On Fatigue Damage Computation in Random Loadings with Threshold Level and Mean Value Influence

D. Benasciutti[1] **and R. Tovo**[1]

Abstract: The probability density functions used to characterize the distribution of fatigue cycles in random loads are usually defined over an infinite domain. This means that they give a non-zero probability to count cycles with an infinitely large peak or valley, which however seems of less physical sense. Moreover, practically all the methods existing in the literature completely neglect the negative effect on fatigue strength produced by fatigue cycles with positive mean values. With theses premises, this work tries to further extending the probabilistic theory used by the frequency-domain methods by addressing to distinct problems. First, it tries to include in cycle distributions the effect of both a threshold level S_L (representing a limit state of the system) and the ultimate static strength S_u. Secondly, it uses the Goodman mean value correction to include the effect of mean values of counted cycles in the fatigue analysis of random loads by frequency-domain methods. The fatigue load is modeled as a stationary random process $X(t)$ with constant mean value m_c; two approaches of increasing complexity are presented: in the first one, only the effect of m_c is considered, while in the second one also the effect of the random mean value m_r, calculated with respect to m_c, is added. The proposed theoretical formulae are applied to two frequency-domain methods, namely the narrow-band approximation and the TB method. Finally, a comparison of the proposed formulae with the results from preliminary numerical simulations is shown.

keyword: Random loading, Rainflow count, Fatigue damage, Mean value effect, Threshold level.

1 Introduction

The fatigue strength of metallic materials under constant amplitude loads is mainly related to the amplitudes of the applied cycles, even if mean values are also important. In fact, we know that positive mean values cause a reduction of fatigue strength, which is usually quantified by proper analytical formulae (e.g. Goodman, Gerber, Smith corrections), synthesizing results from experiments [Łagoda, Macha and Pawliczek 2001]. According to these formulae, a fatigue cycle with amplitude s and positive mean value m is transformed to an equivalent cycle which is thought to cause the same damage as the given one, having a zero mean and greater amplitude (Goodman correction):

$$s_{eq} = \frac{s}{1 - m/S_u} \tag{1}$$

where S_u is the material ultimate static strength.

In random loadings the above situation becomes more complex, since fatigue cycles are not immediately defined and we first need to use proper counting methods (e.g. rainflow count) to identify and to extract them. Fatigue damage is then determined by assuming a suitable damage accumulation law (e.g. Palmgren-Miner linear rule).

Secondly, all cycles counted in a random load are randomly distributed and they should be handled by probabilistic tools; for example, we use probability density functions to characterize the statistical variability of their amplitudes and mean values. Such distributions could be tentatively assumed and then calibrated on observed results [Tovo 2001; Nagode, Klemenc and Fajdiga 2001; Xiong and Shenoi 2005], or alternatively could be correlated to the frequency-domain characteristics of the load, synthesized by a power spectral density function (frequency-domain approach) (for an extended review on frequency-domain methods see [Benasciutti 2005]).

In both cases, all such distributions are generally defined over an infinite domain, which implies that they predict a finite non-zero probability to have cycles with infinitely large peak or valley levels, which however seems rather questionable from a more physical viewpoint.

In fact, all materials have an ultimate static strength, S_u,

[1] ENDIF, ENgineering Department In Ferrara, University of Ferrara, via Saragat 1, 44100 Ferrara (Italy). Corresponding Author. E-mail: dbenasciutti@ing.unife.it Fax. +39 532 974870

above which we have static failures and not fatigue damage.

Furthermore, real systems often possess a threshold level S_L identifying a limit state situation, which once crossed immediately leads to a not-admissible condition. Otherwise, in particular cases, a mechanical device could force the system to work within specified ranges (e.g. a suspension in a vehicle), so that the system can not cross a specified threshold level (self-limited systems) and then all the fatigue cycles are necessary confined within predetermined bounds. For example, in some testing conditions a clipping test may be used to protect the test machine [Liou, Wu and Shin 1999].

Consequently, in the fatigue analysis of real systems we could consider the existence of an ultimate static strength S_u, which discriminates between static failures and fatigue damage, or even of a threshold level S_L, which controls the maximum stress.

More specifically, according to the system description proposed just above, the fatigue cycles counted in a random process becomes events randomly distributed within prescribed bounds correlated to the given threshold, and hence the fatigue assessment framework should characterize their statistical variability by means of truncated distributions.

A second problem is that the great part of frequency-domain methods for the random fatigue analysis often ignores the increment caused on damage by positive mean values of the counted cycles, only considering the statistical variability of their amplitudes. Otherwise, when included, the mean value correction is not explicitly correlated to the cycle distribution, but it is only implicitly included in best-fitting approximated formulae for damage [Petrucci and Zuccarello 2004], or the cycle distribution is truncated at an arbitrarily chosen limit [Khil and Sarkani 1999].

On the other hand, a necessary condition to insert the mean value correction as in Eq. (1) is first to understand and to include the effect of S_u (i.e. a threshold) in the cycle distribution, as well as in damage formulae, as explained just above.

Starting from the above premises, this paper aims to develop a theoretical framework, which including the influence of both the threshold and the cycles' mean value aims to provide a more realistic description of the fatigue damage process under random loadings.

Throughout the paper, symbol S_L will indicate a threshold level and S_u the ultimate static strength (with $S_L \leq S_u$), being $-S_L$ and $-S_u$ the corresponding symmetric values, respectively. Further, in what follows both S_L and S_u are considered as a deterministic value (i.e. not random).

Note, however, that S_L and S_u determine two different system behaviors. For what concerns S_u, its overcoming will result in an immediate system failure. On the other hand, if there exists a system threshold $S_L < S_u$ imposed by some physical device to prevent critical conditions, the event of a static failure will be not possible at all and the extremes (maxima or minima) of all counted fatigue cycles will be bounded by S_L.

The fatigue load is modeled as a stationary random process $X(t)$ with mean value m_c (constant), see Fig. 1; each rainflow cycle counted in $X(t)$ is characterized, besides its amplitude, also by a mean value m, decomposed as the sum of the constant component m_c and the random component m_r, evaluated with respect to m_c.

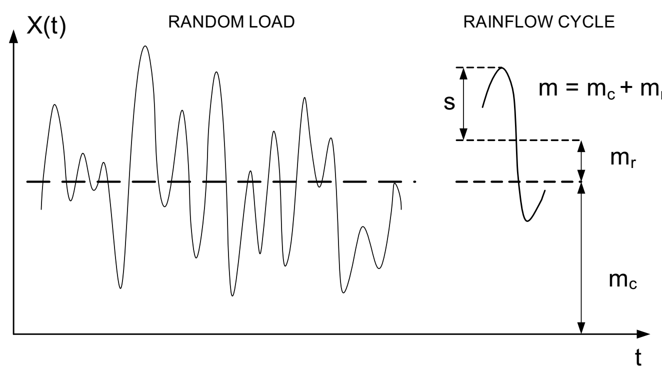

Figure 1 : Amplitude s and mean value m of a rainflow cycle; m_c is the mean value of process $X(t)$.

The cycle distributions and the formulae for damage estimation will be updated by considering the influence of:

- the threshold level S_L;
- the constant mean value m_c (approximate approach);
- both m_c and m_r mean values.

The threshold level S_L identifies in the domain defining the cycle distribution the "exceeding" region, where there are fatigue cycles associated to a threshold crossing occurrence (i.e. their peak or valley exceeds the threshold).

If $S_L = S_u$, a threshold crossing occurrence will produce a static failure and not fatigue damage, hence the probability of the ”exceeding” cycles should be excluded in principle from fatigue damage computation; for instance, an approximate method is proposed in Appendix to estimate the failure probability. At the opposite, if $S_L < S_u$, we assume as a simplifying hypothesis that the threshold is imposed to the system by some device, which prevents a threshold crossing occurrence; then S_L becomes the maximum load admissible in fatigue damage computations. Hence, we propose to not neglect the probability associated to the ”exceeding” cycles, but to shift it to the boundary of the ”exceeding” region, so that also these cycles do contribute to fatigue damage.

Equation (1) is then applied to include in damage computation the effect of positive mean values. First, only m_c influence is considered; the proposed approximated approach, applicable to all those methods which give only the distribution of amplitudes, is asymptotically exact for narrow-band loadings, in which m_r is approximately zero. Subsequently, m_r influence is added, through formulae which however require the joint amplitude-mean distribution.

The proposed formulae are applied, as an example, to two frequency-domain methods, i.e. the narrow-band approximation and the TB method [Tovo 2002; Benasciutti and Tovo 2005].

Finally, results from preliminary numerical simulations are shown, in order to judge about the correctness of the proposed formulae.

2 Spectral characterization of a random process

The fatigue load $X(t)$ is modeled as a stationary random process with mean value m_c (see Fig. 1) and power spectral density (PSD) $S(\omega)$, which is characterized by the spectral moments:

$$\lambda_i = \int_0^{+\infty} \omega^i \, S(\omega)\, \mathrm{d}\omega \tag{2}$$

which represent some important time-domain properties of $X(t)$; for example, the variance is given by the zero-order moment, $\sigma_X^2 = \lambda_0$. Further, if $X(t)$ is Gaussian, the mean upcrossing rate ν_0 and the rate of peak occurrence ν_p are, respectively [Lutes and Sarkani 1997]:

$$\nu_0 = \frac{1}{2\pi}\sqrt{\frac{\lambda_2}{\lambda_0}} \quad , \quad \nu_p = \frac{1}{2\pi}\sqrt{\frac{\lambda_4}{\lambda_2}} \tag{3}$$

Particular combinations of the spectral moments define the bandwidth parameters, as [Lutes and Sarkani 1997]:

$$\alpha_1 = \frac{\lambda_1}{\sqrt{\lambda_0\,\lambda_2}} \quad , \quad \alpha_2 = \frac{\lambda_2}{\sqrt{\lambda_0\,\lambda_4}} \tag{4}$$

Spectral moments and bandwidth parameters are the main spectral parameters used by the frequency-domain methods developed for the analysis of random loadings (see [Benasciutti and Tovo 2005] or, for an extended review, see [Benasciutti 2005]).

3 Theoretical formulae for fatigue damage

Each fatigue cycle counted in $X(t)$ is characterized by an amplitude s and a mean value m:

$$s = \frac{u-v}{2} \quad , \quad m = \frac{u+v}{2} \tag{5}$$

functions of the peak u and valley v.

According to Fig. 1, the mean value m can be thought as the sum of the constant mean value m_c of process $X(t)$ and the random mean value m_r, evaluated with respect to m_c.

Amplitudes and mean values, as well as peaks and valleys, of the cycles counted in random process $X(t)$ are clearly random variables and, if continuously distributed, they could be described by a probability density function (PDF).

Obviously, any counting method (e.g. rainflow, level-crossing, range-mean counting) applied to random load $X(t)$ will define its own set of counted cycles and hence its own cycle distribution. Our attention will mainly focus on the rainflow method, which has been recognized as the best counting procedure [Dowling 1972].

Let $h_{\mathrm{RF}}(u,v)$ be the joint PDF of rainflow cycles counted in $X(t)$, depending on peak u and valley v. In the engineering field, however, we mainly refer to the amplitude-mean PDF:

$$p_{\mathrm{RF}}(s,m) = 2h_{\mathrm{RF}}(m+s,\, m-s) \tag{6}$$

and to the marginal amplitude PDF:

$$p_{\mathrm{RF}}(s) = \int_{-\infty}^{+\infty} p_{\mathrm{RF}}(s,m)\,\mathrm{d}m \tag{7}$$

Known the fatigue behavior under constant amplitude loads as the S-N relation $s^k N = C$, defined for $m = 0$, the damage intensity under the Palmgren-Miner linear rule (neglecting the mean value m of counted cycles) is:

$$\overline{D}^{a}_{\mathrm{RF},nc} = \nu_p \int_{0}^{+\infty} \frac{s^k}{C} p_{\mathrm{RF}}(s)\,\mathrm{d}s \tag{8}$$

where ν_p (i.e. the peak frequency) is taken as the frequency of counted cycles, since the rainflow count gives a one-to-one correspondence between counted cycles and peaks in the process.

The damage in Eq. (8) only considers the statistical variability of the amplitudes of rainflow cycles. In order to formally introduce a dependence of damage on $p_{\mathrm{RF}}(s,m)$, we can use Eq. (7) to rewrite the previous equation also as:

$$\overline{D}^{a}_{\mathrm{RF},nc} = \nu_p \int_{0}^{+\infty} \int_{-\infty}^{+\infty} \frac{s^k}{C} p_{\mathrm{RF}}(s,m)\,\mathrm{d}s\mathrm{d}m \tag{9}$$

It is important to emphasize that, even if damage in Eq. (9) is computed by using the amplitude-mean joint PDF $p_{\mathrm{RF}}(s,m)$, it does not actually depend on mean values, but only on amplitudes, since no mean value correction, as that in Eq. (1), is used. In addition, both equations integrate the cycle distributions $p_{\mathrm{RF}}(s)$ and $p_{\mathrm{RF}}(s,m)$ over an infinite domain, without correcting for the threshold. Hence, they are defined as "no correction" damage estimates.

4 Influence of the threshold level S_L on damage

The cycle distributions, $p_{\mathrm{RF}}(s)$ or $p_{\mathrm{RF}}(s,m)$, used to quantify the statistical variability of the amplitudes and the mean values of counted cycles are defined over infinite domains. Consequently, they give a finite non-zero probability to count cycles in process $X(t)$ having an infinitely large peak or valley level. In addition, all these cycles are assumed to contribute to the total fatigue damage, since integration in Eqs. (8)-(9) is from zero to infinite (i.e. all amplitudes and mean values are admitted). However, once we introduce a threshold level S_L, this assumption is no longer valid.

For example, if $S_L = S_u$ (the material static strength), then all cycles with peak or valley exceeding S_L or $-S_L$ levels respectively (here defined as "exceeding" cycles, see Fig. 2), would actually produce static fracture and not fatigue damage.

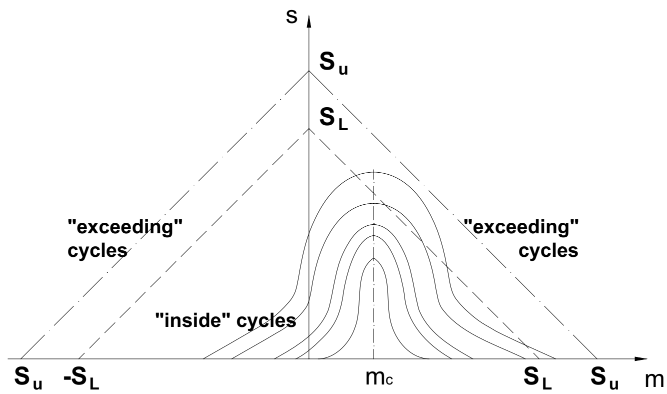

Figure 2 : Schematic representation of the level curves of the function $p_{\mathrm{RF}}(s,m)$. The "inside" and the "exceeding" regions are indicated.

Hence, the "exceeding" cycles should be excluded in principle from the computation of the fatigue damage, and from a mathematical point of view the statistical variability of all fatigue cycles counted in $X(t)$ should be described by means of truncated distributions.

Obviously, simple truncation of a probability density function defined over an infinite domain would not be strictly correct, since the resulting truncated distribution would have a total probability less then one (i.e. the probability of cycles, associated to a threshold crossing occurrence would be lost).

As an example, Fig. 3 plots the truncated cumulative distribution function of amplitudes, $P_{\mathrm{RF}}(s)$, as a function of both the (normalized) threshold and the global mean m_c, for a narrow-band process.

According to its definition, function $P_{\mathrm{RF}}(S_L - |m_c|)$ gives the probability to count a cycle with amplitude equal or lower than $S_L - |m_c|$. Since in a narrow-band process $p_{\mathrm{RF}}(s)$ is a Rayleigh PDF [Lutes and Sarkani 1997], we have:

$$P_{\mathrm{RF}}(S_L - |m_c|) = \gamma\left(1\,,\ \frac{(S_L - |m_c|)}{2\lambda_0}\right) \tag{10}$$

where we use the incomplete gamma function:

$$\gamma(a,\, x) = \int_0^x u^{a-1}\, e^{-u}\, \mathrm{d}u \tag{11}$$

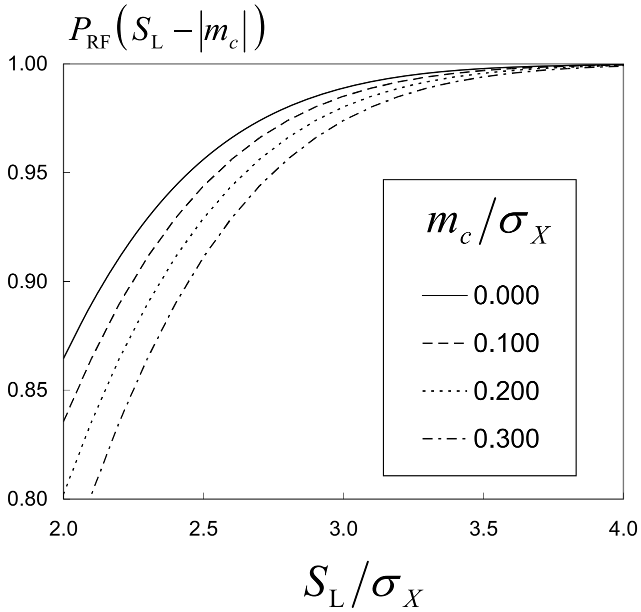

Figure 3 : Cumulative distribution function of amplitudes for a narrow-band process, as a function of m_c and S_L.

As expected, Figure 3 shows that $P_{\mathrm{RF}}\,(S_L - |m_c|)$ correctly converges to unity when S_L goes to infinity (no truncation). Figure 3 also shows how the probability lost due to the truncation greatly depends on S_L, so that the total probability tends to rapidly decrease for S_L just below three standard deviations, even for the lowest m_c value.

For a joint amplitude-mean PDF, instead, we propose in Appendix A two approximations to estimate the probability of threshold crossing occurrence.

On the other hand, the correct way to rescale a given distribution to a total probability equal to one would result in a difficult task, also requiring a re-fitting procedure of the rescaled distribution to existing data, to get accurate damage estimations.

If instead S_L represents a system threshold imposed for example by a physical device, then the extremes of all counted cycles are forced to not exceed $|S_L|$. Hence, the probability of the ”exceeding” cycles will not be lost.

A simpler solution adopted here is to truncate the cycle distribution at the threshold level and to concentrate the occurrence probability of the ”exceeding” cycles to the ”boundary” of the ”inside” region, see Fig. 2.

In this way, we can continue to use the distributions proposed by the existing methods to completely characterize the random counted cycles. Further, in the case of self-limited systems described above, where all fatigue cycles are confined within prescribed bounds related to the threshold, this approach would actually be physically coherent.

The mathematical formalization of the above concepts leads to the following form for the truncated amplitude PDF:

$$p_{\mathrm{RF}}^{\mathrm{thr}}(s) = p_{\mathrm{RF}}(s)\, \mathbf{I}\,(S_L - |m_c| - s) + \delta\,(s + |m_c| - S_L)\,\left[1 - P_{\mathrm{RF}}\,(S_L - |m_c|)\right] \tag{12}$$

where $P_{\mathrm{RF}}(s)$ is the CDF of amplitudes, $\mathbf{I}(x)$ is an indicator function ($\mathbf{I}(x) = 1$ if $x \geq 0$, $\mathbf{I}(x) = 0$ elsewhere) and $\delta(-)$ is the Dirac delta function.

For the truncated joint amplitude-mean PDF we have:

$$p_{\mathrm{RF}}^{\mathrm{thr}}(s,m) = p_{\mathrm{RF}}(s,m)\, \mathbf{I}\,(S_L - |m| - s) + \delta\,(S_L - |m|) \int_{S_L-|m|}^{+\infty} p_{\mathrm{RF}}(s,m)\mathrm{d}s \tag{13}$$

For what concerns fatigue damage the use of Eqs. (8)-(9), which integrating from zero to infinite include all possible amplitudes and mean values, is no longer acceptable.

By using, instead, the truncated cycle distributions as defined just above, our assumption leads to a total damage formed by two separate contributions:

$$\overline{D}_{\mathrm{RF,thr}} = \overline{D}_{\mathrm{RF,in}} + \overline{D}_{\mathrm{RF,exc}} \tag{14}$$

where $\overline{D}_{\mathrm{RF,in}}$ is the damage of the ”inside” cycles and $\overline{D}_{\mathrm{RF,exc}}$ the damage of the ”exceeding” cycles, shifted to the ”boundary” region.

For the damage computed as a function only of amplitudes, Eq. (14) specializes as:

$$\overline{D}^{a}_{\mathrm{RF,thr}} = \nu_p \int_0^{S_L-|m_c|} \frac{s^k}{C} p_{\mathrm{RF}}(s)\mathrm{d}s + \nu_p \frac{(\,S_L - |m_c|\,)^k}{C}\,\left[1 - P_{\mathrm{RF}}\,(S_L - |m_c|)\right] \tag{15}$$

in which the amplitudes of the "exceeding" cycles are transformed in amplitudes equal to $S_L - |m_c|$.

If we refer instead to the joint amplitude-mean PDF $p_{\text{RF}}(s,m)$, fatigue damage including the effect of the threshold can be expressed as:

$$\overline{D}^{\text{a,m}}_{\text{RF,thr}} = \overline{D}^{\text{a,m}}_{\text{RF,in}} + \overline{D}^{\text{a,m}}_{\text{RF,exc}} \tag{16}$$

in which $\overline{D}^{\text{a,m}}_{\text{RF,in}}$ is the damage corresponding to rainflow cycles for which $|m| + s < S_L$ ("inside" region):

$$\overline{D}^{\text{a,m}}_{\text{RF,in}} = \frac{\nu_p}{\text{C}} \int_{-S_L}^{S_L} \int_{0}^{S_L-|m|} s^k p_{\text{RF}}(s,m) \mathrm{d}s\, \mathrm{d}m \tag{17}$$

while $\overline{D}^{\text{a,m}}_{\text{RF,exc}}$ is the damage for rainflow cycles with $|m| + s \geq S_L$ ("exceeding" region):

$$\overline{D}^{\text{a,m}}_{\text{RF,exc}} = \frac{\nu_p}{\text{C}} \int_{-S_L}^{S_L} \int_{S_L-|m|}^{+\infty} (S_L + |m|)^k p_{\text{RF}}(s,m) \mathrm{d}s\, \mathrm{d}m \tag{18}$$

which are transformed into cycles with the same mean value m and amplitude $s = S_L - |m|$.

It becomes now clear that if we discarded from total damage the contribution of the "exceeding" cycles, we would have damage values less than the theoretical ones, given in Eqs. (8)-(9).

5 Effect of the mean value on fatigue damage

The next step in the theoretical analysis consists to increase the fatigue damage of those cycles with a positive mean value, according to the Goodman criterion, see Eq. (1). Two approaches are proposed: besides the correction for m_c, the influence of m_r is either neglected or included.

5.1 *Effect of m_c on damage*

When considering only the statistical variability of amplitudes, fatigue cycles are assumed to have a common mean value, equal to the mean value m_c of process $X(t)$.

Hence, damage only depends on the amplitude PDF and the influence of m_c on damage is simply obtained by inserting in Eq. (15) the Goodman correction given in Eq. (1):

$$\overline{D}^{a}_{\text{RF},m_c} = \frac{\nu_p}{\text{C}} \left(\int_{0}^{S_L-|m_c|} \left(\frac{s}{1 - \mathbf{I}(m_c)\, m_c/S_u} \right)^k p_{\text{RF}}(s) \mathrm{d}s + \left(\frac{S_L - |m_c|}{1 - \mathbf{I}(m_c)\, m_c/S_u} \right)^k \left[1 - P_{\text{RF}}(S_L - |m_c|) \right] \right) \tag{19}$$

The indicator function $\mathbf{I}(x)$ is used to specify that the mean value correction is applied only when $m_c > 0$.

The formula for the correction of m_c, Eq. (19), even if approximated, is easily applicable to all existing spectral methods which provide an estimate of the amplitude distribution $p_{\text{RF}}(s)$ (e.g. narrow-band approximation, TB method [Tovo 2002; Benasciutti and Tovo 2005], Dirlik method [Dirlik 1985], Zhao-Baker method [Zhao and Baker 1982]).

We expect that the error given by Eq. (19) in neglecting the influence of m_r on damage depends on the relative importance of this component with respect to the global mean value m_c and it should diminish as the frequency bandwidth of process $X(t)$ decreases, since in narrow-band processes all fatigue cycles are virtually symmetric with respect to m_c, all with $m_r \cong 0$.

Thus, a simple way to judge the process bandwidth could be to refer to the process bandwidth parameters (e.g. α_1 and α_2) or to their proper combinations; for example, a possibility is to use the b factor defined later on by the TB method: b values near unity indicate a narrow-band process, while values close to zero are for a wide-band process.

5.2 *Effect of m_c and m_r on damage*

The possibility to include also the influence of the random mean value m_r in damage computation only applies to formulae where damage depends on $p_{\text{RF}}(s,m)$, see Eqs. (16)-(18).

The correction for positive mean values is inserted by considering the Goodman correction, Eq. (1), for all cycles with $m > 0$. The formula for rainflow damage still writes as the sum of two contributions:

$$\overline{D}^{\text{a,m}}_{\text{RF},m} = \overline{D}^{\text{a,m}}_{\text{in},m} + \overline{D}^{\text{a,m}}_{\text{exc},m} \tag{20}$$

in which $\overline{D}^{\text{a,m}}_{\text{in},m}$ is the damage calculated for cycles with

$|m|+s<S_L$ ("inside" cycles):

$$\overline{D}_{\mathrm{in},m}^{\mathrm{a,m}}=\frac{\nu_p}{\mathrm{C}}\left(\int_{-S_L}^{0}\int_{0}^{S_L+m}s^k p_{\mathrm{RF}}(s,m)\,\mathrm{d}s\,\mathrm{d}m+\int_{0}^{S_L}\int_{0}^{S_L-m}\left(\frac{s}{1-m/S_u}\right)^k p_{\mathrm{RF}}(s,m)\,\mathrm{d}s\,\mathrm{d}m\right) \tag{21}$$

while $\overline{D}_{\mathrm{exc},m}^{\mathrm{a,m}}$ is the damage calculated for cycles with $|m|+s\geq S_L$ ("exceeding" cycles):

$$\overline{D}_{\mathrm{exc},m}^{\mathrm{a,m}}=\frac{\nu_p}{\mathrm{C}}\left(\int_{-S_L}^{0}\int_{S_L+m}^{+\infty}(S_L+m)^k p_{\mathrm{RF}}(s,m)\,\mathrm{d}s\,\mathrm{d}m+\int_{0}^{S_L}\int_{S_L-m}^{+\infty}\left(\frac{S_L-m}{1-m/S_u}\right)^k p_{\mathrm{RF}}(s,m)\,\mathrm{d}s\,\mathrm{d}m\right) \tag{22}$$

At the opposite of the previous case, Eqs. (20)-(22) are applicable only to those methods which provide an estimate of the joint PDF $p_{\mathrm{RF}}(s,m)$, e.g. [Nagode, Klemenc and Fajdiga 2001; Tovo 2002; Benasciutti and Tovo 2005; Lindgren and Broberg 2005].

6 Examples

In the following two paragraphs we apply the previous formulae to two frequency-domain methods, namely the well-known narrow-band approximation and the TB method.

6.1 Narrow-band approximation

If process $X(t)$ is both Gaussian and narrow-band, amplitudes follow a Rayleigh distribution and the frequency of cycles is approximated with ν_0; hence damage is expressed as:

$$\overline{D}_{\mathrm{NB}}=\frac{\nu_0}{C}\left(\sqrt{2\lambda_0}\right)^k\Gamma\left(1+\frac{k}{2}\right) \tag{23}$$

where $\Gamma(\,\cdot\,)$ is the Gamma function.

The modification for S_L according to Eq. (15) becomes:

$$\overline{D}_{\mathrm{NB,thr}}=\frac{\nu_0}{C}\left[\left(\sqrt{2\lambda_0}\right)^k\gamma\left(1+\frac{k}{2},\frac{(S_L-|m_c|)^2}{2\lambda_0}\right)+(S_L-|m_c|)^k e^{-\frac{(S_L-|m_c|)^2}{2\lambda_0}}\right] \tag{24}$$

where $\gamma(\,\cdot\,,\,\cdot\,)$is the incomplete gamma function.

Note how Eq. (24) reduces to Eq. (23) (i.e. the theoretical damage) as long as $S_L-|m_c|$ tends to infinity.

The correction for m_c is very simple, according to Eq. (19):

$$\overline{D}_{\mathrm{NB},m_c}=\frac{\nu_0}{C}\left[\left(\frac{\sqrt{2\lambda_0}}{1-\mathbf{I}(m_c)\,m_c/S_u}\right)^k\gamma\left(1+\frac{k}{2},\frac{(S_L-|m_c|)^2}{2\lambda_0}\right)+\left(\frac{S_L-|m_c|}{1-\mathbf{I}(m_c)\,m_c/S_u}\right)^k e^{-\frac{(S_L-|m_c|)^2}{2\lambda_0}}\right] \tag{25}$$

Note that in a narrow-band process all cycles are symmetric about m_c, with $m_r\cong 0$; hence there is no correction for m_r and the equations reported above are exact.

6.2 TB method

In wide-band processes, the amplitudes of counted cycles do not follow a Rayleigh distribution; in addition, the fraction of cycles having m_r different from zero could be relevant.

Therefore, a correct estimate of the fatigue damage should include the statistical variability of both amplitudes and mean values, described by the joint amplitude-mean PDF. However, this distribution is very difficult to be estimated and for this reason the great part of frequency-domain methods simply restricts to an amplitude density estimate [Dirlik 1985; Zhao and Baker 1992].

On the other hand, several methods exist, which try to also give an (analytical or numerical) estimate of the joint amplitude-mean distribution, see [Lindgren and Broberg 2005; Nagode, Klemenc and Fajdiga 2001].

Among them, we consider here the TB method [Tovo 2002; Benasciutti 2005; Benasciutti and Tovo 2005], in which a linear combination is used to estimate the joint PDF of rainflow cycles:

$$h_{\mathrm{RF}}(u,v)=b\,h_{\mathrm{LC}}(u,v)+(1-b)\,h_{\mathrm{RM}}(u,v) \tag{26}$$

where b is a proper "weight" (between 0 and 1), depending on α_1 and α_2 parameters of the process PSD:

$$b\cong\frac{(\alpha_1-\alpha_2)\left[\begin{array}{c}1.112\,(1+\alpha_1\alpha_2-(\alpha_1+\alpha_2))\,e^{2.11\,\alpha_2}\\+(\alpha_1-\alpha_2)\end{array}\right]}{(\alpha_2-1)^2} \tag{27}$$

The probability density functions $h_{\text{LC}}(u,v)$ and $h_{\text{RM}}(u,v)$ are the cycle distributions for the level-crossing counting:

$$h_{\text{LC}}(u,v) = \begin{cases} [p_p(u) - p_v(u)]\,\delta(u+v-2m_c) + \\ \qquad p_v(u)\,\delta(u-v) & \text{if } u > m_c \\ p_p(u)\,\delta(u-v) & \text{if } u \le m_c \end{cases} \quad (28)$$

and the range-mean counting (which is only approximated):

$$h_{\text{RM}}(u,v) = \frac{1}{2\sqrt{2\pi}\lambda_0\alpha_2^2} e^{-\frac{u^2+v^2}{4\lambda_0\alpha_2^2(1-\alpha_2^2)}} \cdot e^{-\frac{(u-v)^2}{4\lambda_0\alpha_2^2(1-\alpha_2^2)}\frac{1-2\alpha_2^2}{2\alpha_2^2}} \cdot e^{-\frac{m_c(m_c-u-v)}{2\lambda_0(1-\alpha_2^2)}} \left[\frac{u-v}{\sqrt{4\lambda_0(1-\alpha_2^2)}}\right] \quad (29)$$

By means of Eq. (6), the corresponding amplitude-mean distribution for the level-crossing is:

$$p_{\text{LC}}(s,m) = \begin{cases} [p_p(s) - p_v(s)]\,\delta(m-m_c) + \\ \qquad p_v(m)\,\delta(s) & \text{if } s+m > m_c \\ p_p(m)\,\delta(s) & \text{if } s+m \le m_c \end{cases} \quad (30)$$

and for the range-mean count is:

$$p_{\text{RM}}(s,m) = \frac{1}{\sqrt{2\pi\lambda_0(1-\alpha_2^2)}} e^{-\frac{(m-m_c)^2}{2\lambda_0(1-\alpha_2^2)}} \frac{s}{\lambda_0\alpha_2^2} e^{-\frac{s^2}{2\alpha_2^2\lambda_0}} \quad (31)$$

Note that all the above distributions, which are valid for Gaussian processes, are symmetric with respect to m_c.

The marginal amplitude PDF are found according to Eq. (7).

The resulting rainflow damage is then expressed as a linear combination:

$$\overline{D}_{\text{TB}} = b\,\overline{D}_{\text{LC}} + (1-b)\overline{D}_{\text{RM}} \quad (32)$$

where $\overline{D}_{\text{LC}}$ is the damage from the level-crossing counting (which is shown to be equal to $\overline{D}_{\text{NB}}$ given in Eq. (23) [Tovo 2002]) and $\overline{D}_{\text{RM}}$ is the (approximated) damage from the range-mean counting, see [Madsen, Krenk and Lind 1986].

When considering the correction for m_c, formula for damage as a function only of amplitudes becomes:

$$\overline{D}^a_{\text{TB},m_c} = b\,\overline{D}^a_{\text{LC},m_c} + (1-b)\,\overline{D}^a_{\text{RM},m_c} \quad (33)$$

in which $\overline{D}^a_{\text{LC},m_c}$ and $\overline{D}^a_{\text{RM},m_c}$ are the damage of the level-crossing and range-mean counting, computed as a function of amplitudes.

In particular, $\overline{D}^a_{\text{LC},m_c}$ coincides with $\overline{D}_{\text{NB},m_c}$ given in Eq. (25), while $\overline{D}^a_{\text{RM},m_c}$, computed according to the distribution $p_{\text{RM}}(s)$ derived from [Tovo 2002], is:

$$\overline{D}^a_{\text{RC},m_c} = \frac{\nu_p}{C}\left[\left(\frac{\sqrt{2\,\alpha_2^2\lambda_0}}{1-\mathbf{I}(m_c)\,m_c/S_u}\right)^k \times \gamma\left(1+\frac{k}{2}, \frac{(S_L-|m_c|)^2}{2\alpha_2^2\lambda_0}\right) + \left(\frac{S_L-|m_c|}{1-\mathbf{I}(m_c)m_c/S_u}\right)^k e^{-\frac{(S_L-|m_c|)^2}{2\alpha_2^2\lambda_0}}\right] \quad (34)$$

If we refer, instead, to the damage as a function of amplitudes and mean values, the damage including the mean value correction is formally expressed as:

$$\overline{D}^{\text{a,m}}_{\text{TB},m} = b\,\overline{D}^{\text{a,m}}_{\text{LC},m} + (1-b)\,\overline{D}^{\text{a,m}}_{\text{RM},m} \quad (35)$$

in which $\overline{D}^{\text{a,m}}_{\text{LC},m}$ is the contribution from the level-crossing counting, which is coincident with the damage $\overline{D}_{\text{NB},m_c}$ given in Eq. (25), whereas $\overline{D}^{\text{a,m}}_{\text{RM},m}$ is formed by the sum of two contributions as:

$$\overline{D}^{\text{a,m}}_{\text{RC},m} = \overline{D}^m_{\text{RC,in}} + \overline{D}^m_{\text{RM,exc}} \quad (36)$$

in which $\overline{D}^m_{\text{RM,in}}$ is the damage of the range-counting cycles associated to the condition $|m|+s<S_L$ ("inside" cycles):

$$\overline{D}^m_{\text{RM,in}} = \frac{\nu_p\left(\sqrt{2\,\alpha_2^2\lambda_0}\right)^k}{C\sqrt{2\pi\lambda_0(1-\alpha_2^2)}} \cdot \int_{-S_L}^{S_L} \frac{e^{-\frac{(m-m_c)^2}{2\lambda_0(1-\alpha_2^2)}}}{\left(1-\mathbf{I}(m)m/S_u\right)^k}\gamma\left(1+\frac{k}{2}, \frac{(S_L-|m|)^2}{2\lambda_0\alpha_2^2}\right) \mathrm{d}m \quad (37)$$

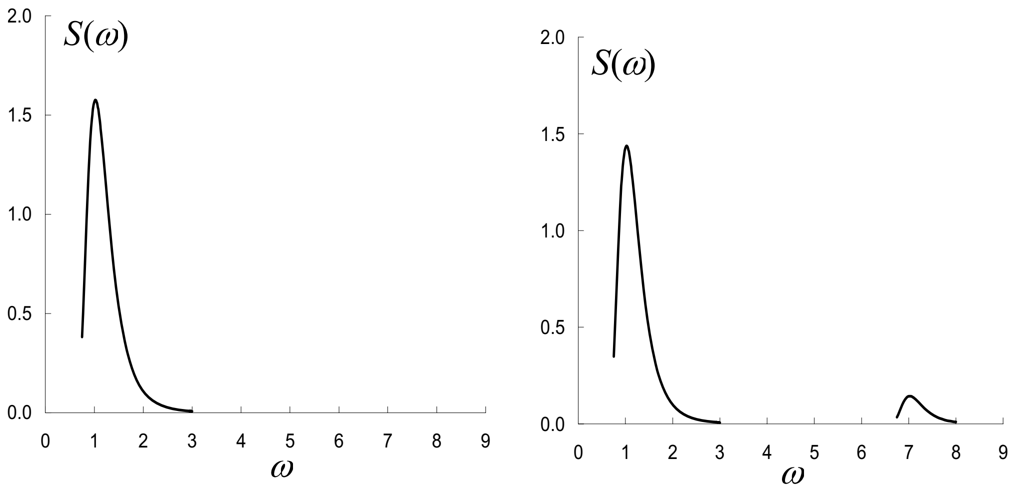

Figure 4 : Wide-band (left) and bimodal PSD (right).

Table 1 : Spectral parameters of the PSDs used in numerical simulations

PSD	λ_0	ν_0	ν_p	α_1	α_2
narrow-band	1.0	1.592	1.593	0.9998	0.9994
wide-band	1.0	0.202	0.240	0.963	0.844
bimodal	1.0	0.390	1.004	0.713	0.389

while $\overline{D}^m_{\text{RM,exc}}$ is the damage of the range-counting cycles associated to the condition $|m| + s \geq S_L$ ("exceeding" cycles):

$$\overline{D}^m_{\text{RM,exc}} = \frac{\nu_p}{C\sqrt{2\pi\lambda_0(1-\alpha_2^2)}} \cdot \int_{-S_L}^{S_L} \left(\frac{S_L - |m|}{1 - \mathbf{I}(m)m/S_u}\right)^k e^{-\frac{(m-m_c)^2}{2\lambda_0(1-\alpha_2^2)}} e^{-\frac{(S_L-|m|)^2}{2\lambda_0\alpha_2^2}} \mathrm{d}m \quad (38)$$

The above integrals can be solved by numerical integration.

7 Numerical simulations

In order to judge the accuracy of all previous formulae, we simulated several time histories corresponding to Gaussian random processes with different PSD and m_c values, choosing also different values of both S_L and S_u (being always $S_L \leq S_u$).

In the following, we refer to the normalized values:

$$\frac{m_c}{\sigma_X} \quad ; \quad \frac{S_L}{\sigma_X} \quad ; \quad \frac{S_u}{\sigma_X} \quad (39)$$

where $\sigma_X^2 = \lambda_0$ is the variance of process $X(t)$.

In order to evaluate the effect of m_c, S_L and S_u on both the cycle distribution and the fatigue damage, we select a wide set possible combinations of all the above parameters.

We mainly considered two sets of simulations: in the first set, S_u was kept fixed (values of $S_u/\sigma_X = 10, 20$ are chosen) and S_L/σ_X varied in the range from 2.6 up to the S_u/σ_X value; in the second set, S_L was kept fixed at values $S_L/\sigma_X = 3, 4, 5$ while S_u/σ_X varied in the range from the S_L/σ_X value up to 20. Finally, for each given combination of S_L and S_u values, fives values of m_c/σ_X are selected in the interval $0 \div 1.5$.

The PSDs investigated in numerical simulations were a narrow-band spectrum (a rectangular one-block PSD centered at 10 rad/sec), and the wide-band and bimodal spectra shown in Fig. 4 (for their equations see [Kihl, Sarkani and Beach 1995]).

Their spectral parameters are summarized in Tab. 1; as can be seen, the largest spectral bandwidth is associated to the bimodal PSD.

For each PSD, we simulated 5 time histories and for each of them we apply first the rainflow count and then the Palmgren-Miner linear rule, so to get the total damage.

The first damage value, say $\hat{D}_{nc}$ (called the "no correction" damage), is computed by completely neglecting the effect of both S_L and S_u (amplitudes are not truncated and there is no mean value correction):

$$\hat{D}_{nc} = C^{-1} \sum_i s_i^k \tag{40}$$

In the second damage value, say $\hat{D}_m$, the amplitudes of the "exceeding" cycles are truncated analogously to that described in Section 4, and the Goodman correction as in Eq. (1) is also applied to cycles with positive mean value:

$$\hat{D}_m = C^{-1} \left(\sum_{\substack{i \\ |m_i| + s_i < S_L}} \left(\frac{s_i}{1 - \mathbf{I}(m_i)\, m_i / S_u} \right)^k + \sum_{\substack{i \\ |m_i| + s_i \geq S_L}} \left(S_L - \mathbf{I}(-m_i)\, m_i \right)^k \right) \tag{41}$$

In the above equations, symbols s_i and m_i denote the amplitude and mean value of the i-th counted cycle.

By dividing the total damage by the total time length we have the damage intensity (damage/sec) for each simulated history. The mean damage intensity for the given random process was finally computed as the average of the five damage values calculated for each simulated history.

As an example, we plot in Figure 5 the ratio $\hat{D}_m/D_{nc}$ of the damage from simulations (calculated with and without corrections for both threshold and mean value) as a function of threshold S_L, for different m_c values and two fixed S_u levels. Similar results are obtained for the two other PSDs.

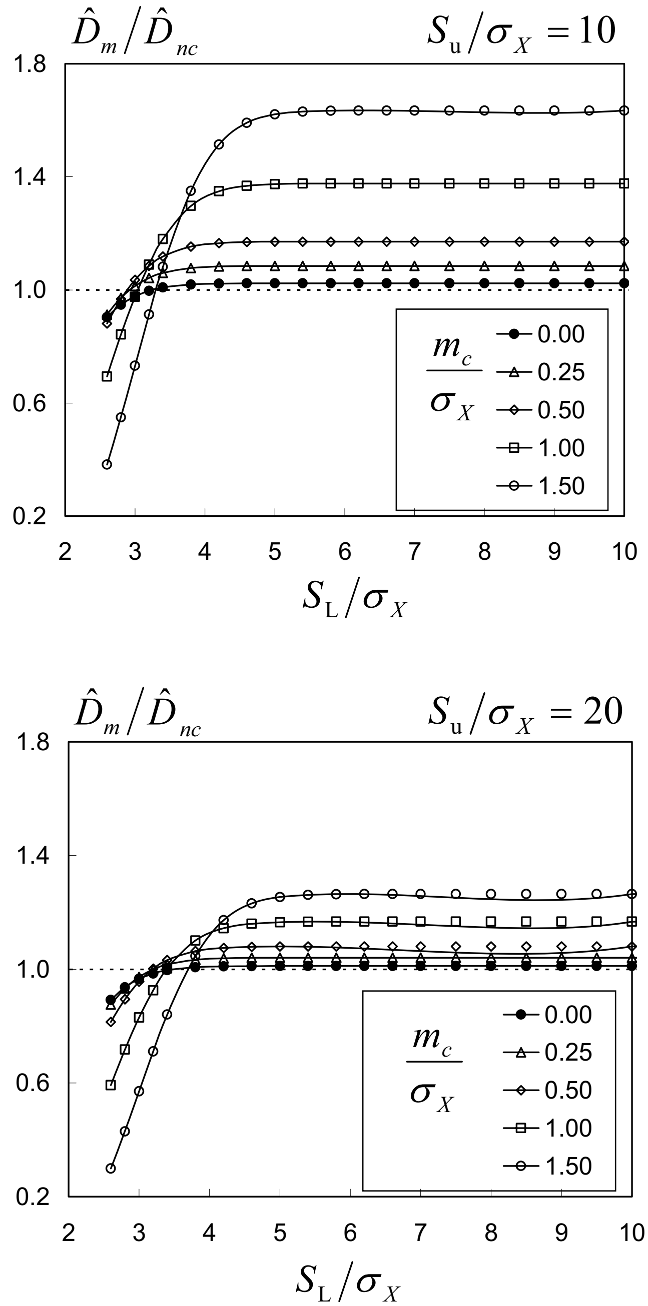

Figure 5 : Effect of threshold S_L on damage from simulations, for different mean values m_c and for S_u/σ_X equal to 10 (top) or 20 (bottom). Bimodal PSD.

The plot captures two distinct trends: the first is related to the amplitude truncation caused by S_L and prevails at low threshold levels, while the second is due to the mean value (Goodman) correction related to S_u and is clearly visible at high thresholds.

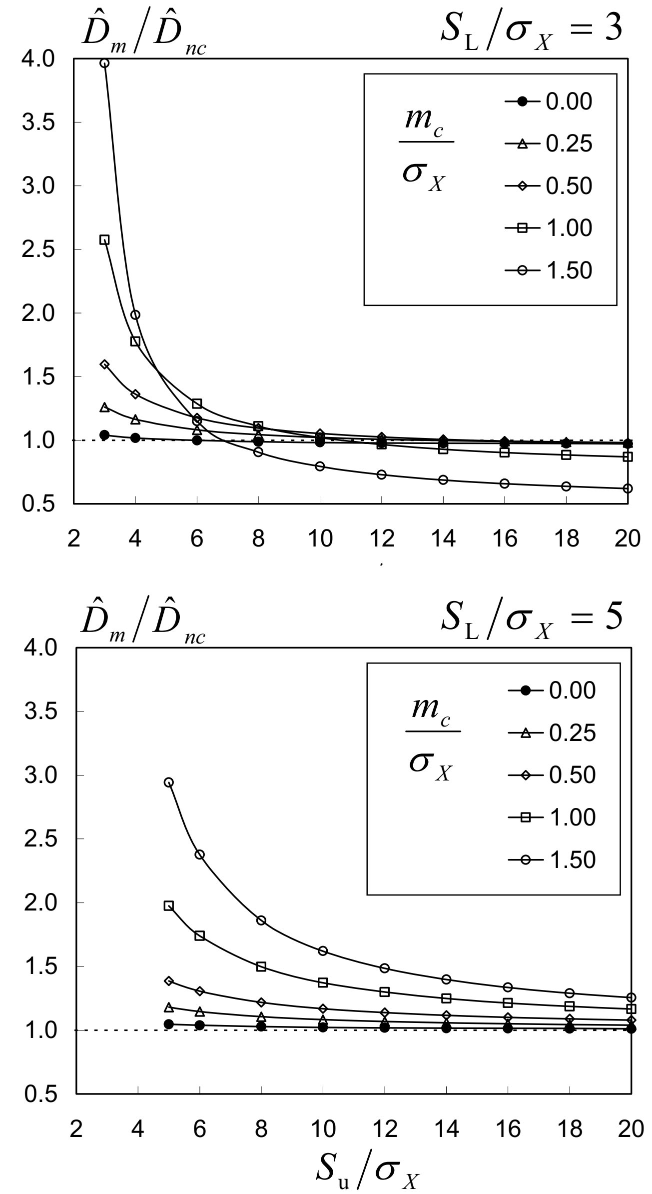

Figure 6 : Effect of S_u on damage from simulations, for different mean values m_c and for S_L/σ_X equal to 3 (top) and 5 (bottom). Wide-band PSD.

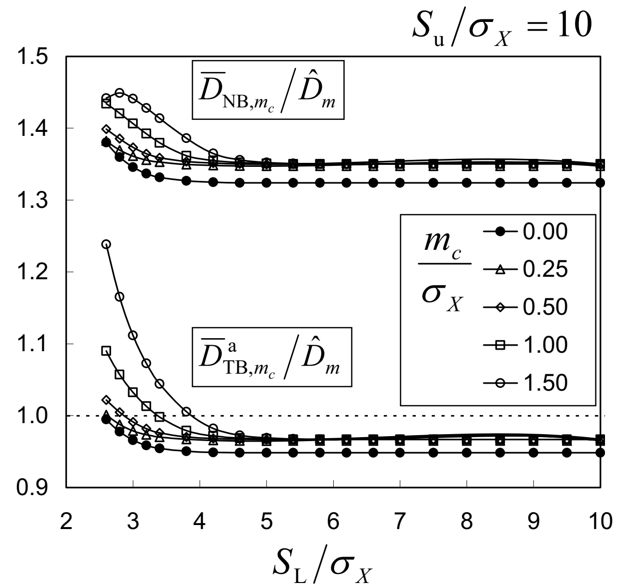

Figure 7 : Comparison of damage estimations $\overline{D}_{\mathrm{NB},m_c}$ and $\overline{D}^a_{\mathrm{TB},m_c}$ (only m_c correction) with damage $\hat{D}_m$ from simulations. Bimodal PSD.

The first trend due to the amplitude truncation appears as a damage reduction related to S_L. For a fixed m_c value, with decreasing S_L levels the amount of damage contributed by the "exceeding" cycles, which in our model are forced to have lower amplitudes, prevails over the damage of the "inside" cycles. As a consequence, the damage $\hat{D}_m$ becomes smaller than $\hat{D}_{nc}$, damage where the amplitude truncation is not applied.

The effect becomes markedly evident at very low thresholds S_L, where the damage contribution lost by truncation becomes important and the ratio $\hat{D}_m/D_{nc}$ could even decrease below unity. Further, at high m_c values the truncation effect is shown to start at higher S_L values and appears as more pronounced.

For what concerns the effect of the mean value correction, it is clearly visible only at high thresholds S_L, where amplitude truncation does not occur and therefore the mean value correction always increments damage $\hat{D}_m$ with respect to $\hat{D}_{nc}$, computed without corrections.

The results for $m_c = 0$ summarize the increment on damage caused by the random mean component m_r alone, while the results for other m_c values adds together the effects of both m_c and m_r.

The two plots in Figure 5 clearly evidence how the damage increment given by the Goodman correction depends primarily on m_c and very little on m_r. Further, according to Eq. (1), it correctly decreases at high S_u values, being inversely proportional to S_u, as also shown in Figure 6.

It should be noted that the effect of m_r on fatigue damage strictly depends on the bandwidth of process $X(t)$. More precisely, we expect that the increment on damage when $m_c = 0$ increases as the process bandwidth increases; this because in narrow-band processes, where all cycles are symmetric about m_c and all have $m_r \cong 0$, the contribution

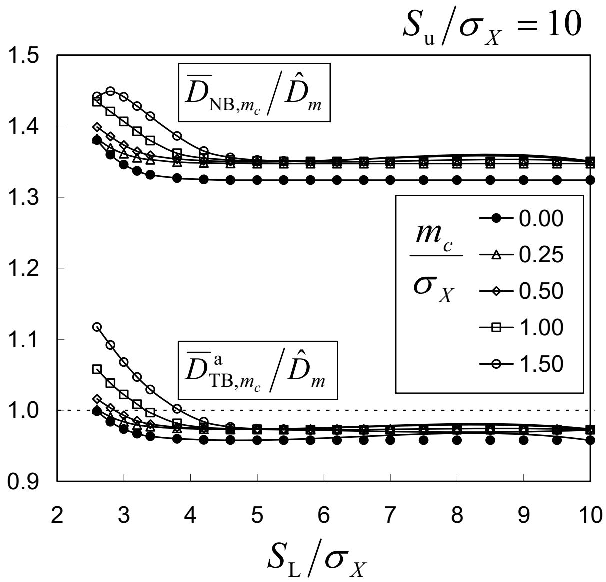

Figure 8 : Comparison of damage estimations $\overline{D}_{NB,m_c}$ ($m_c$correction) and $\overline{D}^{a,m}_{TB,m}$ (m mean value correction) with damage $\hat{D}_m$ from simulations. Bimodal PSD.

of m_r is practically negligible, at the opposite of what happens in wide-band processes (e.g. bimodal), where the fraction of cycles with m_r different from zero could be relevant.

For what concerns the accuracy of the theoretical estimations and the damage from numerical simulations, a comparison is shown in Figure 7 and Figure 8.

Figure 7 confirms that the narrow-band approximation modified with the m_c mean value correction gives too conservative damage estimations with random processes which are actually not narrow-band.

At the opposite, the TB method updated with the m_c correction provides better damage estimates, since it is capable to account with sufficient precision for the effect on damage of the spectral bandwidth of process $X(t)$.

In Figure 8 we finally show the comparison between damage $\overline{D}^{a,m}_{TB,m}$ with correction of both m_c and m_r mean values with damage from simulations. As can be seen, the estimation given by $\overline{D}^{a,m}_{TB,m}$ slightly improves with respected to $\overline{D}^{a}_{TB,m_c}$.

8 Conclusions

This paper proposes theoretical formulae for fatigue damage estimation in a random process by frequency-domain methods, in which the influence of both a threshold level $S_L$and the ultimate static strength S_u, as well as positive mean values of fatigue cycles, are considered.

The fatigue load is modeled as a stationary random process $X(t)$ with mean value m_c; hence, each rainflow cycle counted in $X(t)$ is characterized, besides its amplitude, also by its mean value $m = m_c + m_r$, sum of the global mean value m_c and of the random mean value m_r, computed with respect to m_c.

First, only the influence of S_L on the cycle distributions is considered. Then, the mean value correction, through the Goodman criterion, is included with two different approaches. The first evaluates only the influence of m_c and it is applicable to those frequency-domain methods that provide an estimate of the distribution of the amplitudes of counted cycles (e.g. narrow-band approximation, TB method, Dirlik method, Zhao-Baker method).

The second one considered the effect of both m_c and m_r, using the joint amplitude-mean distribution $p_{RF}(s,m)$, and thus it is applicable only to those methods which estimate $p_{RF}(s,m)$.

As an example, the proposed theoretical formulae are applied to two frequency-domain methods, namely the narrow-band approximation and the TB method.

Results from preliminary numerical simulations are finally shown.

9 Appendix A - Probability of threshold crossing occurrence

Once we introduce the threshold level S_L, the cycle distribution leads to a non-zero probability to count the so-called "exceeding" cycles, that is cycles associated to a threshold crossing occurrence.

In practical applications, it is important to have a rough estimation of this probability, without the need to apply a direct integration of the joint amplitude-mean cycle distribution, which could be unknown.

In this section we provide closed-form approximated formulae which give an answer to the above question.

Let $\mathbf{P}^1_f$ denote the probability of threshold crossing occurrence, which is the probability to count a cycle with peak and/or valley exceeding the $|S_L|$ level. If threshold were equal to the material ultimate static strength (i.e. $S_L = S_u$), $\mathbf{P}^1_f$ would actually represent a fracture probability.

If threshold levels S_L and $-S_L$ are assumed symmetric, the probability $\mathbf{P}_f^1$ can be computed as:

$$\mathbf{P}_f^1 = \int_{u=S_L}^{u=+\infty} \int_{v=-\infty}^{v=u} h_{\mathrm{RF}}(u,v)\,\mathrm{d}u\mathrm{d}v + \int_{v=-\infty}^{v=-S_L} \int_{u=v}^{u=+\infty} h_{\mathrm{RF}}(u,v)\,\mathrm{d}u\mathrm{d}v - \int_{u=S_L}^{u=+\infty} \int_{v=-\infty}^{v=-S_L} h_{\mathrm{RF}}(u,v)\,\mathrm{d}u\,\mathrm{d}v \qquad (42)$$

Now, we use a property of the $h_{\mathrm{RF}}(u,v)$ distribution (called the "completeness condition"), which relates the marginal distribution of $h_{\mathrm{RF}}(u,v)$ to the PDF of peaks and valleys [Tovo 2002]:

$$\begin{cases} p_p(\mathrm{u}) = \int_{-\infty}^{u} h_{\mathrm{RF}}(u,v)\,\mathrm{d}v \\ p_v(v) = \int_{v}^{+\infty} h_{\mathrm{RF}}(u,v)\,\mathrm{d}u \end{cases} \qquad (43)$$

Further, we express the last integral term in Eq. (42) by means of the so-called rainflow count intensity, defined as [Rychlik 1993]:

$$\mu_{\mathrm{RF}}(u,\mathrm{v}) = \nu_p \int_{x=u}^{+\infty} \int_{-\infty}^{y=v} h_{\mathrm{RF}}(x,y)\,\mathrm{d}x\,\mathrm{d}y \qquad (44)$$

which represents the intensity of rainflow cycles with peak equal or higher than u and valley equal or lower than v. From its definition, $\mu_{\mathrm{RF}}(u,v)$ tends to zero as u and v increase towards plus and minus infinity, respectively.

Hence, we can rewrite Eq. (42) as:

$$\mathbf{P}_f^1 = 1 - P_p(S_L) - P_v(-S_L) - \frac{\mu_{\mathrm{RF}}(S_L, -S_L)}{\nu_p} \qquad (45)$$

where $P_p(u)$ and $P_v(v)$ are the CDF of peaks and valleys, respectively. Note that Eq. (45) is very general and holds for any distribution $h_{\mathrm{RF}}(u,v)$ satisfying Eq. (43), when the proper peak and valley distributions are used. Note also how $\mathbf{P}_f^1$ defined by Eq. (45) converges to zero when S_L tends to infinity.

Now, if we assume that process $X(t)$, with mean m_c, is Gaussian, its peak and valley PDFs are symmetric to m_c, i.e. $p_v(\mathrm{v}) = p_p(2m_c - v)$, hence for their CDFs we have $P_v(\mathrm{v}) = 1 - P_p(2m_c - v)$. Consequently, we can further simplify Eq. (45) as:

$$\mathbf{P}_f^1 = 2 - P_p(S_L) - P_p(2m_c + S_L) - \frac{\mu_{\mathrm{RF}}(S_L, -S_L)}{\nu_p} \qquad (46)$$

In addition, the CDF of peaks for a Gaussian process is known explicitly [Lutes and Sarkani 1997]:

$$P_p(u) = \Phi\left(\frac{u - m_c}{\sigma_X\sqrt{1-\alpha_2^2}}\right) - \alpha_2 e^{-\frac{(u-m_c)^2}{2\sigma_X^2}}\,\Phi\left(\frac{\alpha_2(u-m_c)}{\sigma_X\sqrt{1-\alpha_2^2}}\right) \qquad (47)$$

Note, however, that we can not solve explicitly Eq. (46), since we do not know the analytical expression of $\mu_{\mathrm{RF}}(u,v)$. For very high S_L levels, a rough approximation could be to neglect the last term in Eq. (46); in the other cases, the following two approximations are proposed.

9.1 Poisson approximation

The approximation of the rainflow count intensity based on the Poisson convergence of the level upcrossing spectrum is [Johannesson and Thomas 2001]:

$$\frac{\mu_{\mathrm{RF}}^{\mathrm{Pois}}(u,v)}{\nu_p} \approx \frac{\mu(u)\mu(v)}{\nu_p\left(\mu(u)+\mu(v)\right)} \qquad (48)$$

where $\mu(x)$ is the upcrossing spectrum, which gives the number of upcrossings of the level x per time unit. In Gaussian processes, $\mu(x)$ is given by the well-known Rice's formula [Lutes and Sarkani 1997]:

$$\mu(x) = \nu_0 e^{-\frac{(x-m_c)^2}{2\sigma_X^2}} \qquad (49)$$

Note that Eq. (48) is asymptotically exact only when $u >> m_c$ and $v << m_c$.

Hence, in a Gaussian process the rainflow count intensity can be approximated as:

$$\frac{\mu_{\mathrm{RF}}^{\mathrm{Pois}}(u,v)}{\nu_p} \approx \frac{\alpha_2}{e^{\frac{(u-m_c)^2}{2\sigma_X^2}} + e^{\frac{(v-m_c)^2}{2\sigma_X^2}}} \qquad (50)$$

since $\alpha_2 = \nu_0/\nu_p$, see Eqs. (3) and (4).

Finally, calculating the cumulative distribution for $u = S_L$ and $v = -S_L$ gives:

$$\frac{\mu_{\mathrm{RF}}^{\mathrm{Pois}}(S_L,-S_L)}{\nu_p} \approx \frac{\alpha_2}{e^{\frac{(S_L-m_c)^2}{2\sigma_X^2}} + e^{\frac{(S_L+m_c)^2}{2\sigma_X^2}}} \qquad (51)$$

9.2 Linear approximation

From Eqs. (26) and (44), we have that:

$$\mu_{\mathrm{RF}}^{\mathrm{TB}}(u,v) = b\mu_{\mathrm{LC}}(u,v) + (1-b)\mu_{\mathrm{RM}}(u,v) \tag{52}$$

where $\mu_{\mathrm{LC}}(u,\, v)$ and $\mu_{\mathrm{RM}}(u,\, v)$ are the count intensity of the level-crossing and range-mean counting, respectively. By using the distribution given in Eq. (28), we have that:

$$\frac{\mu_{\mathrm{LC}}(u,\, v)}{\nu_p} = \alpha_2 \left[e^{-\frac{(u-m_c)^2}{2\sigma_X^2}} \mathbf{I}(u+v-2m_c) + e^{-\frac{(v-m_c)^2}{2\sigma_X^2}} \mathbf{I}(-(u+v-2m_c)) \right] \tag{53}$$

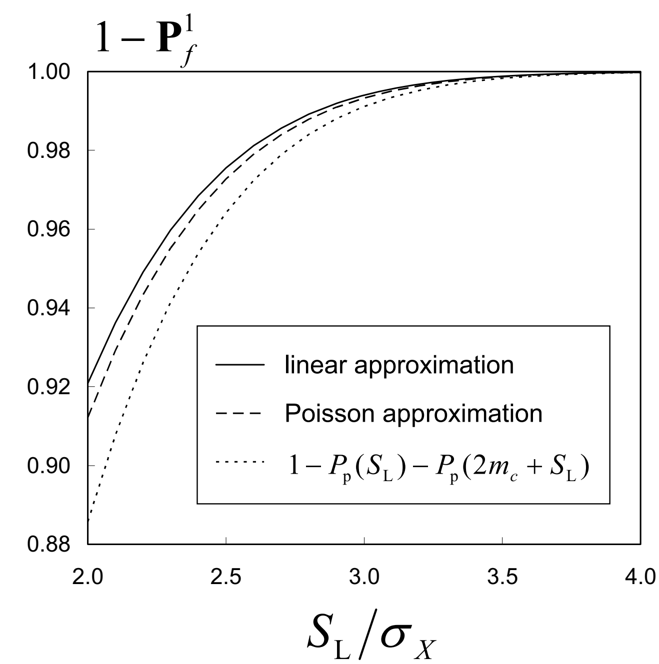

Figure 9 : Cumulative probability $1-\mathbf{P}_f^1$ according to Poisson and linear approximation. Bandwidth parameters are $\sigma_X^2 = 1$, $\alpha_1 = 0.713$ and $\alpha_2 = 0.389$ (see bimodal PSD in Tab. 1).

and from Eq. (29):

$$\frac{\mu_{\mathrm{RM}}(u,\, v)}{\nu_p} = \alpha_2 \left\{ e^{-\frac{(v-m_c)^2}{2\sigma_X^2}} \left[1 - \Phi\left(\frac{u-v+2\alpha_2^2(v-m_c)}{2\alpha_2\sigma_X\sqrt{1-\alpha_2^2}} \right) \right] + e^{-\frac{(u-m_c)^2}{2\sigma_X^2}} \Phi\left(\frac{v-m_c-(u-m_c)(1-2\alpha_2^2)}{2\alpha_2\sigma_X\sqrt{1-\alpha_2^2}} \right) \right\} \tag{54}$$

Calculating the above formulae for $u = S_L$ and $v = -S_L$:

$$\frac{\mu_{\mathrm{LC}}(S_L,\, -S_L)}{\nu_p} = \alpha_2\, e^{-\frac{(S_L+m_c)^2}{2\sigma_X^2}} \tag{55}$$

and:

$$\frac{\mu_{\mathrm{RM}}(S_L,\, -S_L)}{\nu_p} = \alpha_2 \left\{ e^{-\frac{(S_L+m_c)^2}{2\sigma_X^2}} \left[1 - \Phi\left(\frac{S_L(1-\alpha_2^2)-m_c\alpha_2^2}{\alpha_2\sigma_X\sqrt{1-\alpha_2^2}} \right) \right] + e^{-\frac{(S_L-m_c)^2}{2\sigma_X^2}} \left[1 - \Phi\left(\frac{S_L(1-\alpha_2^2)+m_c\alpha_2^2}{\alpha_2\sigma_X\sqrt{1-\alpha_2^2}} \right) \right] \right\} \tag{56}$$

The two approximations just proposed provide in general quite similar results for a wide range of combinations of α_1 and α_2 bandwidth parameters (although the Poisson approximation does not actually depends on α_1), where the largest differences are observed for the highest $\alpha_1 - \alpha_2$ values.

As an example, Fig. 9 plots the probability $1-\mathbf{P}_f^1$ (which could be interpreted as a cumulative distribution for the joint amplitude-mean PDF), using the bandwidth parameters of the bimodal PSD. In Fig. 9 we also plot the approximation that we would obtain by neglecting the contribution of the last term (i.e. the count intensity) in Eq. 46.

As can be seen, all the proposed approximations show a trend which is very similar to that already plotted in Fig. 3 for the CDF of amplitudes.

Finally, Fig. 10 shows the influence of m_c on $1-\mathbf{P}_f^1$ probability, according to the linear approximation.

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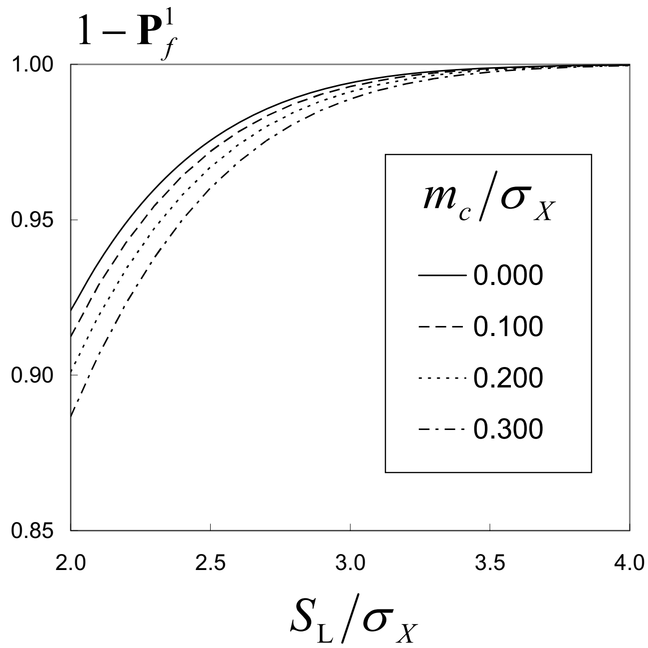

Figure 10 : Cumulative probability $1-\mathbf{P}_f^1$ according to the linear approximation, for different m_c values. Bandwidth parameters are $\sigma_X^2 = 1$, $\alpha_1 = 0.713$ and $\alpha_2 = 0.389$ (see bimodal PSD in Tab. 1).

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