An Alternative BEM for Fracture Mechanics

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Abstract: An alternative single domain boundary element formulation and its numerical implementation are presented for the analysis of two-dimensional cracked bodies. The problem is formulated employing the classical displacement boundary integral representation and a novel integral equation based on the stress or Airy's function. This integral equation written on the crack provides the relations needed to determine the problem solution in the framework of linear elastic fracture mechanics. Results are presented for typical problems in terms of stress intensity factors and they show the accuracy and efficiency of the approach.

keyword: Fracture mechanics, Stress function, Stress intensity factor, Boundary element method.

1 Introduction

Computational analysis and simulation of fracture mechanics problems is a generally established practice to ensure reasonable costs in design and maintenance of aerospace structures. Indeed numerical methods represent the only acceptable alternative for many fracture mechanics applications because of the analytical difficulties related to the solid body complex forms. The finite element method has been widely used to determine fracture mechanics parameters as stress intensity factors (SIFs), crack opening displacements (COD), energy release rate or J-integral. In case of standard finite element method, a significant refinement of the mesh near the crack tip is necessary to obtain accurate results and this leads to high computational costs [Ingraffea and Wawrzynek (2003)]. When special finite elements with correct stress singularity [Ingraffea and Wawrzynek (2003), Pian (1975)] are employed the computational effort is reduced nevertheless some complications are introduced in the method. The boundary element method (BEM) is particularly well suited and efficient to analyse problems characterised by stress concentrations. The main advantages of BEM are given by the pointwise representation of the solution and the computational gain associated with the reduction in dimensionality due to boundary discretization [Banerjee and Butterfield (1981), Hong and Chen (1988), Aliabadi (2002)]. These features give meaningful advantages in the modelization of crucial problems in structures like those involved in fracture mechanics, e.g. presence of inclusions or cracks. In the framework of fracture mechanics, different BEM approaches have been proposed: Green functions, multidomain approach and the Dual Boundary Element Method (DBEM) [Aliabadi (1997), Aliabadi, (2003)]. The Green function method is very accurate but it can be employed only for simple cases whereas the other two approaches are general methods. The multidomain boundary element method (MBEM) allows to model any crack problem but its implementation scheme gives rise to additional degrees of freedom and then the resolving system has a greater order with the consequent computational effort [Blandford, Ingraffea and Ligget (1981), Davì and Milazzo (2001)]. The Dual Boundary Element Method (DBEM) is based on the use of the traction integral equations and this leads to a system of integral equations involving hypersingular integrals [Portela, Aliabadi and Rooke (1992), Gray, Martha and Ingraffea (1990)]. In this paper, a novel single domain approach is presented to overcome the drawbacks of MBEM and DBEM. It is based on the employment of the Airy stress function to educe the integral equations needed to the solution of the crack problem. The approach preserves the computational advantages of single domain formulations without involving hypersingular kernels so that the treatment of the integral equations of the model requires no particular care. The numerical results show the accuracy and effectiveness of the proposed approach.

2 The Boundary Element Method for Fracture Mechanics

The Somigliana identity is the fundamental relation giving the boundary integral representation of the elastic re-

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sponse in the elastic domain Ω having contour Γ. Indeed it links the displacements at the point P_0 to the displacements $\mathbf{u}$ and tractions $\mathbf{p}$ on the boundary through a fictitious elastic system due to a concentrated body force acting at the point P_0. Denoting by $\mathbf{u}_j$and $\mathbf{p}_j$the displacements and tractions of the fictitious system, respectively, the Somigliana identity is written [Banerjee and Butterfield (1981), Hong and Chen (1988)]

$$\mathbf{u}(P_0) = \int_\Gamma \left(\mathbf{u}_j^T \mathbf{p} - \mathbf{p}_j^T \mathbf{u}\right) d\Gamma + \int_\Omega \mathbf{u}_j^T \mathbf{f} \, d\Omega \tag{1}$$

where $\mathbf{f}$ are the body forces applied in the domain. The Eq. (1) is the boundary integral representation of the displacement field inside the continuum Ω. If the point P_0 belongs to the boundary Γ, by a suitable limit procedure [Banerjee and Butterfield (1981)], one obtains the boundary integral equation which, taking the prescribed boundary conditions into account, allows the solution of the elastic problem in terms of displacements and tractions on the boundary [Banerjee and Butterfield (1981), Hong and Chen (1988)]. One has

$$\mathbf{c}\,\mathbf{u}(P_0) = \int_\Gamma \left(\mathbf{u}_j^T \mathbf{p} - \mathbf{p}_j^T \mathbf{u}\right) d\Gamma + \int_\Omega \mathbf{u}_j^T \mathbf{f} \, d\Omega \tag{2}$$

where the coefficient matrix $\mathbf{c}$ is given by

$$\mathbf{c} = -\int_\Gamma \mathbf{p}_j^T \, d\Gamma \tag{3}$$

The Eq. (2) is the basis for the numerical solution of the problem by the Boundary Element Method; However, in the framework of Fracture Mechanics when cracks are located in the domain, the Eq. (2) needs to be revised by taking the unknown relative displacements along the cracks into account; assuming a traction free crack the Eq. (2) becomes [Aliabadi (2002), Aliabadi (1997), Snyder and Cruse (1975)]

$$\mathbf{c}\,\mathbf{u}(P_0) = \int_\Gamma \left(\mathbf{u}_j^T \mathbf{p} - \mathbf{p}_j^T \mathbf{u}\right) d\Gamma - \int_{\Gamma_f} \mathbf{p}_j^T \Delta\mathbf{u} \, d\Gamma + \int_\Omega \mathbf{u}_j^T \mathbf{f} \, d\Omega \tag{4}$$

where Γ_f is the boundary representative of the crack and $\Delta\mathbf{u}$ are the relative displacements along it. It straight away appears that Eq. (4) in its numerical application originates a system with more unknowns than equations. To overcome this drawback many approaches have been proposed among which there are the Green's function, the multidomain method and the Dual Boundary Element Method (DBEM) [Aliabadi (1997)]. The Green's function method even is very accurate is limited to very simple problems [Snyder and Cruse (1975)], whereas the other two approaches are general and therefore they are the most employed. The multidomain method requires a partition of the investigated domain into suitable subregions so that each face of the crack belongs to the boundary of distinct subregions. Restoring the continuity conditions between the considered subregions the number of integral equations written is equal to the number of unknowns and the problem can be modelled without limitations [Blandford, Ingraffea and Ligget (1981), Davì and Milazzo (2001)]. Nevertheless the resolving system arising from the multidomain approach has higher order than that strictly needed to solve the problem with the consequent higher computational effort required. On the other hand the Dual Boundary Element Method (DBEM) does not require any partition of the investigated domain [Portela, Aliabadi and Rooke (1992)]; it recovers the further equations for the problem solution by expressing the tractions acting on the crack faces by means of the relative boundary integral representations. The main difficulty of this single domain approach is due to the hypersingular kernels occurring in the traction integral equation which need particular care in their numerical integration [Gray, Martha and Ingraffea (1990)].

3 Stress function approach

For an homogeneous, isotropic two-dimensional body the stress field can be derived from a single function, the so-called stress function or Airy function $\Phi = \Phi(x, y)$, so that the equilibrium equations are trivially fulfilled. Assuming that the body force field $\mathbf{f}$ is conservative there exists a potential function Ψ such that

$$\mathbf{f} = \begin{bmatrix} \frac{\partial \Psi}{\partial x} & \frac{\partial \Psi}{\partial y} \end{bmatrix}^T \tag{5}$$

and one has [Chou and Pagano (1992)]

$$\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{yy} & \sigma_{xy} \end{bmatrix}^T = \mathbf{C}\Phi - \tilde{\mathbf{I}}\Psi \tag{6}$$

where

$$\mathbf{C}^T = \begin{bmatrix} \frac{\partial^2}{\partial y^2} & \frac{\partial^2}{\partial x^2} & -\frac{\partial^2}{\partial x \partial y} \end{bmatrix} \tag{7}$$

$$\tilde{\mathbf{I}}^T = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \tag{8}$$

Besides, the compatibility condition requires that the stress function $\mathbf{\Phi}$ satisfies the following governing equation

$$\mathbf{C}^T\mathbf{E}^{-1}\mathbf{C}\,\Phi - \mathbf{C}^T\mathbf{E}^{-1}\tilde{\mathbf{I}}\Psi = 0 \tag{9}$$

where $\mathbf{E}$ denotes the elasticity matrix. For problems with no body forces from Eq. (9) one deduces that the stress function $\mathbf{\Phi}$ is biharmonic. Once the stress function is introduced the displacement field can be expressed by

$$\mathbf{u} = \mathbf{v} - \frac{1}{2G}\mathbf{S}\,\Phi \tag{10}$$

where $\mathbf{S} = \begin{bmatrix} \partial/\partial x & \partial/\partial y \end{bmatrix}^T$ is the gradient operator; G is the shear modulus and $\mathbf{v}$ is a vector, whose components v_1 and v_2 are conjugate harmonic functions. The boundary tractions $\mathbf{p}$ are expressed by the following relationship

$$\mathbf{p} = \frac{\partial}{\partial s}\mathbf{H}\mathbf{S}\,\Phi - \mathbf{n}\Psi \tag{11}$$

where $\partial/\partial s$ indicates the tangent derivative, $\mathbf{n}$ is the boundary unit normal vector, whereas the matrix $\mathbf{H}$ is defined by

$$\mathbf{H} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{12}$$

In terms of stress function the integral equation (4) becomes

$$\mathbf{c}\,\mathbf{v}(P_0) - \frac{1}{2G}\mathbf{c}\,\mathbf{S}\,\Phi(P_0) = \int_\Gamma \left(\mathbf{u}_j^T\,\mathbf{p} - \mathbf{p}_j^T\,\mathbf{u}\right)d\Gamma - \int_{\Gamma_f}\mathbf{p}_j^T\,\Delta\mathbf{u}\,d\Gamma + \int_\Omega \mathbf{u}_j^T\,\mathbf{f}\,d\Omega \tag{13}$$

Given that the components of $\mathbf{v}$ are harmonic functions, applying the Green theorem one has

$$\mathbf{c}\,\mathbf{v}(P_0) = \int_\Gamma \left(\varphi\frac{\partial\mathbf{v}}{\partial n} - \frac{\partial\varphi}{\partial n}\mathbf{v}\right)d\Gamma \tag{14}$$

where

$$\varphi = \ln r(P, P_0) \tag{15}$$

and $r(P, P_0)$ is the distance between the domain point P and the point P_0. Remembering that v_1 and v_2 are conjugate and taking Eq. (10) into account, Eq. (12) becomes

$$\mathbf{c}\,\mathbf{v}(P_0) = \int_\Gamma (\mathbf{u}^*\,\mathbf{p} - \mathbf{p}^*\,\mathbf{u})\,d\Gamma - \frac{1}{2G}\int_\Gamma \mathbf{S}\Phi\frac{\partial\varphi}{\partial n}d\Gamma + \int_\Gamma \mathbf{u}^*\mathbf{n}\Psi d\Gamma \tag{16}$$

where

$$\mathbf{u}^* = \frac{1}{2G}\begin{bmatrix} \varphi & 0 \\ 0 & \varphi \end{bmatrix} \tag{17}$$

$$\mathbf{p}^* = \begin{bmatrix} \frac{\partial\varphi}{\partial n} & -\frac{\partial\varphi}{\partial s} \\ \frac{\partial\varphi}{\partial s} & \frac{\partial\varphi}{\partial n} \end{bmatrix} \tag{18}$$

Finally, by using Eq. (14), the integral equation (11) is written as

$$\frac{1}{2G}\mathbf{c}\,\mathbf{S}\Phi(P_0) = \int_\Gamma \left(\mathbf{p}_j^T\,\mathbf{u} - \mathbf{u}_j^T\,\mathbf{p}\right)d\Gamma + \int_{\Gamma_f}\mathbf{p}_j^T\,\Delta\mathbf{u}\,d\Gamma + \int_\Gamma (\mathbf{u}^*\,\mathbf{p} - \mathbf{p}^*\,\mathbf{u})\,d\Gamma - \int_{\Gamma_f}\mathbf{p}^*\,\Delta\mathbf{u}\,d\Gamma - \frac{1}{2G}\int_\Gamma \mathbf{S}\Phi\frac{\partial\varphi}{\partial n}d\Gamma + \int_\Gamma \mathbf{u}^*\mathbf{n}\,\Psi\,d\Gamma - \int_\Omega \mathbf{u}_j^T\mathbf{f}\,d\Omega \tag{19}$$

Recalling that the components of the resultant of the tractions applied between the point P_0 and a generic point P_A are defined as

$$\mathbf{R} = \int_{P_0}^{P_A}\mathbf{p}\,d\Gamma \tag{20}$$

by integration of the Eq. (11) one obtains

$$\mathbf{S}\Phi = \mathbf{H}^{-1}\left(\mathbf{R} + \int_{P_0}^{P_A}\mathbf{n}\Psi\,d\Gamma\right) + \mathbf{k} = \mathbf{H}^{-1}\mathbf{F} + \mathbf{k} \tag{21}$$

where $\mathbf{k}$ is a vector whose components are arbitrary constants. For a point P_0 belonging to the crack line the integral equation (19) becomes

$$\frac{1}{G}\mathbf{c}\,\mathbf{k} = \int_\Gamma \left(\mathbf{p}_j^T\,\mathbf{u} - \mathbf{u}_j^T\,\mathbf{p}\right)d\Gamma + \int_{\Gamma_f}\mathbf{p}_j^T\,\Delta\mathbf{u}\,d\Gamma + \int_\Gamma (\mathbf{u}^*\,\mathbf{p} - \mathbf{p}^*\,\mathbf{u})\,d\Gamma - \int_{\Gamma_f}\mathbf{p}^*\,\Delta\mathbf{u}\,d\Gamma - \frac{\mathbf{H}^{-1}}{2G}\int_\Gamma \mathbf{F}\frac{\partial\varphi}{\partial n}d\Gamma + \int_\Gamma \mathbf{u}^*\mathbf{n}\,\Psi\,d\Gamma - \int_\Omega \mathbf{u}_j^T\mathbf{f}\,d\Omega \tag{22}$$

This equation allows the problem solution through the boundary element method. After the discretization by boundary elements of the boundaries Γ and Γ_f [Aliabadi (2002)], one obtains the resolving system by collocating the Eq. (4) at the nodes on the boundary Γ and the Eq. (22) at the points belonging to Γ_f needed to determine the unknowns $\Delta\mathbf{u}$ and the constants $\mathbf{k}$. Once the displacements $\mathbf{u}$ and the tractions $\mathbf{p}$ on the boundary Γ and the relative displacements $\Delta\mathbf{u}$ along Γ_f are determined, the stress intensity factors are calculated by the displacement extrapolation method [Aliabadi (2002), Blandford, Ingraffea and Ligget (1981), Cruse (1972)].

4 Applications

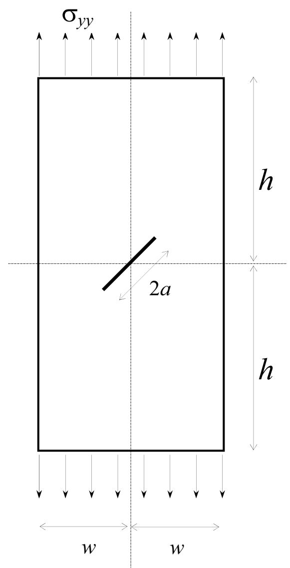


Figure 1 : Finite rectangular plate with central crack.

To validate the proposed approach and prove its efficacy and potentiality some numerical results are presented for classical fracture mechanics problems. The analyses have been performed with a discretization consisting of 32 linear boundary elements for the contour Γ and 18 linear boundary elements for the crack line Γ_f. The collocation point for Eq. (20) on Γ_f have been set at the element mid-point. The first application deals with the computation of the stress intensity factors for a crack of length $2a$ embedded in an infinite domain. The results for horizontal and 45° inclined crack are shown in Table 1 where the comparison with the analytical solution is also presented. This comparison clearly shows the accuracy of the proposed approach to compute the stress intensity factor. In the second example a rectangular panel having $h/w = 2$ with a central crack inclined of 45° is analysed (see Figure 1). The results obtained in terms of stress

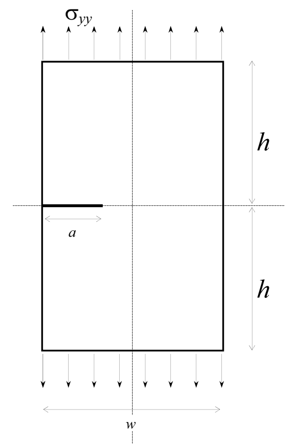


Figure 2 : Finite rectangular plate with edge crack.

Table 1 : $K_I/\left(\sigma_{yy}/\sqrt{\pi a}\right)$ for horizontal and 45° inclined crack in an infinite domain.

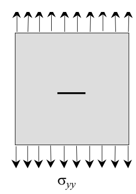
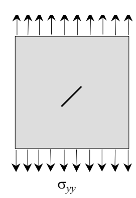
	Stress Intensity Factor (SIF)		
	SIF	Present	Analytic
	$K_I/\sigma_{yy}\sqrt{\pi a}$	1.00	1.00
	$K_I/\sigma_{yy}\sqrt{\pi a}$	0.50	0.50
	$K_{II}/\sigma_{yy}\sqrt{\pi a}$	0.50	0.50

Table 2 : $K_I/\left(\sigma_{yy}/\sqrt{\pi a}\right)$ for a 45° central crack embedded in a finite rectangular plate with $h/w = 2$.

a/w	Presente	Aliabadi (2002)	Civelek and Erdogan (1982)
0.2	0.51	0.53	0.52
0.3	0.52	0.55	0.54
0.4	0.56	0.59	0.57
0.5	0.60	0.63	0.61
0.6	0.65	0.69	0.66

Table 3 : $K_{II}/\left(\sigma_{yy}/\sqrt{\pi a}\right)$ for a 45 $^\circ$ central crack embedded in a finite rectangular plate with $h/w = 2$.

a/w	Present	Aliabadi (2002)	Civelek and Erdogan (1982)
0.2	0.49	0.52	0.51
0.3	0.50	0.53	0.52
0.4	0.52	0.54	0.53
0.5	0.54	0.56	0.55
0.6	0.56	0.58	0.57

Table 4 : $K_I/\left(\sigma_{yy}/\sqrt{\pi a}\right)$ for finite rectangular plate having $h/w = 0.5$ with edge crack.

a/w	Present	Aliabadi (2002)	Civelek and Erdogan (1982)
0.2	1.48	1.57	1.49
0.3	1.86	1.96	1.85
0.4	2.34	2.23	2.32
0.5	3.04	3.27	3.01

intensity factor for different crack length are given in Tables 2 and 3. Again the comparison of the present results with those found in the literature shows the accuracy and efficiency of the proposed method. Finally a finite rectangular plate $h/w = 0.5$ with an edge crack of length a has been analyzed (see Fig. 2). The calculated stress intensity factors are given in Table 4. Once again the comparison between the present results and those found in the literature confirms the soundness of the method for both its accuracy and efficiency.

5 Conclusions

A single domain boundary element method for two dimensional elastic solids has been presented with the aim of overcoming the computational drawbacks of classical BEM approaches for fracture mechanics. The method rests on the use of additional integral equations deduced in terms of stress function which collocated on the crack provide the relations needed to determine the solution. These integral equations do not involve hypersingular integrals with the resulting simplification in numerical implementation. The numerical results obtained show the accuracy, efficiency and usefulness of the proposed approach to determine the characteristic parameters of fracture mechanics

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