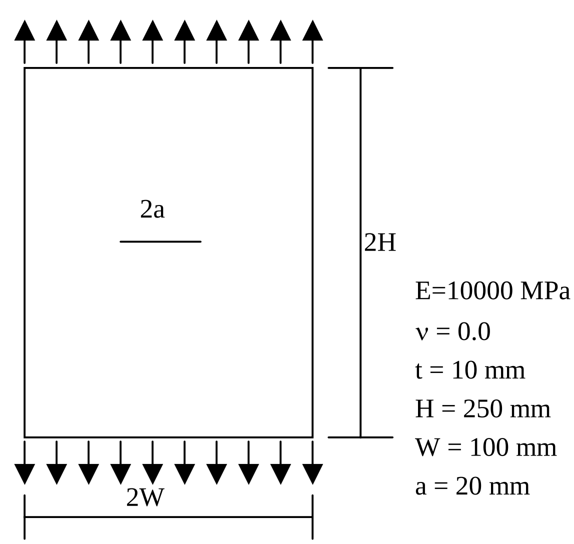
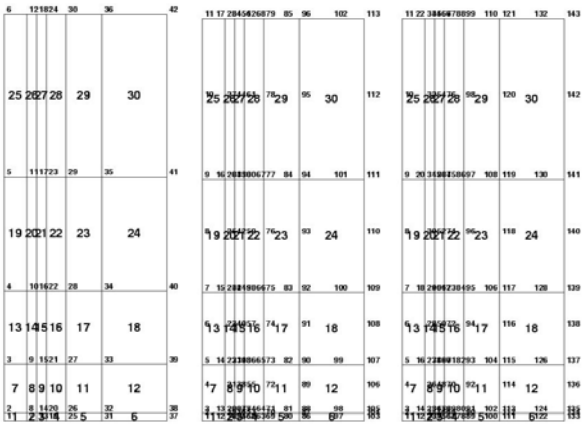
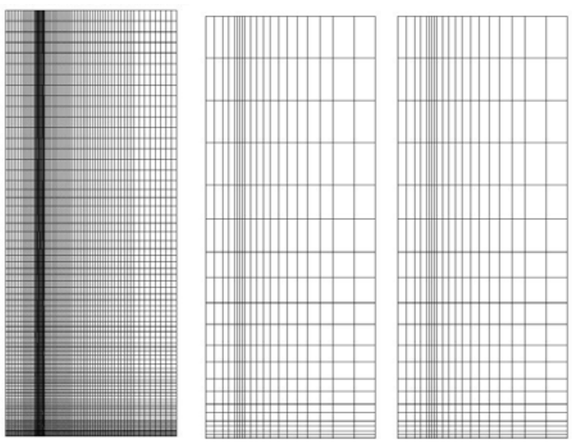


Figure 3 : Rectangular Plate with Center Crack



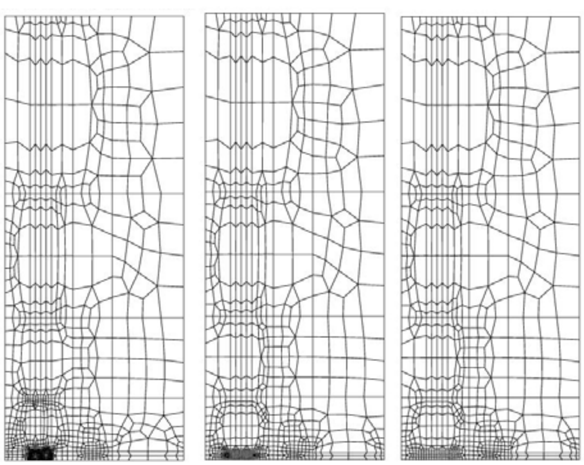
(a) 4-noded (b) 8-noded (c) 9-noded

Figure 4 : Basic Mesh Employed in the Studies



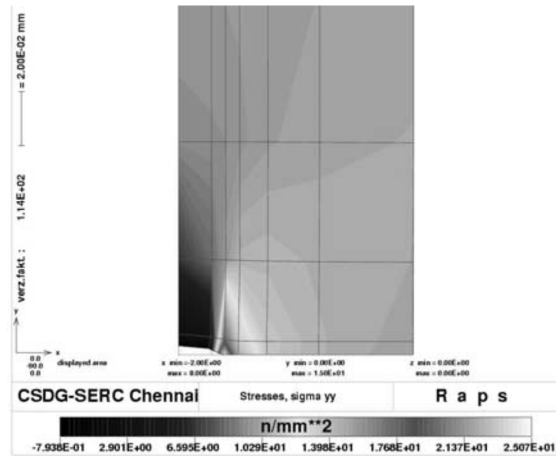
(a) 4-noded (b) 8-noded (c) 9-noded

Figure 5 : Final Level of Uniform Mesh Employed in the Studies

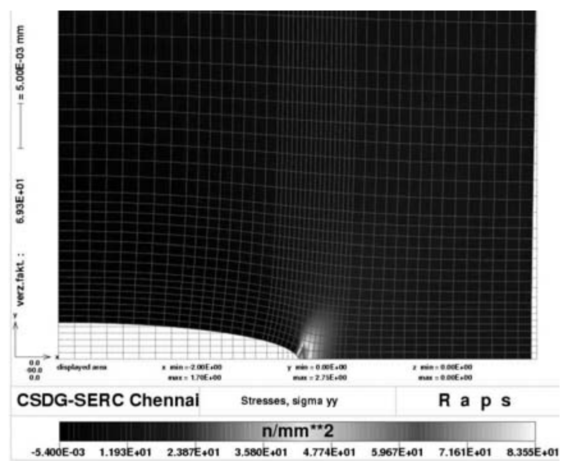


(a) 4-noded (b) 8-noded (c) 9-noded

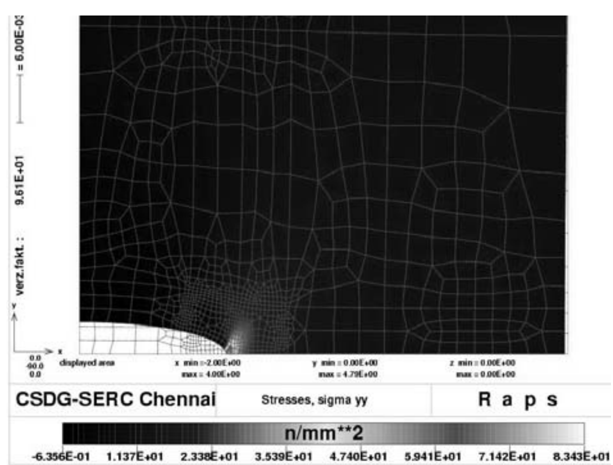
Figure 6 : Adaptive Mesh Employed in the Studies (Mode I – Center Crack)



(a) Basic Mesh



(b) Uniform Mesh (Lev. IV)



(c) Adaptive Mesh

Figure 7 : Stress Contour Superposed with Deformed Shape (Mode I – Center Crack)

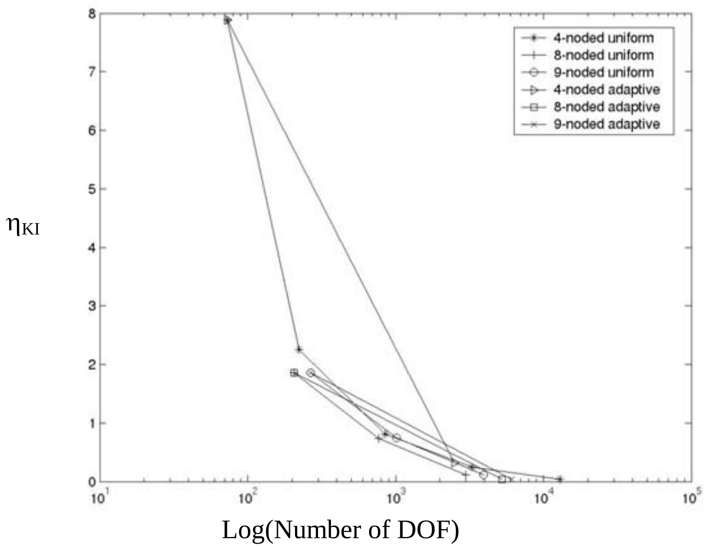
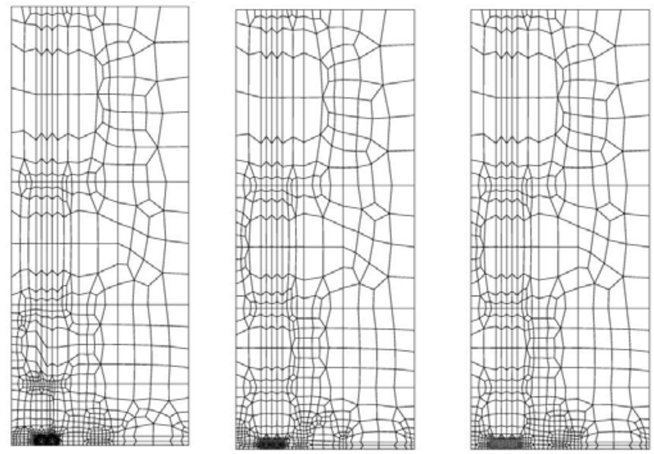


Figure 8 : Convergence of Uniform and Adaptive Meshes (Mode I - Centre Crack)



(a) 4-noded (b) 8-noded (c) 9-noded

Figure 10 : Adaptive Mesh Employed in the Studies (Mode I - Edge Crack)

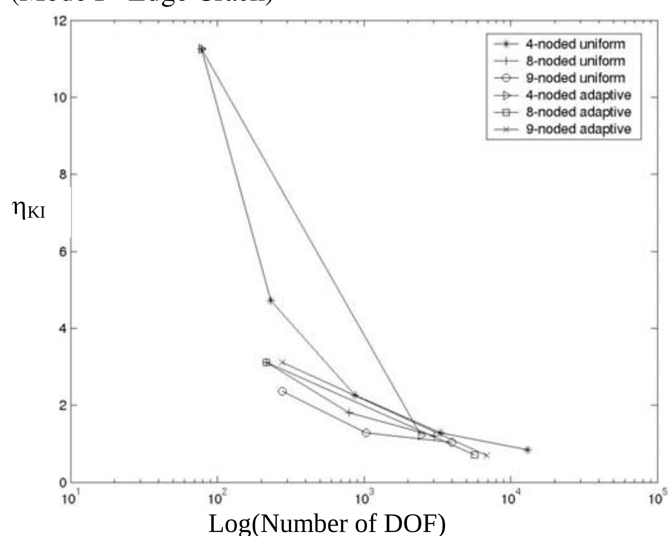


Figure 11 : Convergence of Uniform and Adaptive Meshes (Mode I - Edge Crack)

Example-2: Rectangular Plate with Edge Crack under Uniaxial Tension

A rectangular plate with an edge crack subjected to uniaxial tensile loading (mode I) as shown in Fig. 9 has been analysed to compute SERR and SIF at the crack tip. One half of the plate with appropriate changes for the boundary conditions has been idealized using 4-noded, 8-noded, and 9-noded plane stress finite elements. The basic and the uniform refinement meshes employed for each of these elements are same as those shown in Figs. 4 and 5. The graded adaptive meshes for these elements are shown in Fig. 10. The convergence of K-errors obtained by using uniform and adaptive meshes is shown in Fig. 11. Table 2 presents the details of FE mesh topology, SERR and SIF values and the K-S domain error estimates obtained in the present study using 4-noded, 8-noded and 9-noded elements along with the finite plate solution for SIF available in the literature [Rooke and Cartwright (1976)]. The plot of the deformed shape superposed with σ_{xx} stress contour is shown in Fig. 12 for basic mesh, level IV of uniform mesh and adaptive mesh

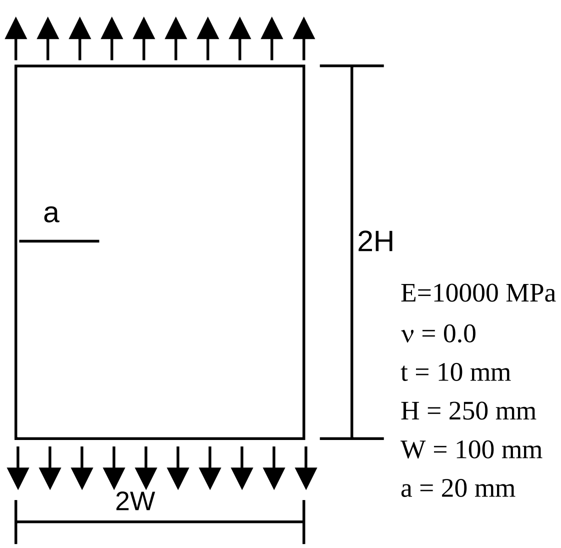
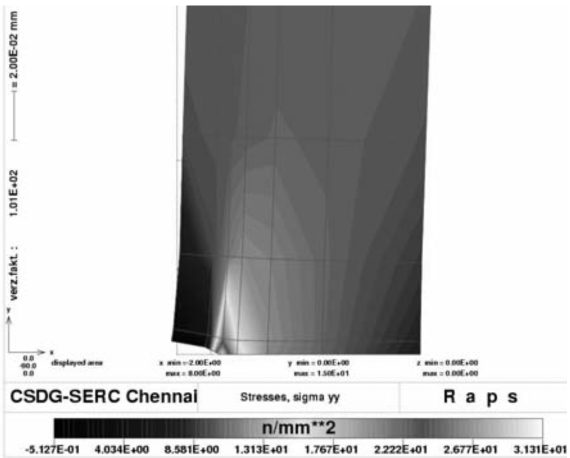
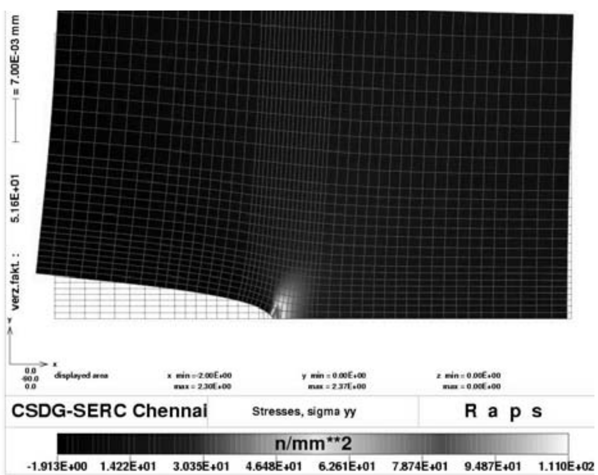


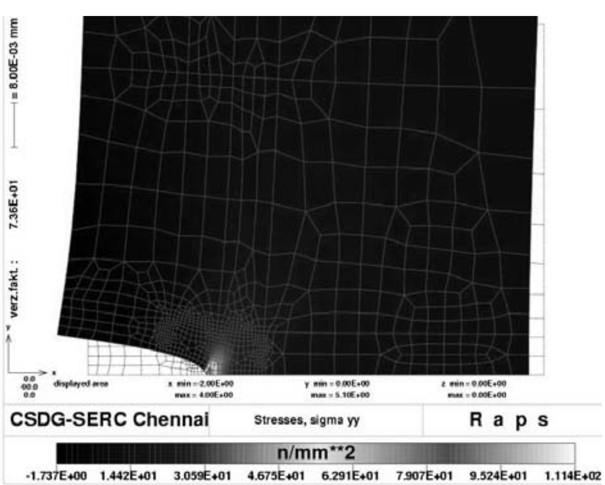
Figure 9 : Rectangular Plate with Edge Crack



(a) Basic Mesh

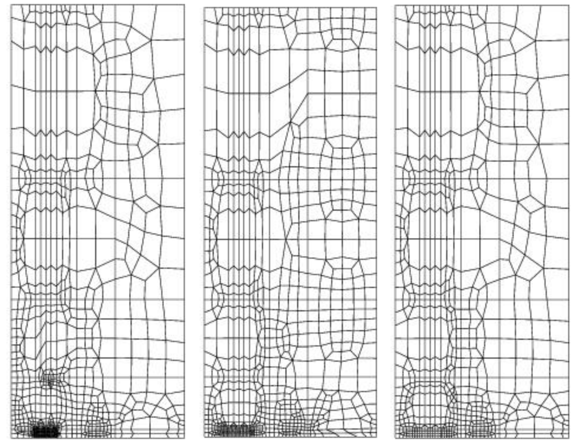


(b) Uniform Mesh (Lev. IV)



(c) Adaptive Mesh

Figure 12 : Stress Contour Superposed with Deformed Shape (Mode I - Edge Crack)



(a) 4-noded (b) 8-noded (c) 9-noded

Figure 13 : Adaptive Mesh Employed in the Studies (Mode II – Centre Crack)

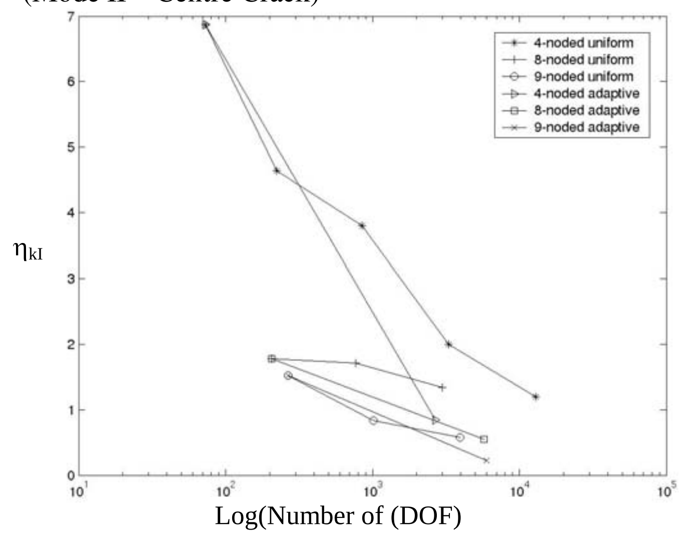


Figure 14 : Convergence of Uniform and Adaptive Meshes (Mode II - Centre Crack)

for 4-noded element.

Example-3: Rectangular Plate with Center Crack under Shear Load

A rectangular plate with a center crack subjected to shear load (mode II) has been analysed to compute SERR and SIF at the crack tip. The plate has been idealized considering quarter symmetry with using 4-noded, 8-noded, and 9-noded plane stress finite elements with appropriate changes for the loading and boundary conditions. The basic and the uniform refinement meshes employed for each of these elements are same as those shown in Figs. 4 and 5. The graded adaptive meshes for these elements

Table 1 : K-S Error Estimates for Rectangular Plate with Centre Crack (Mode I)

Mesh	#nodes	#DOF	G_1	K_I	K_I error	η_{kI} (%)	θ_{kI}	η_s (%)
4-noded element								
Basic	42	73	0.0540	23.24	2.0191	7.89	0.859	7.19
Uni. lev. I	121	223	0.0621	24.93	0.5758	2.25	0.872	5.00
Uni. lev. II	441	846	0.0643	25.36	0.2073	0.81	0.904	3.06
Uni. lev. III	1681	3292	0.0651	25.52	0.0640	0.25	0.910	2.00
Unif. lev. IV	6561	12984	0.0655	25.60	0.0102	0.04	0.911	1.36
Adaptive	1285	2498	0.0650	25.50	0.0819	0.32	0.920	1.64
8-noded element								
Basic	113	206	0.0627	25.04	0.4762	1.86	0.864	5.48
Uni. lev. I	405	772	0.0644	25.38	0.1894	0.74	0.907	4.21
Uni. lev. II	1529	2984	0.0653	25.56	0.0281	0.11	0.915	3.09
Adaptive	2686	5240	0.0655	25.60	0.0102	0.04	0.917	2.58
9-noded element								
Basic	143	266	0.0634	25.18	0.4760	1.86	0.867	5.65
Uni. lev. I	525	1012	0.0650	25.49	0.1919	0.75	0.911	4.59
Uni. lev. II	2009	3944	0.0654	25.57	0.0281	0.11	0.919	3.88
Adaptive	3111	6094	0.0655	25.60	0.0102	0.04	0.927	3.37

Analytical K_I: 25.59 [Rooke and Cartwright (1976)]

Table 2 : K-S Error Estimates for Rectangular Plate with Edge Crack (Mode I)

Mesh	#nodes	#DOF	G_1	K_I	K_I error	η_{kI} (%)	θ_{kI}	η_s (%)
4-noded element								
Basic	42	78	0.0885	29.75	3.8723	11.25	0.829	7.19
Uni. lev. I	121	233	0.1057	32.52	1.6246	4.72	0.855	5.00
Uni. lev. II	441	866	0.1125	33.53	0.7779	2.26	0.874	3.06
Uni. lev. III	1681	3332	0.1151	33.92	0.4440	1.29	0.891	2.00
Unif. lev. IV	6561	13064	0.1163	34.10	0.2891	0.84	0.901	1.36
Adaptive	1248	2461	0.1153	33.96	0.4199	1.22	0.912	1.64
8-noded element								
Basic	113	217	0.1097	33.12	1.0739	3.12	0.826	5.48
Uni. lev. I	405	793	0.1137	33.72	0.6196	1.80	0.887	4.21
Uni. lev. II	1529	3025	0.1153	33.96	0.4130	1.20	0.895	3.09
Adaptive	2866	5671	0.1196	34.15	0.2444	0.71	0.907	2.58
9-noded element								
Basic	143	277	0.1118	33.44	0.8123	2.36	0.829	5.65
Uni. lev. I	525	1033	0.1151	33.92	0.4440	1.29	0.891	4.59
Uni. lev. II	2009	3985	0.1158	34.03	0.3545	1.03	0.909	3.88
Adaptive	3461	6857	0.1167	34.16	0.2409	0.70	0.917	3.37

Analytical K_I: 34.42 [Rooke and Cartwright (1976)]

Table 3 : K-S Error Estimates for Rectangular Plate with Centre Crack (Mode II)

Mesh	#nodes	#DOF	G_2	K_{II}	K_{II} error	η_{kII} (%)	θ_{kII}	η_s (%)
4-noded element								
Basic	42	73	0.0527	22.95	1.7223	6.87	0.812	5.99
Uni. lev. I	121	223	0.0562	23.71	1.1632	4.64	0.857	4.87
Uni. lev. II	441	846	0.0575	23.98	0.9527	3.80	0.874	3.26
Uni. lev. III	1681	3292	0.0599	24.48	0.5014	2.00	0.851	1.94
Unif. lev. IV	6561	12984	0.0612	24.74	0.3008	1.20	0.906	1.44
Adaptive	1374	2671	0.0617	24.84	0.2106	0.84	0.912	1.26
8-noded element								
Basic	113	206	0.0602	24.54	0.4462	1.78	0.844	5.26
Uni. lev. I	405	772	0.0605	24.59	0.4287	1.71	0.897	4.14
Uni. lev. II	1529	2984	0.061	24.70	0.3359	1.34	0.905	3.03
Adaptive	2937	5742	0.0621	24.92	0.1379	0.55	0.912	2.15
9-noded element								
Basic	143	266	0.0606	24.62	0.3811	1.52	0.848	5.57
Uni. lev. I	525	1012	0.0617	24.84	0.2081	0.83	0.898	4.19
Uni. lev. II	2009	3944	0.0621	24.91	0.1454	0.58	0.909	3.08
Adaptive	3041	5958	0.0626	25.01	0.0552	0.22	0.917	2.37

Analytical K_{II}: 25.07 -infinite plate solution.

are shown in Fig. 13. The convergence of K-errors obtained by using uniform and adaptive meshes is shown in Fig. 12. Table 3 presents the details of FE mesh topology, SERR and SIF values and the K-S domain error estimates obtained in the present study using 4-noded, 8-noded and 9-noded elements along with the solutions for an infinite plate.

6 Discussion of results

It can be observed from Figs. 7, 10 and 12 and Tables 1 to 4 that K-error as well as S-error computed in the present study using the proposed K-S error estimator for 4-noded, 8-noded and 9-noded quadrilateral finite elements exhibit monotonic convergence for the all the example problems solved. The S-error conveges to less than 5% error for all the example problems both for the uniform and adaptive meshes. The K-error converges to less than about 1% for all the example problems. The effectivity index, θ_k computed for K-error estimator with respect to the corresponding analytical solution is generally found to be converging from around 0.8 to 0.9 for uniform as well as adaptive meshes obtained by employing 4-noded, 8-noded and 9-noded quadrilateral finite elements. As already pointed out the effectivity index is a measure of the reliability of the error estimator [Zienkiewicz and Taylor (2000)]. The error estimator is asymptotically correct if this index converges to unity when the errors converge to zero. It is observed that the K- and S-errors for the graded adaptive meshes obtained in single level of refinements are within the specified limits (5% for $\overline{\eta}_s$ and 1% for $\overline{\eta}_K$). It is observed from Figs. 6, 8 and 11 that in general the adaptive graded meshes are more refined in the region around crack tip and in the other regions the mesh pattern remains almost the same for all the example problems. The number of DOFs for these meshes are higher than the uniform meshes especially for 8-noded and 9-noded elements. It may be noted that the solution for the adaptive meshes has been obtained in single level of refinements.

7 Summary and conclusions

A new *a posteriori* K-S error estimator and h-adaptive refinement strategy has been proposed for fracture analysis of 2-D crack problems. SERR and SIF have been computed by using NI-MVCCI technique.The error estimates are computed as a post-processing approach to FEA and are based on SIF and stresses. For the elements meeting at the crack tip, which are the ones contributing to SERR, SIF based approach is proposed for computing K-error estimator. As rest of the elements in the domain are still in the linear and static state, stress based error estimator is used for computing the domain as well as element errors for these elements. SIF based error estimator and stress based error estimator together forms the K-S error estimator. It may be noted that K-error is used in conjunction with S-error during the solution. The efficacy of the error estimator and the adaptive refinement strategy has been demonstrated for 4-noded bilinear, 8-noded Serendipity and 9-noded Lagrangian isoparametric finite elements. Based on the numerical studies conducted on cracked plates the following conclusions are drawn:

1. The proposed *a posteriori* K-S error estimator exhibit monotonic convergence when 4-noded, 8-noded and 9-noded quadrilateral finite elements are employed.

2. The effectivity index, θ_k computed for K-error estimator with respect to the corresponding analytical solution is generally found to be converging from around 0.8 to 0.9 for uniform as well as adaptive meshes obtained by employing 4-noded, 8-noded and 9-noded quadrilateral finite elements. This demonstrates the reliability of the proposed K-error estimator.

3. In general, S-error converges to less than 5% and K-error converges to less than about 1% both for uniform and adaptive meshes.

4. The adaptive graded meshes are more refined in the region around crack tip and in the other regions the mesh pattern remains almost the same.

5. The proposed K-S error estimator and the adaptive refinement strategy can be easily implemented in to any existing FEA software as a post-processing module. This will facilitate automation of fracture analysis process with reliable estimates of SERR and SIF.

Acknowledgement: Thanks are due to Shri J. Rajasankar, and Shri A. Rama Chandra Murthy, Scientists, SERC, Chennai, for the useful discussions and suggestions during the course of the investigations. This paper is being published with the kind permission of the Director, SERC, Chennai.

References

Aliabadi,M.H.; Rooke,D.P. (1991): *Numerical Fracture Mechanics*, Kluwer Academic Publishers.

Atluri, S.N. (2005): Applications of DTALE: damage Tolerance Analysis and Life Enhancement [3-D Non-plannar Fatigue Crack Growth], *Structural Integrity and Durability*, vol 1, pp. 1-20.

Blacker, T.; Belytschko, T. (1994): Superconvergent patch recovery with equilibrium and conjoint enhancements, *Intl. Jl. Num. Meth. Engg.*, vol. 37, pp. 517-536.

Byrd, D.E. (1988): *Identification and elimination of errors in finite element analysis*, *Ph. D. Thesis*, Univ. of Colorado at Boulder, USA.

Iyer, Nagesh R. (1993): *Error estimation and adaptive refinements for finite element analysis of structures*, *Ph. D. Thesis*, Indian Institute of Science, Bangalore, India.

Giner, E.; Fhenmayor, F.J.; Tarancon, J.E. (2004): An improvement of the EDI method in linear elastic fracture mechanics by means of an *a posteriori* error estimator in G, *Intl. Jl. Num. Meth. Engg.*, vol. 59, pp. 533-558.

Koenke, C.; Harte, R.; Kratzig, W.B.; Rosenstein, O. (1998): On adaptive remeshing techniques for crack simulation problems, *Engg. Computat.*, vol. 15, pp. 74-88.

Meshii, T.; Watanabe, K. (2003): Stress intensity factor error index for finite element analysis with singular elements, *Engg. Fract. Mech.*, vol. 70, pp. 257-669.

Min, J.B.; Bass, J.M.; Spradley, L.W. (1994): Adaptive finite element methods for two-dimensional problems in computational fracture mechanics, *Comput. Struct.*, vol. 50, pp. 433-445.

Palani, G.S.; Iyer, Nagesh R.; Appa Rao, T.V.S.R. (1998): *A posteriori error estimation for static and dynamic analysis of unstiffened and stiffened plate/shells*, Report No. SST-RR-98-04, SERC, Chennai.

Palani, G.S.; Iyer, Nagesh R.; Dattaguru, B. (2004): Numerically Integrated Modified Virtual Crack Closure Integral technique for 2-D crack problems. *Intl. Jl. Struct. Engg. Mech.*, 18, pp. 731-744.

Rooke, D.P.; Cartwright, D.J. (1976): *Compendium of stress intensity factors*, Her Majesty's Stationery Office, London.

Schlenpen, A.; Ramm, E. (2000): Local and global error estimations in linear structural dynamics, *Comput. Struct.*, vol. 76, pp. 741-756.

Wiberg, N.E.; Abdulwahab, F. (1993): Patch recovery based on superconvergent derivatives and equilibrium, *Intl. Jl. Num. Meth. Engg.*, vol. 36, pp. 2703-2724.

Wiberg, N.E.; Abdulwahab, F.; Ziukas, S. (1994): Enhanced superconvergent patch recovery and incorporating equilibrium and boundary conditions, *Intl. Jl. Num. Meth. Engg.*, vol. 37, pp. 3417-3340.

Wiberg, N.E.; Li, X.D. (1999): Adaptive finite element procedures for linear and non-linear dynamics, *Intl. Jl. Num. Meth. Engg.*, vol. 46, pp. 1781-1802.

Wiberg, N.E.; Li, X.D.; Abdulwahab, F. (1996): Adaptive finite element procedures in elasticity and plasticity, *Engg. with Computat.*, vol. 12, pp. 120-141.

Zienkiewicz, O.C.; Zhu, J.Z. (1987): A simple error estimator and adaptive procedure for practical engineering analysis, *Intl. Jl. Num. Meth. Engg.*, vol. 24, pp. 337-357.

Zienkiewicz, O.C.; Zhu, J.Z. (1989): Error estimates and adaptive refinement for plate bending problems, *Intl. Jl. Num. Meth. Engg.*, vol. 26, pp. 2839-2854.

Zienkiewicz, O.C.; Taylor, R.L. (2000): *The Finite element method, Vol. I: The basis, Vol. II: Solid mechanics*, Butterworth-Hienemann Ltd.

Zienkiewicz, O.C.; Zhu, J.Z. (1992): The superconvergent patch recovery and *a posteriori* error estimators. Part I: The recovery technique; Part II: Error estimates and adaptivity, *Intl. Jl. Num. Meth. Engg.,* vol. 33, pp. 1131-1382.