# Shear Deformation Effect in Second-Order Analysis of Composite Frames Subjected in Variable Axial Loading by BEM 

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keyword: Transverse shear stresses, Shear center, Shear deformation coefficients, Beam, Second Order Analysis, Boundary element method


#### Abstract

In this paper a boundary element method is developed for the second-order analysis of frames consisting of composite beams of arbitrary constant cross section, taking into account shear deformation effect. The composite beam consists of materials in contact, each of which can surround a finite number of inclusions. The materials have different elasticity and shear moduli with same Poisson's ratio and are firmly bonded together. Each beam is subjected in an arbitrarily concentrated or distributed variable axial loading, while the shear loading is applied at the shear center of the cross section, avoiding in this way the induction of a twisting moment. To account for shear deformations, the concept of shear deformation coefficients is used. Three boundary value problems are formulated with respect to the beam deflection, the axial displacement and to a stress function and solved employing a pure BEM approach, that is only boundary discretization is used. The evaluation of the shear deformation coefficients is accomplished from the aforementioned stress function using only boundary integration. Numerical examples with great practical interest are worked out to illustrate the efficiency, the accuracy and the range of applications of the developed method. The influence of both the shear deformation effect and the variableness of the axial loading are remarkable.


## 1 Introduction

An important consideration in the analysis of the components of plane and space frames or grid systems is the influence of the action of axial, lateral forces and end moments on the deformed shape of a beam. Lateral loads and end moments generate deflection that is further amplified by axial compression loading. The presence of

[^0]axial forces on the deformed shape of a beam leads to second-order effects, which have been the subject of research for many years.
Over the past twenty years, many researchers have developed and validated various methods of performing second-order analyses on structures. Early efforts led to methods based on accounting for the aforementioned effect by using magnification factors applied to the results obtained from first-order analyses [Chen and Atsura. (1977), Rutenberg, (1981)]. An example of such a method is the " $B_{1}$ and $B_{2}$ factor approach" provided in the AISC-LRFD specification [American Institute of Steel Construction (AISC). (1994)]. Since the modifications used in this method are only applied to the moments of the columns and not of the beams, the results obtained from this method are often unsatisfactory especially for cases involving moderate to large deformations [Goto and Chen. (1987)]. Consequently, due to the demand of more rigorous and accurate second-order analysis of structural components several research papers have been published including a non-linear incremental stiffness method [Chajes and Churchill, (1987)], closedform stiffness methods [Liew and Chen. (1994), Goto. (1994)], the analysis of non-linear effects by treating every element as a"beam-column" one [Vasek. (1993)], a design method for space frames using stability functions to capture second-order effects associated with P- $\delta$ and P- $\Delta$ effects [Kim, Park and Choi. (2001)], uniform formulae restricted to a single bar of a skeletal structure and to only a few loadings [Rubin. (1997)], the finite element method using cubic and linear shape functions [Torkamani, Sommez and Cao. (1997)] and a 3-D second-order plastic-hinge analysis accounting for material and geometric non-linearities [Kim, Lee and Park (2003), Kim and Choi. (2005)]. In all these studies shear deformation effect is ignored. Though these deformations are quite small in most civil engineering applications, they may be dominant in some situations, where bending moments are small compared to shear forces acting on the mem-
ber. This is normally true in short span beams or in structural plate systems stiffened by beams [Wen, Aliabadi, and Young. (2002)].
Recently, Kim et al. presented a practical second-order inelastic static [Kim, Lee and Kim, (2004)] and dynamic [Kim, Ngo-Huu, and Lee. (2005)] analysis for 3-D steel frames and Machado and Cortinez [2005], a geometrically non-linear beam theory for the lateral buckling problem of bisymmetric thin-walled composite simply supported or cantilever beams, taking into account shear deformation effect. Nevertheless, in all of the aforementioned research efforts the axial loading of the structural components is assumed to be constant.
In this paper a boundary element method is developed for the second-order analysis of frames consisting of composite beams of arbitrary constant cross section, taking into account shear deformation effect. The composite beam consists of materials in contact, each of which can surround a finite number of inclusions. The materials have different elasticity and shear moduli with same Poisson's ratio and are firmly bonded together. Each beam is subjected in an arbitrarily concentrated or distributed variable axial loading, while the shear loading is applied at the shear center of the cross section, avoiding in this way the induction of a twisting moment. To account for shear deformations, the concept of shear deformation coefficients is used. Three boundary value problems are formulated with respect to the beam deflection, the axial displacement and to a stress function and solved employing a pure BEM approach, that is only boundary discretization is used. The evaluation of the shear deformation coefficients is accomplished from the aforementioned stress function using only boundary integration. The essential features and novel aspects of the present formulation compared with previous ones are summarized as follows.

1. The beam is subjected in an arbitrarily concentrated or distributed variable axial loading.
2. The beam is supported by the most general linear boundary conditions including elastic support or restrain.
3. Shear deformation effect is taken into account.
4. The shear deformation coefficients are evaluated using an energy approach [Bach and Baumann
(1924), Stojek. (1964)], instead of Timoshenko's [Timoshenko and Goodier. (1984)], and Cowper's [(1966)], definitions, for which several authors [Schramm, Kitis, Kang and Pilkey, (1994), Schramm, Rubenchik, and Pilkey. (1997)] have pointed out that one obtains unsatisfactory results or definitions given by other researchers [Stephen. (1980), Hutchinson. (2001)], for which these factors take negative values.
5. The effect of the material's Poisson ratio $v$ is taken into account.
6. The proposed method employs a pure BEM approach (requiring only boundary discretization) resulting in line or parabolic elements instead of area elements of the FEM solutions (requiring the whole cross section to be discretized into triangular or quadrilateral area elements), while a small number of line elements are required to achieve high accuracy.

Numerical examples with great practical interest are worked out to illustrate the efficiency, the accuracy and the range of applications of the developed method. The influence of both the shear deformation effect and the variableness of the axial loading are remarkable.

## 2 Statement of the problem

Consider a component of a plane frame of length $L$ with a cross section of arbitrary shape, occupying the two dimensional multiply connected region $\Omega$ of the $y, z$ plane bounded by the $K+1$ curves $\Gamma_{1}, \Gamma_{2}, \ldots, \Gamma_{K}, \Gamma_{K+1}$, as shown in Fig.1. These boundary curves are piecewise smooth, i.e. they may have a finite number of corners. The material of the prismatic beam, with shear modulus $G$ and Poisson's ratio $v$ is assumed homogeneous, isotropic and linearly elastic. Without loss of generality, it may be assumed that the $x$-axis of the beam principal coordinate system is the line joining the centroids of the cross sections. The beam is subjected to an arbitrarily distributed axial loading $p_{x}$ and to torsionless bending arising from arbitrarily distributed transverse loading through the shear center $p_{z}$ and bending moment $m_{y}$ along $z$ and $y$ axes, respectively (Fig.1a).
According to the linear theory of beams (small deflections), the angle of rotation of the cross-section in the $x-z$ plane of the beam subjected to the aforementioned

(a)

(C: centroid, S: shear center)
(b)

Figure 1 : Prismatic beam subjected to torsionless bending (a) with a cross-section of arbitrary shape occupying the two dimensional region $\Omega$ (b)
loading and taking into account shear deformation effect satisfies the following relations
$\cos \omega_{y} \simeq 1$
$\sin \omega_{y} \simeq \omega_{y}=-\frac{d w}{d x}=\theta_{y}-\gamma_{z}$
where $w=w(x)$ is the beam deflection, while the corresponding curvature is given as
$k_{y}=\frac{d \theta_{y}}{d x}=-\frac{d^{2} w}{d x^{2}}+\frac{d \gamma_{z}}{d x}=-\frac{d^{2} w}{d x^{2}}-\frac{p_{z}}{G_{1} A_{z}}$
where $\gamma_{y}$ is the additional angle of rotation of the crosssection due to shear deformation and $G_{1} A_{z}$ is its shear rigidity of the Timoshenko's beam theory, where
$A_{z}=\kappa_{z} A=\frac{1}{a_{z}} A$
is the shear area with respect to $z$ axis with $\kappa_{z}$ the shear correction factor, $a_{z}$ the shear deformation coefficient and $A$ the composite cross section area, which due to the common Poisson's ratio of the materials is given as
$A=\sum_{j=1}^{K} \frac{G_{j}}{G_{1}} \int_{\Omega_{j}} d \Omega_{j}=\sum_{j=1}^{K} \frac{E_{j}}{E_{1}} \int_{\Omega_{j}} d \Omega_{j}$
Referring to Fig. 2b, the stress resultants $R_{x}, R_{z}$ acting in the $x, z$ directions, respectively, are related to the axial $N$ and shear $Q_{z}$ forces as
$R_{x}=N \cos \omega_{y}+Q_{z} \sin \omega_{y}$
$R_{z}=Q_{z} \cos \omega_{y}-N \sin \omega_{y}$
which by virtue of eqns. (1) become
$R_{x}=N-Q_{z} \frac{d w}{d x}$


Figure 2 : Displacements (a) and forces (b) acting on the deformed element in the $x z$ plane.
$R_{z}=Q_{z}+N \frac{d w}{d x}$
The second term in the right hand side of eqn. (6a), expresses the influence of the shear force $Q_{z}$ on the horizontal stress resultant $R_{x}$. However, this term can be neglected since $Q_{z}$ is much smaller than $N$ and thus eqn. (6a) can be written as
$R_{x} \simeq N$
The governing equation of the beam transverse displacement $w=w(x)$ will be derived by considering the equi-
librium of the deformed element in the $x-z$ plane. Thus, referring to Fig. 2b we obtain
$\frac{d R_{x}}{d x}+p_{x}=0$
$\frac{d R_{z}}{d x}+p_{z}=0$
$\frac{d M_{y}}{d x}-Q_{z}+m_{y}=0$
Substituting eqns. (7), (6b) into eqns. (8a,b), using eqn. (8c) to eliminate $Q_{z}$, employing the well-known relation
$M_{y}=E_{1} I_{y} k_{y}$
where the moment of inertia of the composite cross section with respect to $y$ axis is given as
$I_{y}=\sum_{j=1}^{K} \frac{E_{j}}{E_{1}} \int_{\Omega_{j}} z^{2} d \Omega_{j}$
and utilizing eqn. (2) we obtain the expressions of the angle of rotation due to bending $\theta_{y}$ and the stress resultants $M_{y}, R_{z}$ as

$$
\begin{align*}
\theta_{y} & =-\frac{d w}{d x}+\frac{1}{G_{1} A_{z}}\left(-E_{1} I_{y} \frac{d^{3} w}{d x^{3}}\right. \\
& \left.-\frac{E_{1} I_{y}}{G_{1} A_{z}}\left(\frac{d p_{z}}{d x}+N \frac{d^{3} w}{d x^{3}}-2 p_{x} \frac{d^{2} w}{d x^{2}}-\frac{d p_{x}}{d x} \frac{d w}{d x}\right)+m_{y}\right) \tag{11}
\end{align*}
$$

$$
\begin{align*}
M_{y} & =-E_{1} I_{y} \frac{d^{2} w}{d x^{2}}-\frac{E_{1} I_{y}}{G_{1} A_{z}}\left(p_{z}+\frac{d N}{d x} \frac{d w}{d x}+N \frac{d^{2} w}{d x^{2}}\right)  \tag{12a}\\
R_{z} & =-E_{1} I_{y} \frac{d^{3} w}{d x^{3}} \\
& -\frac{E_{1} I_{y}}{G_{1} A_{z}}\left(\frac{d p_{z}}{d x}+N \frac{d^{3} w}{d x^{3}}-2 p_{x} \frac{d^{2} w}{d x^{2}}-\frac{d p_{x}}{d x} \frac{d w}{d x}\right) \\
& +m_{y}+N \frac{d w}{d x} \tag{12b}
\end{align*}
$$

and the governing differential equation as

$$
\begin{align*}
& E_{1} I_{y}\left(1+\frac{N}{G_{1} A_{z}}\right) \frac{d^{4} w}{d x^{4}}=p_{z}-p_{x} \frac{d w}{d x}+N \frac{d^{2} w}{d x^{2}}+\frac{d m_{y}}{d x} \\
& \quad-\frac{E_{1} I_{y}}{G_{1} A_{z}}\left(\frac{d^{2} p_{z}}{d x^{2}}-3 p_{x} \frac{d^{3} w}{d x^{3}}-3 \frac{d p_{x}}{d x} \frac{d^{2} w}{d x^{2}}-\frac{d^{2} p_{x}}{d x^{2}} \frac{d w}{d x}\right) \tag{13}
\end{align*}
$$

Moreover, the pertinent boundary conditions of the problem are given as
$\alpha_{1} w(x)+\alpha_{2} R_{z}(x)=\alpha_{3}$
$\beta_{1} \frac{d w(x)}{d x}+\beta_{2} M_{y}(x)=\beta_{3}$ at the beam ends $x=0, l$
where $\alpha_{i}, \beta_{i}(i=1,2,3)$ are given constants. Eqns. (14) describe the most general boundary conditions associated with the problem at hand and can include elastic support or restrain. It is apparent that all types of the conventional boundary conditions (clamped, simply supported, free or guided edge) can be derived form these equations by specifying appropriately the functions $\alpha_{i}$ and $\beta_{i}$ (e.g. for a clamped edge it is $\alpha_{1}=\beta_{1}=1, \alpha_{2}=\alpha_{3}=\beta_{2}=\beta_{3}=0$ ).
In the aforementioned boundary value problem the axial force $N$ inside the beam or at its boundary can be evaluated from the solution of the following boundary value problem
$E_{1} A \frac{d^{2} u}{d x^{2}}=-p_{x} \quad$ inside the beam
$c_{1} u(x)+c_{2} N(x)=c_{3} \quad$ at the beam ends $\mathrm{x}=0,1$
where $u=u(x)$ is the beam axial displacement, $A$ the composite cross section area given from eqn.(4) and $c_{i}$ ( $i=1,2,3$ ) given constants.
It is worth here noting that the reduction of eqns.(4), (9), (10), (11), (12), (13) and (15) using the modulus of elasticity $E_{1}$ and of eqns. (4), (11) using the shear modulus $G_{1}$ of the first material, could be achieved using any other material, considering it as reference material.
The solution of both of the aforementioned boundary value problems, presumes the evaluation of the shear deformation coefficient $a_{z}$ corresponding to the principal centroidal system of axes Cyz. This coefficient is established equating the approximate formula of the shear strain energy per unit length [Schramm, Rubenchik and Pilkey.(1997)]
$U_{\text {appr. }}=\frac{a_{z} Q_{z}^{2}}{2 A G_{1}}$
with the exact one given from
$U_{\text {exact }}=\sum_{j=1}^{K} \frac{E_{1}}{E_{j}} \int_{\Omega_{j}} \frac{\left(\tau_{x z}\right)_{j}^{2}}{2 G_{1}} d \Omega_{j}$
and is obtained as [Sapountzakis and Mokos. (2005)]

$$
\begin{align*}
a_{z} & =\frac{1}{\kappa_{z}} \\
& =\frac{A}{E_{1} \Delta^{2}} \sum_{j=1}^{K} \int_{\Omega_{j}} E_{j}\left((\nabla \Phi)_{j}-d\right) \cdot\left((\nabla \Phi)_{j}-d\right) d \Omega_{j} \tag{19}
\end{align*}
$$

where $\left(\tau_{x z}\right)_{j}$ is the transverse (direct) shear stress component, $(\nabla) \equiv i_{y}(\partial / \partial y)+i_{z}(\partial / \partial z)$ is a symbolic vector with $i_{\tilde{y}}, i_{z}$ the unit vectors along $y$ and $z$ axes, respectively, $\Delta$ is given from
$\Delta=2(1+v) \mathrm{I}_{y} \mathrm{I}_{z}$
$v$ is the Poisson ratio of the cross section materials, $d$ is a vector defined as
$d=\left(v I_{z} y z\right) i_{y}+\left(v I_{z} \frac{z^{2}-y^{2}}{2}\right) i_{z}$
and $\Phi(y, z)$ is a stress function evaluated from the solution of the following Neumann type boundary value problem [Sapountzakis and Mokos. (2005)]
$\nabla^{2} \Phi=-2 I_{z} z \quad$ in $\Omega$
$\frac{\partial \Phi}{\partial n}=n \cdot d \quad$ on $\Gamma=\bigcup_{j=1}^{K+1} \Gamma_{j}$
where $n$ is the outward normal vector to the boundary $\Gamma$. In the case of negligible shear deformations $a_{z}=0$. It is also worth here noting that the boundary condition (22b) has been derived from the physical consideration that the traction vector in the direction of the normal vector $n$ vanishes on the free surface of the beam.
The analysis of the plane frame requires the construction of the $6 \times 6$ local stiffness matrix for each beam component written as
$\left[k^{i}\right]=\left[\begin{array}{cccccc}E_{1}^{i} A^{i} / L^{i} & 0 & 0 & -E_{1}^{i} A^{i} / L^{i} & 0 & 0 \\ 0 & k_{2,2}^{i} & k_{2,3}^{i} & 0 & k_{2,5}^{i} & k_{2,6}^{i} \\ 0 & k_{3,2}^{i} & k_{3,3}^{i} & 0 & k_{3,5}^{i} & k_{3,6}^{i} \\ -E_{1}^{i} A^{i} / L^{i} & 0 & 0 & E_{1}^{i} A^{i} / L^{i} & 0 & 0 \\ 0 & k_{5,2}^{i} & k_{5,3}^{i} & 0 & k_{5,5}^{i} & k_{5,6}^{i} \\ 0 & k_{6,2}^{i} & k_{6,3}^{i} & 0 & k_{6,5}^{i i} & k_{6,6}^{i i}\end{array}\right]$
(23)
relating the nodal displacement vector in the local coordinate system, as shown in Fig. 1

$$
\left\{D^{i}\right\}^{T}=\left\{\begin{array}{llllll}
u_{j}^{i} & w_{j}^{i} & \left(\theta_{y}\right)_{j}^{i} & u_{k}^{i} & w_{k}^{i} & \left(\theta_{y}\right)_{k}^{i} \tag{24}
\end{array}\right\}
$$

with the respective nodal load vector

$$
\left\{F^{i}\right\}^{T}=\left\{\begin{array}{llllll}
N_{j}^{i} & \left(Q_{z}\right)_{j}^{i} & \left(M_{y}\right)_{j}^{i} & N_{k}^{i} & \left(Q_{z}\right)_{k}^{i} & \left(M_{y}\right)_{k}^{i} \tag{25}
\end{array}\right\}
$$

where the $k_{l, m}^{i}(l, m=2,3,5,6)$ coefficients are evaluated from the solution of the boundary value problem (13), (14) for appropriate values of the $\alpha_{i}, \beta_{i}(i=1,2,3)$ constants. The construction of these matrices is followed by their composition leading to the final equations of equilibrium of the plane frame. The arising system of equations is nonlinear due to the presence of the unknown axial forces $N$.

## 3 Integral Representations - Numerical Solution

According to the precedent analysis, the nonlinear analysis of a beam including shear deformation reduces in establishing the deflection $w=w(x)$ with respect to zaxis, the axial displacement $u=u(x)$ and the stress function $\Phi(y, z)$.

### 3.1 For the transverse displacement $w$.

The numerical solution of the boundary value problem described by eqns $(13),(14 a, b)$ is accomplished using the Analog Equation Method [Katsikadelis. (2002)]. This method is applied for the problem at hand as follows.
Let $w$ be the sought solution of the boundary value problem described by eqns (13) and (14a,b). Differentiating this function four times yields

$$
\begin{equation*}
\frac{d^{4} w}{d x^{4}}=q_{z}(x) \tag{26}
\end{equation*}
$$

Eqn (26) indicates that the solution of the original problem can be obtained as the deflection of a beam with unit flexural rigidity and infinite shear rigidity subjected to a flexural fictitious load $q_{z}(x)$ under the same boundary conditions. The fictitious load is unknown. However, it can be established using BEM as follows.

The solution of eqn (26) is given in integral form as [Sapountzakis. (2001)]

$$
\begin{align*}
w(x) & =\int_{0}^{l} q_{z} \Lambda_{4 z}(r) d x \\
& -\left[\Lambda_{4 z}(r) \frac{d^{3} w}{d x^{3}}+\Lambda_{3 z}(r) \frac{d^{2} w}{d x^{2}}+\Lambda_{2 z}(r) \frac{d w}{d x}+\Lambda_{1 z}(r) w\right]_{0}^{l} \tag{27}
\end{align*}
$$

where the kernels $\Lambda_{i z}(r),(i=1,2,3,4)$ are given as
$\Lambda_{1 z}(r)=-\frac{1}{2} \operatorname{sgn} \rho$
$\Lambda_{2 z}(r)=-\frac{1}{2} l(1-|\rho|)$
$\Lambda_{3 z}(r)=-\frac{1}{4} l^{2}|\rho|(|\rho|-2) \operatorname{sgn} \rho$
$\Lambda_{4 z}(r)=\frac{1}{12} l^{3}\left(2+|\rho|^{3}-3|\rho|^{2}\right)$
Notice that in eqn.(27) for the line integral it is $r=x-\xi$, $x, \xi$ points inside the beam, whereas for the rest terms $r=x-\zeta, x$ inside the beam, $\zeta$ at the beam ends $0, l$.
Differentiating eqn (27) results in the integral representations of the derivatives of the deflection $w$ as

$$
\begin{align*}
& \frac{d w(x)}{d x}=\int_{0}^{l} q_{z} \Lambda_{3 z}(r) d x \\
& \quad-\left[\Lambda_{3 z}(r) \frac{d^{3} w}{d x^{3}}+\Lambda_{2 z}(r) \frac{d^{2} w}{d x^{2}}+\Lambda_{1 z}(r) \frac{d w}{d x}\right]_{0}^{l} \tag{29a}
\end{align*}
$$

$$
\begin{align*}
& \frac{d^{2} w(x)}{d x^{2}}=\int_{0}^{l} q_{z} \Lambda_{2 z}(r) d x-\left[\Lambda_{2 z}(r) \frac{d^{3} w}{d x^{3}}+\Lambda_{1 z}(r) \frac{d^{2} w}{d x^{2}}\right]_{0}^{l}  \tag{29b}\\
& \frac{d^{3} w(x)}{d x^{3}}=\int_{0}^{l} q_{z} \Lambda_{1 z}(r) d x-\left[\Lambda_{1 z}(r) \frac{d^{3} w}{d x^{3}}\right]_{0}^{l} \tag{29c}
\end{align*}
$$

The integral representations (27), (29a) written for the beam ends 0, ltogether with the boundary conditions (14a,b) can be employed to express the unknown boundary quantities $w, w^{\prime}, w^{\prime \prime}$ and $w^{\prime \prime \prime}$ in terms of $q_{z}$. This is accomplished numerically. If Lis the number of the nodal points along the beam axis, this procedure yields
the following set of linear equations

$$
\begin{align*}
& {\left[\begin{array}{cccc}
{\left[E_{11}\right]} & {\left[E_{12}\right]} & {\left[E_{13}\right]} & {\left[E_{14}\right]} \\
{[0]} & {\left[E_{22}\right]} & {\left[E_{23}\right]} & {[0]} \\
{\left[E_{31}\right]} & {\left[E_{32}\right]} & {\left[E_{33}\right]} & {\left[E_{34}\right]} \\
{[0]} & {\left[E_{42}\right]} & {\left[E_{43}\right]} & {\left[E_{44}\right]}
\end{array}\right]\left\{\begin{array}{c}
\{w\} \\
\left\{w^{\prime}\right\} \\
\left\{w^{\prime \prime}\right\} \\
\left\{w^{\prime \prime \prime}\right\}
\end{array}\right\}} \\
& \quad=\left\{\begin{array}{c}
\left\{a_{3}^{z}\right\} \\
\left\{\begin{array}{l}
\left.\beta_{3}^{z}\right\} \\
\{0\} \\
\{0\}
\end{array}\right\}+\left[\begin{array}{c}
{[0]} \\
{[0]} \\
{\left[F_{3}\right]} \\
{\left[F_{4}\right]}
\end{array}\right\}\left\{q_{z}\right\}
\end{array}\right. \tag{30}
\end{align*}
$$

in which $\left[E_{22}\right],\left[E_{23}\right],\left[E_{1 i}\right],(i=1,2,3,4)$ are $2 \times 2$ matrices including the nodal values of the functions $a_{1}, \mathrm{a}_{2}, \beta_{1}, \beta_{2}$ of eqns (14a,b) and $\left[E_{i j}\right],(i=3,4, j=$ $1,2,3,4$ ) are square $2 \times 2$ known coefficient matrices resulting from the values of the kernels $\Lambda_{i z}$ at the beam ends; $\left\{a_{3}\right\},\left\{\beta_{3}\right\}$ are $2 \times 1$ known column matrices including the boundary values of the functions $a_{3}, \beta_{3}$ in eqns $(14 \mathrm{a}, \mathrm{b})$ and $\left[F_{i}\right](i=3,4)$ are $2 \times L$ rectangular known matrices originating from the integration of the kernels on the axis of the beam. Finally, $\{w\},\left\{w^{\prime}\right\},\left\{w^{\prime \prime}\right\}$ and $\left\{w^{\prime \prime \prime}\right\}$ are vectors including the two unknown nodal values of the respective boundary quantities and $\left\{q_{z}\right\}$ is a vector including the $L$ unknown nodal values of the fictitious load.
The discretized counterpart of eqn (27) when applied to all nodal points in the interior of the beam yields

$$
\begin{align*}
& \{W\}=[F]\left\{q_{z}\right\} \\
& \quad-\left(\left[E_{1}\right]\{w\}+\left[E_{2}\right]\left\{w^{\prime}\right\}+\left[E_{3}\right]\left\{w^{\prime \prime}\right\}+\left[E_{4}\right]\left\{w^{\prime \prime \prime}\right\}\right) \tag{31}
\end{align*}
$$

where $[F]$ is an $L x L$ known matrix and $\left[E_{i}\right],(i=1,2,3,4)$ are $L \times 2$ also known matrices. Elimination of the boundary quantities from eqn (31) using eqn (30) for homogeneous boundary conditions $(14 \mathrm{a}, \mathrm{b})\left(a_{3}=\beta_{3}=0\right)$ yields

$$
\begin{equation*}
\{W\}=\left[B_{z}\right]\left\{q_{z}\right\} \tag{32}
\end{equation*}
$$

where $[B]$ is an $L x L$ matrix.
Moreover, the discretized counterpart of eqns (29a,b,c) when applied to all nodal points in the interior of the beam, after elimination of the boundary quantities using eqn (30) yields
$\left\{W^{\prime}\right\}=\left[B_{z}^{\prime}\right]\left\{q_{z}\right\}$
(33a)
$\left\{W^{\prime \prime}\right\}=\left[B_{z}^{\prime \prime}\right]\left\{q_{z}\right\}$
$\left\{W^{\prime \prime \prime}\right\}=\left[B_{z}^{\prime \prime \prime}\right]\left\{q_{z}\right\}$
where $\left[B_{z}^{\prime}\right],\left[B_{z}^{\prime \prime}\right],\left[B_{z}^{\prime \prime \prime}\right]$ are known $L x L$ coefficient matrices. Note that eqns (32) and (33a,b,c) are valid for homogeneous boundary conditions ( $a_{3}=\beta_{3}=0$ ). For non-homogeneous boundary conditions, an additive constant vector will appear in the right hand side of these equations.
Finally, applying eqn (13) to the $L$ nodal points in the interior of the beam we obtain the following linear system of equations with respect to $q_{z}$

$$
\begin{align*}
& {\left[\left[D_{z}^{\prime \prime \prime \prime}\right]-\left[D_{z}^{\prime \prime \prime}\right]\left[B_{z}^{\prime \prime \prime}\right]-\left[D_{z}^{\prime \prime}\right]\left[B_{z}^{\prime \prime}\right]-\left[D_{z}^{\prime}\right]\left[B_{z}^{\prime}\right]\right]\left\{q_{z}\right\}} \\
& \quad=\left\{p_{z}\right\}+\left\{m_{y}^{\prime}\right\}-\left[D_{z}\right]\left\{p_{z}^{\prime \prime}\right\} \tag{34}
\end{align*}
$$

where $\left[D_{z}^{\prime \prime \prime}\right],\left[D_{z}^{\prime \prime \prime}\right],\left[D_{z}^{\prime \prime}\right],\left[D_{z}^{\prime}\right],\left[D_{z}\right]$ are diagonal $L x L$ matrices whose elements are given from
$\left(D_{z}^{\prime \prime \prime \prime}\right)_{i i}=E_{1} I_{y}\left(1+\frac{N_{i}}{G_{1} A_{z}}\right)$
$\left(D_{z}^{\prime \prime \prime}\right)_{i i}=\frac{3 E_{1} I_{y}}{G_{1} A_{z}}\left(p_{x}\right)_{i}$
$\left(D_{z}^{\prime \prime}\right)_{i i}=\frac{3 E_{1} I_{y}}{G_{1} A_{z}}\left(p_{x}^{\prime}\right)_{i}+N_{i}$
$\left(D_{z}^{\prime}\right)_{i i}=\frac{E_{1} I_{y}}{G_{1} A_{z}}\left(p_{x}^{\prime \prime}\right)_{i}-\left(p_{x}\right)_{i}$
$\left(D_{z}\right)_{i i}=\frac{E_{1} I_{y}}{G_{1} A_{z}}$
at the $L$ nodal points in the interior of the beam; $\left\{q_{z}\right\}$, $\left\{p_{z}\right\},\left\{m_{y}^{\prime}\right\}$ and $\left\{p_{z}^{\prime \prime}\right\}$ are vectors with Lelements including the values of the fictitious loading, the transverse loading, the first derivative of the bending moment distributed loading and the second derivative of the transverse loading at the $L$ nodal points in the interior of the beam. The values of the $\left(m_{y}\right)^{\prime},\left(p_{z}\right)^{\prime \prime},\left(p_{x}\right)^{\prime}$ and $\left(p_{x}\right)^{\prime \prime}$ quantities result after approximating the corresponding derivatives with appropriate central, forward or backward finite differences. Following the estimation of the nodal values of the axial force $N_{i}$, the solution of the linear system of equations (34) and the evaluation of the fictitious load $q_{z}$, the transverse deflection $w$ and its derivatives in the interior of the beam are obtained using eqns (32) and (33a,b,c).

### 3.2 For the axial displacement $u$.

The solution of eqn (15) is given in integral form as
$u(x)=-\frac{1}{E_{1} A} \int_{0}^{l} p_{x} \Lambda_{2 x}(r) d x-\left[\Lambda_{2 x}(r) \frac{d u}{d x}-\Lambda_{1 x}(r) u\right]_{0}^{l}$
where $r=x-\xi, x, \xi$ points of the beam and the kernels $\Lambda_{i x}(r),(i=1,2)$ are given as
$\Lambda_{1 x}(r)=\frac{1}{2} s g n r$
$\Lambda_{2 x}(r)=\frac{1}{2}|r|$
Notice that in eqn.(36) for the line integral it is $r=x-\xi$, $x, \xi$ points inside the beam, whereas for the rest terms $r=x-\zeta, x$ inside the beam, $\zeta$ at the beam ends $0, l$.
Differentiating eqn (36) with respect to $x$ results in the following integral representation

$$
\begin{equation*}
\frac{d u(x)}{d x}=\frac{1}{E_{1} A} \int_{0}^{l} p_{x} \Lambda_{1 x}(r) d x+\left[\Lambda_{1 x}(r) \frac{d u}{d x}\right]_{0}^{l} \tag{38}
\end{equation*}
$$

Eqn (36) can give the beam axial displacement at any interior point if the two unknown quantities, i.e. $u, d u / d x$ at the beam ends, are first established. Eqn (38) written for the boundary points of the beam $z=0, l$ and the boundary condition (16) constitute a system of two simultaneous linear equations. These equations can be solved to yield the aforementioned required unknown quantities.
Subsequently, using the discretized form of eqn (36), the axial displacement at any interior point of the beam is computed as
$u(x)=B_{x}+\left\{\begin{array}{ll}\left\{A_{x 1}\right\} & \left\{A_{x 2}\right\}\end{array}\right\}\left\{\begin{array}{ll}\{u\} & \left\{u^{\prime}\right\}\end{array}\right\}^{T}$
while the beam axial force which at any interior point is given as
$N(x)=E_{1} A \frac{d u}{d x}$
is computed using the discretized form of eqn (38) as
$N(x)=D_{x}+\left\{A_{x 2}\right\}\left\{u^{\prime}\right\}^{T}$
where $\left\{A_{x i}\right\}(i=1,2)$ are $1 x 2$ known coefficient row matrices originating from the values of the kernels at the
beam ends, $\{u\},\left\{u^{\prime}\right\}$ are $1 x 2$ row matrices including the values of the boundary quantities $u, d u / d x$, respectively and $B_{x}, D_{x}$ are known coefficients arising from the axial loading of the beam $p_{x}$.
As it was already mentioned, the final equations of equilibrium of the plane frame constitute a nonlinear system of equations due to the presence of the unknown axial forces $N$. For the solution of this system an initial vector $\mathbf{N}^{(0)}=\mathbf{0}$ including zero nodal values of the axial forces is assumed. Using this vector and eqns. (34), (39), (41) the nodal values of the deflection $w$, of the axial deformation $u$ and of the axial force $N$ are obtained leading to the computation of the vector $\mathbf{N}^{(1)}$ (solution of the first-order theory). Subsequently the vector $\mathbf{N}^{(k)}, k \geq 2$ is obtained as
$N_{i}^{(k)}=a N_{i}^{(k-1)}+\beta N_{i}^{(k-2)}$
where $a+\beta=1$. The procedure converges to the solution vector $\mathbf{N}$ by choosing appropriately the weight factors $a$ and $\beta$.

### 3.3 For the stress function $\Phi(y, z)$

The evaluation of the stress function $\Phi(y, z)$ is accomplished using BEM as this is presented in Sapountzakis and Mokos [27].
Moreover, since the torsionless bending problem of beams is solved by the BEM, the domain integrals for the evaluation of the area (eqn. 4), the bending moments of inertia (eqn. 10) and the shear deformation coefficient (eqn. 19) have to be converted to boundary line integrals, in order to maintain the pure boundary character of the method. This can be achieved using integration by parts, the Gauss theorem and the Green identity. Thus, the moments of inertia and the cross section area can be written as
$I_{y}=\frac{1}{E_{1}} \sum_{j=1}^{K} \int_{\Gamma_{j}}\left(E_{j}-E_{i}\right)\left(y z^{2} \cos \beta\right) d s$
$I_{z}=\frac{1}{E_{1}} \sum_{j=1}^{K} \int_{\Gamma_{j}}\left(E_{j}-E_{i}\right)\left(z y^{2} \sin \beta\right) d s$
$A=\frac{1}{2 G_{1}} \sum_{j=1}^{K} \int_{\Gamma_{j}}\left(G_{j}-G_{i}\right)(y \cos \beta+z \sin \beta) d s$
while the shear deformation coefficient $a_{z}$ is obtained from the relation
$a_{z}=\frac{A}{E_{1} \Delta^{2}}\left((4 v+2) I_{z} I_{\Phi z}+\frac{1}{4} v^{2} I_{z}^{2} I_{e d}-I_{\Phi d}\right)$

(a)

(b)

Figure 3 : Structural model (a) and transverse span section (b) of the two span plane frame of example 1.
where

$$
\begin{align*}
& I_{\Phi d}=\sum_{j=1}^{K} \int_{\Gamma_{j}}\left(E_{j}-E_{i}\right)(\Phi)_{j}(n \cdot d) d s  \tag{45a}\\
& I_{e d}=\sum_{j=1}^{K} \int_{\Gamma_{j}}\left(E_{j}-E_{i}\right) \\
& \quad\left(y^{4} z \sin \beta+z^{4} y \cos \beta+\frac{2}{3} y^{2} z^{3} \sin \beta\right) d s  \tag{45b}\\
& I_{\Phi z}=\frac{1}{6} \sum_{j=1}^{K} \int_{\Gamma_{j}}\left(E_{j}-E_{i}\right) \\
& \quad\left[-2 I_{z z} z^{4} y \cos \beta+\left(3 \Phi_{j} \sin \beta-z(\mathbf{n} \cdot \mathbf{d})\right) z^{2}\right] d s \tag{45c}
\end{align*}
$$

## 4 Numerical examples

On the basis of the analytical and numerical procedures presented in the previous sections, a computer program has been written and representative examples have been studied to demonstrate the efficiency and the range of applications of the developed method. In all the examined cross sections steel with $E_{s}=210 \mathrm{GPa}, G_{s}=81 \mathrm{GPa}$ and concrete $\mathrm{C} 20 / 25$ with $E_{c}=29 \mathrm{GPa}, G_{c}=12.08 \mathrm{GPa}$ materials have been used.

### 4.1 Example 1

A two span plane frame of a composite span cross section ( $a_{z}=9.4128$ ) consisting of a steel IPE200 ( $A_{s}=28.49 \mathrm{~cm}^{2}, \quad I_{y}^{s}=1943.0 \mathrm{~cm}^{4}, \quad I_{z}^{s}=142.42 \mathrm{~cm}^{4}$ ) firmly bonded with a rectangular concrete C20/25 ( $A_{c}=$ $\left.2700.0 \mathrm{~cm}^{2}, I_{y}^{c}=50630.0 \mathrm{~cm}^{4}, I_{z}^{c}=7290000.0 \mathrm{~cm}^{4}\right)$ and of a steel HEB200 ( $a_{z}=4.690, A_{s}=78.08 \mathrm{~cm}^{2}, I_{y}^{s}=$ $\left.5695.0 \mathrm{~cm}^{4}, I_{z}^{s}=2004.0 \mathrm{~cm}^{4}\right)$ column cross section subjected to concentrated loading, as shown in Fig. 3 has been studied. In Table 1 the maximum displacements $u_{\text {max }}$ and $w_{\text {max }}$ along $x$ and $z$ axes, respectively for various column heights are presented as compared with those obtained ignoring or taking into account shear deformation and second-order effects. Moreover, in Table 2 the reactions $R_{x}, R_{z}$ and $M_{y}$ at $A, B, \quad C$ supports for two different column heights are presented as compared with those obtained ignoring or taking into account shear deformation and second-order effects. From both of the aforementioned tables the discrepancy of the results arising from the ignorance of shear deformation or the second-order effect is remarkable.

Table 1 : Maximum displacements (cm) for various heights of the two span plane frame of Example 1.

|  | $1^{\text {st }}$-order analysis $1^{\text {st }}$-order analysis ignoring shear def. with shear defor. |  | $2^{\text {nd }}$-order analysis $2^{\text {nd }}$-order analysis ignoring shear def. with shear defor. |  |
| :---: | :---: | :---: | :---: | :---: |
|  | $\mathrm{h}=3.0 \mathrm{~m}$ |  |  |  |
| $u_{\text {max }}$ | 0.524 | 0,581 | 0.535 | 0.595 |
| $w_{\text {max }}$ | -0.589 | -0,696 | -0.591 | -0.698 |
|  | $\mathrm{h}=5.0 \mathrm{~m}$ |  |  |  |
| $u_{\text {max }}$ | 2.33 | 2.43 | 2.46 | 2.56 |
| $w_{\text {max }}$ | -0.0410 | -0.0413 | -0.0413 | -0.0416 |
|  | $\mathrm{h}=7.0 \mathrm{~m}$ |  |  |  |
| $u_{\text {max }}$ | 6.11 | 6.26 | 6.80 | 6.98 |
| $w_{\text {max }}$ | -0.0633 | -0.0638 | -0.0644 | -0.0649 |

### 4.2 Example 2

A plane frame subjected to concentrated loading and having a composite span cross section ( $a_{z}=12.02689$ ) consisting of a steel HEA500 $\left(A_{s}=197.53 \mathrm{~cm}^{2}, I_{y}^{s}=\right.$ $86930.0 \mathrm{~cm}^{4}, I_{z}^{s}=10370.0 \mathrm{~cm}^{4}$ ) firmly bonded with a rectangular concrete $\mathrm{C} 20 / 25\left(A_{c}=6000.0 \mathrm{~cm}^{2}, I_{y}^{c}=\right.$ $0.002 \mathrm{~m}^{4}, I_{z}^{c}=0.45 \mathrm{~m}^{4}$ ) and a composite column cross section ( $a_{z}=1.502$ ) consisting of a steel HEB300 ( $A_{s}=$ $\left.149.07 \mathrm{~cm}^{2}, I_{y}^{s}=25160.0 \mathrm{~cm}^{4}, I_{z}^{s}=8564.0 \mathrm{~cm}^{4}\right)$ totally encased in a circular concrete C20/25 ( $A_{c}=2315.0 \mathrm{~cm}^{2}$, $\left.I_{y}^{c}=458500.0 \mathrm{~cm}^{4}, I_{z}^{c}=474180.0 \mathrm{~cm}^{4}\right)$, as shown in Fig. 4 has been studied. In Table 3 the maximum displacements $u_{\text {max }}$ and $w_{\text {max }}$ along $x$ and $z$ axes, respectively for various span lengths are presented as compared with those obtained ignoring or taking into account shear deformation and second-order effects. As it was expected from this plane frame the influence of second-order effect can be ignored, while that of shear deformation effect is remarkable, especially in short span length frames. This is clearly presented in Fig.5, where the error aris-
ing from the ignorance of the shear deformation effect is plotted with respect to the span length. Moreover, in Table 4 the reactions $R_{x}, M_{y}$ at $A, B$ supports for various span lengths are presented as compared with those obtained ignoring or taking into account shear deformation and second-order effects. From this table the aforementioned conclusions are once more verified.

### 4.3 Example 3

A two-storey plane frame of a composite span cross section ( $a_{z}=12.428$ ) consisting of a steel HEA320 $\left(A_{s}=124.36 \mathrm{~cm}^{2}, I_{y}^{s}=22920.0 \mathrm{~cm}^{4}, I_{z}^{s}=6987.0 \mathrm{~cm}^{4}\right)$ firmly bonded with a rectangular concrete $\mathrm{C} 20 / 25$ ( $A_{c}=$ $\left.4800.0 \mathrm{~cm}^{2}, I_{y}^{c}=0.0096 \mathrm{~m}^{4}, I_{z}^{c}=0.23 \mathrm{~m}^{4}\right)$ and of a steel HEB240 $\left(a_{z}=4.7533, A_{s}=105.99 \mathrm{~cm}^{2}, I_{y}^{s}=\right.$ $\left.11260.0 \mathrm{~cm}^{4}, I_{z}^{s}=3923.0 \mathrm{~cm}^{4}\right)$ column cross section subjected to concentrated loading multiplied by a magnification factor $\mu$, as shown in Fig. 6 has been studied. In Table 5 the maximum displacements $u_{\text {max }}$ and $w_{\text {max }}$ along $x$ and $z$ axes, respectively for various values of the load magni-

Table 2 : Reactions at A, B, C supports for various heights of the two span plane frame of Example 1.

|  | $1^{\text {st }}$-order analysis ignoring shear def. | $1^{\text {st }}$-order analysis with shear defor. | $2^{\text {nd }}$-order analysis ignoring shear def. | $2^{\text {nd }}$-order analysis with shear defor. |
| :---: | :---: | :---: | :---: | :---: |
|  | $\mathrm{h}=3.0 \mathrm{~m}$ |  |  |  |
| $R_{x}^{A}(\mathrm{kN})$ | 6.20 | 4.69 | 6.03 | 4.51 |
| $R_{x}^{B}(\mathrm{kN})$ | -26.37 | -26.12 | -26.03 | -25.75 |
| $R_{x}^{C}(\mathrm{kN})$ | -49.80 | -48.50 | -50.00 | -48.70 |
| $R_{z}^{A}(\mathrm{kN})$ | 119.10 | 119.50 | 118.80 | 119.20 |
| $R_{z}^{B}(\mathrm{kN})$ | 437.30 | 436.50 | 437.50 | 436.70 |
| $R_{z}^{C}(\mathrm{kN})$ | 143.60 | 144.00 | 143.80 | 144.10 |
| $M_{y}^{A}(\mathrm{kNm})$ | -7.78 | -11.09 | -8.34 | -11.72 |
| $M_{y}^{B}(\mathrm{kNm})$ | -40.26 | -39.99 | -40.89 | -40.70 |
| $M_{y}^{C}(\mathrm{kNm})$ | -63.60 | -61.00 | -64.40 | -61.90 |
| $\mathrm{h}=7.0 \mathrm{~m}$ |  |  |  |  |
| $R_{x}^{A}(\mathrm{kN})$ | -15.81 | -15.70 | -17.34 | -17.24 |
| $R_{x}^{B}(\mathrm{kN})$ | -25.05 | -25.02 | -22.73 | -22.68 |
| $R_{x}^{C}(\mathrm{kN})$ | -29.13 | -29.28 | -29.93 | -30.08 |
| $R_{z}^{A}(\mathrm{kN})$ | 93.68 | 94.99 | 90.53 | 91.75 |
| $R_{z}^{B}(\mathrm{kN})$ | 453.20 | 450.70 | 453.70 | 451.20 |
| $R_{z}^{C}(\mathrm{kN})$ | 153.10 | 154.30 | 155.70 | 157.00 |
| $M_{y}^{A}(\mathrm{kNm})$ | -66.58 | -66.65 | -75.16 | -75.42 |
| $M_{y}^{B}(\mathrm{kNm})$ | -88.13 | -88.15 | -95.27 | -95.53 |
| $M_{y}^{C}(\mathrm{kNm})$ | -97.63 | -97.97 | -106.20 | -106.80 |

fication factor $\mu$ are presented as compared with those obtained ignoring or taking into account shear deformation and second-order effects. Moreover, in Fig. 7 the influence of the second-order effect taking into account shear deformation and in Fig. 8 the influence of the shear deformation effect in both the first- and the second-order analysis in the maximum horizontal displacement $u_{\text {max }}$ is presented for various values of the load magnification factor $\mu$. From the aforementioned table and figures the
discrepancy of the results arising from the ignorance of shear deformation or the second-order effects is remarkable.

## 5 Concluding remarks

In this paper a boundary element method is developed for the second-order analysis of frames consisting of composite beams of arbitrary constant cross section, taking into account shear deformation effect. The composite

(a)
(b)
(c)

Figure 4 : Structural model (a) and transverse span (b) and column (c) sections of the plane frame of example 2.


Figure 5 : Error (\%) arising from the ignorance of the shear deformation effect for various span lengths of the plane frame of Example 2.
beam consists of materials in contact, each of which can surround a finite number of inclusions. The materials have different elasticity and shear moduli with same Poisson's ratio and are firmly bonded together. Each beam is subjected in an arbitrarily concentrated or distributed variable axial loading, while the shear loading is applied at the shear center of the cross section, avoiding in this way the induction of a twisting moment. To account for shear deformations, the concept of shear deformation coefficients is used. Three boundary value problems are formulated with respect to the beam deflection, the axial displacement and to a stress function and solved employing a pure BEM approach, that is only boundary discretization is used. The evaluation of the shear deformation coefficients is accomplished from the aforementioned stress function using only boundary integration. The main conclusions that can be drawn from this inves-


Figure 6 : Structural model (a) and transverse span (b) and column (c) sections of the two-storey plane frame of example 3.

## tigation are

1. The numerical technique presented in this investigation is well suited for computer aided analysis for composite beams of arbitrary cross section.
2. The significant influence of second-order analysis in plane frames subjected in intense axial loading is verified.
3. The discrepancy between the results of the first- and the second-order analysis demonstrates the significant influence of the axial loading.
4. The discrepancy in the obtained deformations and stress resultants arising from the ignorance of shear deformation effect is remarkable.
5. The developed procedure retains the advantages of a BEM solution over a pure domain discretization method since it requires only boundary discretization.

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Table 3 : Maximum displacements (cm) for various span lengths of the plane frame of Example 2.

|  | $1^{\text {st }}$-order analysis $\quad 1^{\text {st }}$-order analysis ignoring shear def. with shear defor. |  | $2^{\text {nd }}$-order analysis $2^{\text {nd }}$-order analysis ignoring shear def. with shear defor. |  |
| :---: | :---: | :---: | :---: | :---: |
|  | 1=3.0m |  |  |  |
| $u_{\text {max }}$ | 0.437 | 0.553 | 0.437 | 0.554 |
| $w_{\text {max }}$ | -0.225 | -0.569 | -0.225 | -0.569 |
|  | $1=5.0 \mathrm{~m}$ |  |  |  |
| $u_{\text {max }}$ | 0.495 | 0.569 | 0.496 | 0.570 |
| $w_{\text {max }}$ | -0.784 | -1.370 | -0.784 | -1.370 |
|  | 1=7.0m |  |  |  |
| $u_{\text {max }}$ | 0.570 | 0.633 | 0.571 | 0.634 |
| $w_{\text {max }}$ | -1.880 | -2.730 | -1.880 | -2.730 |



Figure 7 : Maximum horizontal displacement $u_{\text {max }}$ from first- and second-order analysis taking into account shear deformation for various values of the load magnification factor $\mu$ of the plane frame of Example 3 .

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Table 4 : Reactions at A, B supports for various span lengths of the plane frame of Example 2.


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Table 5 : Maximum displacements (cm) for various values of the load magnification factor $\mu$ of the two-storey plane frame of Example 3.


| $u_{\max }$ | 12.9 | 14.6 | 13.8 | 15.6 |
| :--- | :--- | :--- | :--- | :--- |
| $w_{\max }$ | 0.587 | 0.756 | 0.588 | 0.757 |

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(a)
(b)

Figure 8 : Shear deformation effect in maximum horizontal displacement $u_{\max }$ arising from first- (a) and secondorder (b) analysis for various values of the load magnification factor $\mu$ of the plane frame of Example 3 .

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