 SDHM, vol.8, no.1, pp.1-29, 2012

Development of a Modal Approach for the Fatigue Damage Evaluation of Mechanical Components Subjected to Random Loads

F.Cianetti[1]

Abstract: This research activity refers to the problem of fatigue damage evaluation of mechanical components subjected to random loads. In detail, the present paper describes a procedure, developed by the author, that, starting from component modal modelling, can very quickly gives an answer to the request not only of a qualitative evaluation of its stress state but also of a quantitative and very reliable estimation of the component damage. This estimation is available (both in time and in frequency domain), regardless of the stress state recovery, only by an appropriate elaboration of *lagrangian coordinates* and elements stress mode shapes. This allows for a quick assessment of the fatigue behaviour upon the whole model or just on an elements subset, prior to the exact evaluation of the damage that always requires a very high computational burden.

Keywords: Fatigue, random loads, dynamic simulation, modal modelling, modal combination, finite element analysis (FEA), multibody simulation (MBS).

1 Introduction

The present paper represents a further development of a well-established procedure for the evaluation of the fatigue behaviour of mechanical components subjected to random loads, focus of earlier author and co-authors reports [Braccesi and Cianetti (2004-2011)].

The already existing procedure, developed and usually applied by the author [Braccesi, Cianetti and Silvioni (2010), Braccesi, Cianetti, Lori and Pioli (2005)] for linear or non-linear mechanical system, is usable in both time and frequency domain [Braccesi and Cianetti (2011), Braccesi and Cianetti (2008)], starting from system operating conditions modelled and simulated both by multibody and by finite element approach. In both cases the fundamental hypotheses are: linear behaviour of

[1] Machine Design, Industrial Engineering Department, University of Perugia, via G.Duranti – 67, 06125 Perugia (Italy), Tel. +39 075 5853728, E-mail: cianfi@unipg.it

the analysed component and modal modelling approach of component itself [Braccesi and Cianetti (2004)] starting from its finite element modelling; any possible non-linear component behaviour (that the procedure could evaluate and analyse) are exclusively due to the model one, that is to the analysed mechanical system.

The focus of this memory is to improve an evaluation method of the stress state of components under generic random loads. The previous method [Braccesi, Cianetti and Landi (2005)] allows a fast but qualitative identification of the mostly stressed locations.

In the present work, a new method was developed in order to obtain not only a qualitative evaluation of the stress state of the elements but a very reliable evaluation of the component damage condition, starting from the modal response of *lagrangian coordinates* [Braccesi and Cianetti (2004)]. This is obtainable, regardless of the single element stress state recovery (both in time and in frequency), only by an appropriate elaboration of *lagrangian coordinates* and of the elements modal stress shapes. This allows for a quick assessment of the fatigue behaviour upon the whole model or just on an elements subset, prior to the exact evaluation of the damage, that always requires a very high computational burden. This problem is quite relevant when the analysed models have huge computational dimensions, with a great number of nodes and elements and subjected to multiple random load conditions.

The described method has been verified by analysing, as a test case, a complex multibody model of a military device subjected to random loads condition. In particular the fatigue behaviour of one of the components was analysed. The input loads were defined by power spectral density functions (PSD). The goodness of method was validated, both in time and in frequency domain, by comparing the results obtained through the proposed approach versus the results obtained through the standard method [Braccesi, Cianetti, Lori and Pioli (2008), Braccesi, Cianetti and Landi (2005), Braccesi, Cianetti, Lori and Pioli (2005)].

2 State of the art in the simulation of random load conditions

The procedures for the fatigue behaviour evaluation of components, developed by the author and usually adopted in basic research activities [Braccesi and Cianetti (2011), Braccesi, Cianetti, Lori and Pioli (2009), Braccesi, Cianetti, Lori and Pioli (2008), Braccesi, Cianetti and Landi (2005), Braccesi and Cianetti (2005), Braccesi, Cianetti, Lori and Pioli (2005)] and in industrial applications [Braccesi, Cianetti and Silvioni (2010), Braccesi, Cianetti, Lori and Pioli (2005)], need, as aforesaid in introduction of the paper, a modal modelling of the component, that is its linear one, starting from its finite element modelling.

Even if component could be inserted into a complex model of non-linear behaviour, as multibody one (MBS) [Schiehlen (1997), Shabana (1997), Shabana (1998), Braccesi and Cianetti (2011), Braccesi, Cianetti and Silvioni (2010), Braccesi, Cianetti, Lori and Pioli (2009), Braccesi, Cianetti and Landi (2005), Braccesi and Cianetti (2005), Braccesi, Cianetti, Lori and Pioli (2005)], or just into a finite element model (FEM), and analyzable in static and/or dynamic conditions (FEA), the recovery of the stress state, but not solely of the stress state, is obtainable, by definition, through an element by element linear combination of the modal stress tensor and *lagrangian coordinates* amplitude, expressed both in time (time histories) and in frequency (Power Spectral Density functions) domain: *modal combination* [Shabana (1997), Braccesi and Cianetti (2005), Braccesi and Cianetti (2004)]. The components modal modelling typically means the realization of the component FE model and the generation of the modal model by finite elements calculations (i.e. Modal Analysis, *Component Mode Synthesis* [Craig and Bampton (1968), Shabana (1997), Braccesi and Cianetti (2004)]).

In general, the modal model is used in several ways and in two principal computing environments: finite element analysis (FEA) and dynamic multibody analysis (MBS). It must be noticed that the time domain analysis, even if it allows to analyse the system (MBS) or the single component (FEA), allowing to consider the non linear behaviour of the system and/or of the component and to neglect the *Gaussian* hypothesis of the stress processes, typical of the fatigue frequency analysis approach, has an high level of burden from a computational point of view. It needs input processes characterized by a time sample of a very high numerousness to obtain a statistically significant sample of the outputs. This implies long computational time for the dynamic transient analysis and for the post processing of the results (i.e. *RainFlow counting* [Murakami (1992), Rychlik (1987), Collins (1992)]).

As concerns FE analysis, the simplest way, but of difficult managing if seen from the durability (fatigue) analysis point of view, is that to perform the whole analysis into this environment [Bishop and Sherratt (2000)]. It is possible to run both transient dynamic analysis (in time domain, with loads and/or imposed motions time histories as inputs) and spectral dynamic analysis (in frequency domain, with a power spectral density functions matrix as input, of dimensions equal to the considered inputs number and expressed in terms of loads and/or imposed motions). These two approaches are based on modal combination rule, in order to obtain, for each element and for each component of the stress tensor, the stress state in time domain (time histories) or in frequency domain (power spectral density functions) [Braccesi, Cianetti and Landi (2005), Braccesi and Cianetti (2005)]. In this way, the obtained results files have very big size and are difficult to manage. However, it is possible to export from this environment the results (even if with an heavy work) and to evaluate for each element the fatigue damage by using any approach [Socie (2000), You and Lee (1996), Susmel (2002), Pioli (2005), Lori (2005)].

Another possibility, not findable in every FE commercial codes, is the one that allows the user to export simulations results in terms of *lagrangian coordinates* (time histories or power spectral density functions). The possibility to externally manage the results allows to evaluate the multiaxial stress conditions (i.e. evaluation in frequency domain of the *Preumont* equivalent stress [Pitoiset and Preumont (2000), Braccesi, Cianetti and Landi (2005)], evaluation in time domain of *Braccesi* et *al.* equivalent stress [Braccesi, Cianetti, Lori and Pioli (2008)], use of the classical criterion of multiaxial fatigue [Socie (2000), You and Lee (1996), Susmel (2002), Pioli (2005), Lori (2005)]) and the fatigue behaviour with high accuracy. This kind of calculation is easy to manage during the fatigue verification process and it gives back exclusively the component response in terms of *lagrangian coordinates* (always fewer than the elements number); in this way the analyst is free to carry out the modal combination for all or just for an elements subset, for all or just for a stress tensor subset, and to synthesized or to analyze the stress tensor through any possible fatigue evaluation criterion.

The most light analysis approach is, from the author point of view, the one that externally creates the model modal image in a *state-space* (SS) form [Braccesi and Cianetti (2011), Braccesi, Cianetti and Landi (2005), Ogata (2002), Harris and Piersol (2002), Bendat and Piersol (2000)]. This analysis approach assumes, as hypotheses, that the input variables were the model inputs (such as loads or/and imposed motions), the outputs were the *lagrangian coordinates* of the modal model and that the system behaviour was linear. This approach is manageable by using the same numerical calculation environment (FEA) through the exporting of few and simple parameters as the model natural frequencies, the modal participation factors [Bendat and Piersol (2000)] in case of imposed motions as input, the displacement modal shapes in case of loads as inputs (evaluated in correspondence of the degrees of freedom where the loads will be applied); all the parameters are automatically obtained from the analysis previous step (building of the modal model). In this case, according to the sub structuring logic [Bendat and Piersol (2000)], the construction of the standard *state-space* matrices of the model (**A**, **B**, **C**, **D** [Braccesi and Cianetti (2011), Braccesi, Cianetti and Landi (2005), Ogata (2002), Bendat and Piersol (2000), Harris and Piersol (2002)]) is needed in an external numerical calculation environment (i.e. MATLAB). This *state-space* system is easily analyzable both in time (transient integration of linear system) and in frequency domain (simple matrices products) by commonly used numerical calculation environments (i.e. MATLAB) or by compiled codes such as that developed by the author and others researchers in previous research activities [Braccesi, Cianetti, Lori and Pioli

(2005)].

The multibody simulation (MBS) environment has the greatest potentialities in order to analyze a linear or non-linear system. Under the implicit hypotheses of linear behaviour of the flexible body (component) and generic behaviour, also strongly non-linear, of the system, the author, in previous activities, has developed some simulations techniques. These are based on the Multi Input and Multi Output (MIMO) theory [Braccesi and Cianetti (2011), Braccesi, Cianetti and Landi (2005), Ogata (2002), Bendat and Piersol (2000), Harris and Piersol (2002)], adapted to *state-space* systems, and allow to easily obtain the previously defined *state-space* matrices (**A**, **B**, **C**, **D**) directly from this calculation environment. These matrices are exported both as one-off and as statistical sample [Braccesi and Cianetti (2011), Braccesi, Cianetti, Lori and Pioli (2009)] in order to obtain the component response in terms of modal coordinates both in time and in frequency domain. The exporting procedure of the matrices is present in almost all of the commercial numerical multibody codes and, in part, it was developed by the author, as an innovative modality, in the commercial code ADAMS/View [Braccesi, Cianetti and Landi (2005)], in particular, for the part concerning the definition of the system outputs as the *lagrangian coordinates* of the flexible component.

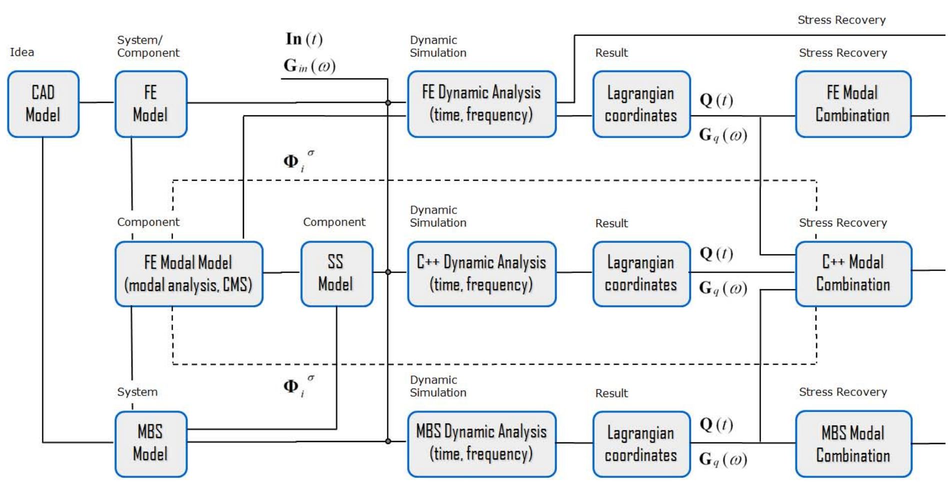

Figure 1: Dynamic simulation scenario

The dynamic simulation scenario is shown in figure 1.

3 Development of the proposed damage evaluation method

The base assumption of this research activity is the knowledge of the component response in terms of modal coordinates (*lagrangian*) expressed in time or in frequency domain. Regardless of how they were obtained, the procedure, developed by the author, is based on an elaboration of these parameters, opportunely combined with the associated stress modal shapes.

By considering a modal model characterized by *m lagrangian coordinates* under a random load condition, if this condition is expressed in the multi input form (MI) of a power spectral density functions matrix $\mathbf{G}_{in}(\omega)$ $(n\times n)$, it is possible to obtain, in the multi output form (MO), an output matrix as $\mathbf{G}_q(\omega)$ $(m\times m)$, i.e. the so called power spectral density functions matrix of the *lagrangian coordinates* (*Hermitian* matrix). Otherwise, if the inputs are defined in time domain, through a multi input form (MI) as a time histories matrix $\mathbf{In}(t)$ $(t\times n)$ of t instants, it is possible to obtain, in the multi output form (MO), a time histories matrix $\mathbf{Q}(t)$ $(t\times m)$ of the *lagrangian coordinates*. For $\mathbf{G}_{in}$, $\mathbf{G}_q$ and for the matrices subsequently defined in the frequency domain, the third dimension, deliberately not indicated, is associated to the frequency vector dimension.

In paragraph 3.1 the damage calculation methodologies considered as reference are shown. In the following paragraph (3.2) the methodology proposed by the author is illustrated.

3.1 Description of the reference damage evaluation method

The reference damage evaluation procedure can be considered as a cycle of operations (it will be subsequently described) able to iterate and analyze all the considered elements set. In frequency domain, being available the modal matrix of the modal stress tensor of the model, the power spectral density functions matrix of the stress tensor for the *i-th* element $\mathbf{S}_i(\omega)$ (6×6) will be obtained by a simple matrices product:

$$\mathbf{S}_i(\omega) = \mathbf{\Phi}_i^{\sigma} \cdot \mathbf{G}_q(\omega) \cdot \mathbf{\Phi}_i^{\sigma^t} \tag{1}$$

Where $\mathbf{\Phi}_i^{\sigma}$ is the modal matrix of the *i-th* element, expressed in stress terms (with dimensions $(6\times m)$), and the superscript ^{t} is the matrix transpose operator.

About the *j-th* mode, the element stress modal shape, composed by the 6 classical components of stress state, is expressed as below:

$$\mathbf{\Phi}_{i,j}^{\sigma} = \{s_x\ s_y\ s_z\ s_{xy}\ s_{xz}\ s_{yz}\}_{i,j}^{t} \tag{2}$$

Otherwise, if the analysis is developed in time domain, being available the modal matrix of the modal stress tensor of the model, the stress state will be expressed by a matrix $\mathbf{S}_i(t)$ ($t\times6$) derived from the linear combination of the $\mathbf{Q}(t)$ ($t\times m$) and $\mathbf{\Phi}_i^\sigma$ ($6\times m$):

$$\mathbf{S}_i(t) = \mathbf{Q}(t)\cdot\mathbf{\Phi}_i^{\sigma'} \tag{3}$$

In the methodology considered as reference by the author, the stress state, generically multiaxial, is synthesized, both in time and frequency domain, as an equivalent uniaxial stress state. In time domain the *Braccesi* et *al.*'s synthesis [Braccesi, Cianetti, Lori and Pioli (2008), Lori (2005)] is considered. It gives back a single time history $s_B(t)$ representative of the multiaxial stress state time history. Instead, in frequency domain the *Preumont* et *al.*'s approach is taken into account. It gives back an equivalent stress state expressed through a single power spectral density function $\mathrm{G}_\sigma(\omega)$ called "*equivalent von Mises stress* (*EVMS*)" [Pitoiset and Preumont (2000), Braccesi, Cianetti and Landi (2005)]. For a detailed description of the two mentioned approaches it is advised to see the cited papers in references.

Even if the author has chosen to adopt the equivalent uniaxial stress state hypothesis other multiaxial criteria (e.g. critical plane or invariants-based approaches) could be also applied in principle as "reference". The choice to use this hypothesis and the above two approaches is only due to the author's desire to obtain results more easily comparable to those of the proposed approach.

Starting from $s_B(t)$ and $\mathrm{G}_\sigma(\omega)$ it is possible to obtain the alternating stress state load spectrum [Collins (1992)] in time domain, through the classical cycles counting method called *Rain Flow* [Murakami (1992), Rychlik (1987), Collins (1992)] (*RFC*), and in frequency domain, by the *Dirlik*'s [Dirlik (1985), Braccesi, Cianetti, Lori and Pioli (2005), Lori (2005)] or by the others, so called, *direct approaches* [Braccesi, Cianetti, Lori and Pioli (2005), Lori (2005)]. However, in this work, only the *Dirlik*'s approach has been taken in to account. About the load spectrum evaluation of the alternating stress state, derived from the *Rain Flow Counting* matrix, the *Goodman* simplified criterion [Collins (1992)], to consider the effect of the stress mean component s_{m_k} of the single *k-th* cycle, has been adopted. No correction was adopted for Dirlik's method. The influence of mean stress was considered only in time domain approach.

For each *i-th* element it is possible to obtain a different load spectrum expressed in terms of alternating stress $(\mathbf{s}_a, \mathbf{n})_i$ and characterized by a different total cycles number n_{t_i}. $\mathbf{s}_a$ and $\mathbf{n}$ are respectively the vectors of dimensions ($R\times1$) of alternating stresses and of the cycles, counted for each class k of the spectrum.

Then, the "true" damage D_i^t for the *i-th* element is evaluated by using *Palmgren-*

Miner's rule [Collins (1992)] for both calculation domains, through the equation:

$$D_i^t = \sum_{k=1}^{R} \left[(n_{a_k})_i / \sqrt[\beta]{\frac{(s_{a_k})_i}{\alpha}} \right] \tag{4}$$

The fatigue strength curve is defined through the following equations, respectively in terms of alternating strength stress and in terms of endurance cycles number:

$$S_f = \alpha \cdot n^\beta \quad N = \sqrt[\beta]{\frac{s_a}{\alpha}} \tag{5}$$

where n is the applied cycles number, S_f is the alternating strength stress, N is the endurance cycles number related to a given value of alternating stress s_a and α, β are the classical parameters of the curve, respectively, curve intercept for $N = 1$ and curve slope.

In this way the *Corten* and *Dolan's* hypothesis [Corten and Dolan (1956)] is adopted. For material as steel, characterized by a strength curve with double slope, this hypothesis considers the slope constant for all the cycles range, allowing a conservative damage evaluation in regard to the classical *Miner's* rule [Collins (1992)] and to the *Haibach's* hypothesis [Haibach (2002)].

Thus, it is possible to define, element by element, an equivalent stress condition characterized by a "flat" load spectrum (i.e. characterized by a single class of constant alternating stress amplitude for all the cycles (6), (7)) in order to realize the same "true" damage D_i^t . This equivalent stress value will be a function of the considered cycles numbers. The latter value can be evaluated as the cycles number effective "counted" for that element, $\bar{n}_{t_i}$ (different for each element) (6), or equal to a reference cycles number arbitrary defined by the user $\tilde{n}_t$ (for example equal for all the elements) (7).

$$\bar{\sigma}_{a_i} = \alpha \cdot \left[\bar{n}_{t_i} / D_i^t\right]^\beta \tag{6}$$

$$\tilde{\sigma}_{a_i} = \alpha \cdot \left[\tilde{n}_t / D_i^t\right]^\beta \tag{7}$$

In upper flow chart of figure 2, a synthetic block diagram of the reference method is shown.

3.2 Development and description of the proposed method

If the modal simulation (modal combination) is adopted, each variation in the element stress state (i.e. each stress state alternation) is exclusively consequence of the component *langrangian coordinates* variation. The stress modal shapes are

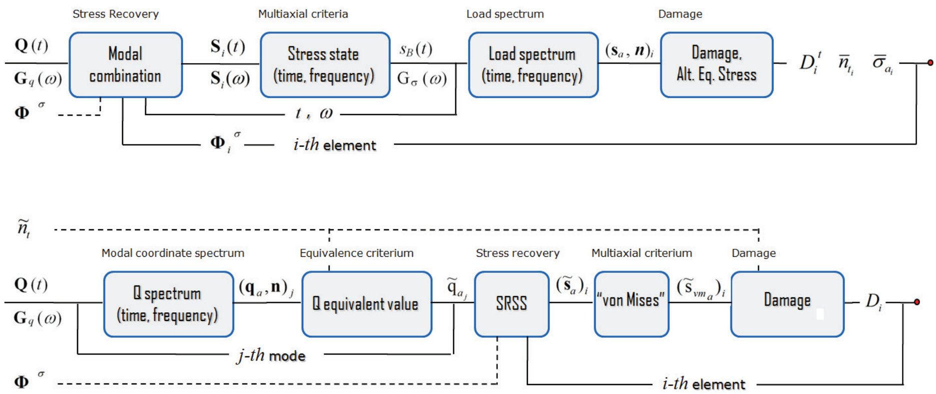

Figure 2: Methods flow charts. Reference method (upper diagram) vs. proposed method (lower diagram)

scalar objects with sign and time invariant, instead the modal coordinates are time variables with sign. For instance, if a simple modal model, endowed with only one normal mode (j) and with a single element (i) characterized by only one component of modal stress tensor different from zero (i.e. s_{x_i}), is considered, the previous assertion is so clear that it is possible to say, according to definition (3), that the stress alternating component in x direction, S_{x_a}, or its root mean square value, is equal to the *lagrangian coordinate* alternating component Q_{j_a}, or its root mean square value, multiply by the modal stress magnitude (1), (3).

The basic idea was to try to elaborate the *lagrangian coordinates* both in time and in frequency domain with the purpose of to obtain, for each coordinate, an its equivalent value in terms of damaging potential. These values should opportunely be combined with the stress modal shapes to obtain a stress value, that was as equivalent alternating stress value, comparable to the result of (6) and (7).

By considering the previous limit case, that is a modal model with only one normal mode and a single element with only one modal stress tensor component unequal to zero, it is possible to hypothesize that the load spectrum $(\mathbf{s}_a, \mathbf{n})_i$ has been calculated (load spectrum for alternating stresses) starting from the single stress time history $\mathrm{S}_{x_i}(t)$ or from the single power spectral density function $\mathrm{S}_{x_i}(\omega)$. When the spectrum $(\mathbf{s}_a, \mathbf{n})_i$ is well-known, it is possible to write:

$$(\mathbf{s}_a, \mathbf{n})_i = |s_{x_i}| \cdot (\mathbf{q}_a, \mathbf{n})_j \tag{8}$$

Where $\mathbf{q}_a$ is the alternating components vector of the "counted" cycles of the *lagrangian coordinates*.

The cycles can be obtained both through *RFC* in time domain and through *Dirlik's* theory in frequency domain. Under these conditions: $(\mathbf{n})_i = (\mathbf{n})_j$.

Without the necessity that the damage has to be evaluated, an equivalence can be done between the "true" load (stress) spectrum and a dummy load spectrum, for example a flat one, that is a spectrum characterized by a single class, using a constant alternating stress value for all cycles.

$$D_i^t = \sum_{k=1}^{R} \left[(n_k)_i / \sqrt[\beta]{\frac{(\mathrm{s}_{a_k})_i}{\alpha}} \right] = \bar{n}_{t_i} / \sqrt[\beta]{\frac{\bar{\sigma}_{a_i}}{\alpha}} = \tilde{n}_t / \sqrt[\beta]{\frac{\tilde{\sigma}_{a_i}}{\alpha}} \tag{9}$$

For example if a reference cycles number $\tilde{n}_t$ is considered:

$$\sum_{k=1}^{R} \left[(n_k)_i / \sqrt[\beta]{(\mathrm{s}_{a_k})_i} \right] = \tilde{n}_t / \sqrt[\beta]{\tilde{\sigma}_{a_i}} \tag{10}$$

$$\tilde{\sigma}_{a_i} = \left\{ \tilde{n}_t \cdot \sum_{k=1}^{R} \left[\sqrt[\beta]{(\mathrm{s}_{a_k})_i} / (n_{a_k})_i \right] \right\}^{\beta} \tag{11}$$

If we rewrite the previous equations in terms of modal approach, analogous expressions will be obtained in which D_i is the "modal" damage:

$$D_i^t = \sum_{k=1}^{R} \left[(n_k)_i / \sqrt[\beta]{\frac{(\mathrm{s}_{a_k})_i}{\alpha}} \right] = \sum_{k=1}^{R} \left[(n_k)_j / \sqrt[\beta]{\frac{|s_{x_i}|\,(\mathrm{q}_{a_k})_j}{\alpha}} \right] = \tilde{n}_t / \sqrt[\beta]{\frac{|s_{x_i}|\,\tilde{\mathrm{q}}_{a_j}}{\alpha}} = D_i \tag{12}$$

$$\sum_{k=1}^{R} \left[(n_k)_j / \sqrt[\beta]{|s_{x_i}|\,(\mathrm{q}_{a_k})_j} \right] = \tilde{n}_t / \sqrt[\beta]{|s_{x_i}|\,\tilde{\mathrm{q}}_{a_j}} \tag{13}$$

$$\tilde{\mathrm{q}}_{a_j} = \left\{ \tilde{n}_t \cdot \sum_{k=1}^{R} \left[\sqrt[\beta]{(\mathrm{q}_{a_k})_j} / (n_{a_k})_j \right] \right\}^{\beta} \tag{14}$$

and by comparing equations (14) and (11) the following equation can be written:

$$\tilde{\sigma}_{a_i} = |s_{x_i}| \cdot \tilde{\mathrm{q}}_{a_j} = \tilde{\mathrm{s}}_{a_i} \tag{15}$$

It is important to highlight that these results are obtained for a given cycles number equal to $\tilde{n}_t$.

This result, even if obtained for a limit case, is very important, because it asserts that in case of a w elements model (i.e. 200.000) and a single *lagrangian coordinate* the standard procedure imposes to extract w time histories or PSD functions and for each one of them to obtain, through w counting operations both in time and in frequency domain, w load spectra, necessary to extract the damage through linear accumulation rule (first part of equation number 9). Instead, the proposed procedure asks to only extract a single time history or a single PSD function and, as consequence, it needs just a single counting operation both in time and in frequency domain; in this way a single load spectrum is obtained. If a reference cycles number is given, it is possible to obtain the *single* combination factor through equation (14). When this factor is known, with a simple product (limit case of linear combination (3)), in a similar way of the time domain stress recovery operation for a single load step, it is possible to obtain the equivalent alternating stress state (15).

Thus the damage will be evaluated through:

$$D_i = \tilde{n}_t / \sqrt[\beta]{\frac{\tilde{s}_{a_i}}{\alpha}} \tag{16}$$

As concerns computational time and data storage, the advantage of this approach is clear.

Now, if a generalization logic process was followed from the limit case to the usual one, that is, if a component with a single element, with the whole modal tensor different from zero and characterized by m normal modes (*lagrangian coordinates*) was considered, which conclusion can we draw ?

The stress load spectra $(\mathbf{S}_a, \mathbf{n}_s)_i$, obtained, component by component, by the *Rain Flow* counting of each tensor component time history or obtained by the extraction of the probability density functions (*Dirlik*) starting from the power spectral density functions matrix of the stress tensor, is clearly different from those obtainable, by opportunely combining (*comb*) the *lagrangian coordinates* spectra with the relative modal stress tensor:

$$(\mathbf{S}_a, \mathbf{n}_s)_i \neq comb\left(\{s_x\ s_y\ s_z\ s_{xy}\ s_{xz}\ s_{yz}\}_{i,j} \cdot (\mathbf{q}_a, \mathbf{n})_j\right) \tag{17}$$

In the previous equation $(\mathbf{S}_a)_i$ represents the $(R\times 6)$ matrix of the alternating components of the stress tensor components and $(\mathbf{n}_s)_i$ represents the $(R\times 6)$ matrix of the cycles number associated to the stress alternating component, in general different component by component.

But, how much these two results are different ? Eventually, which kind of combination criterion minimizes this difference ? These are the questions which the author attempted to answer and meanwhile attempting to verify the partial conclusions extracted until now.

Now, if the limit case, in which a single stress component is considered, is extended to a general one, in which the whole modal stress tensor is different from zero, the result doesn't change. For each component of the stress tensor the equivalent stress can be obtained through (15).

By analyzing a different extension of the limit case, that considers a set of modal coordinates with a numerousness greater than one, it is possible to highligth the first justification of the inequality sign present in (17). To extract load spectrum from each modal coordinates and to linearly combine all these by using equation (3) is not equivalent to obtain the load spectrum starting from the stress history or from its PSD.

To justify the use of the linear combination of the modal coordinates spectra with the related stress mode shapes, the following hypothesis can be assumed: that the "true" load spectrum is independent from the mutual phase relations which, in a random process, characterize the frequency components of the signal and, moreover, that it is independent from the mutual phase relations of the *lagrangian coordinates*. This is always true when the modes are decoupled. In this way the damage accumulation is similar to a *sine sweep-test,* making the accumulation itself conservative. Another justification for the proposed approach, which is in tune with the paper principal aim, is that this operating modality allows to obtain a simple and fast fatigue damage calculation method, useful for the component design phase or for its preliminary verification.

If we accept to do a linear combination between the spectra and the modal shapes, we can do it by using their equivalent value $\tilde{\mathrm{q}}_{a_j}$(14). In order to realize this, a reference cycles number $\tilde{n}_t$, equal for all the coordinates, must be considered, thus to have modes contributes to the "true" stress that are congruent each other.

The combination method (*comb*), who better agrees to the hypothesis of modes decoupling and that has been adopted, is that proposed by *Gupta* [Gupta and Chen (1983)], also known as *SRSS* (Square Root of Sum of Square) and already adopted in previous work by the author [Braccesi, Cianetti and Landi (2005)].

$$(\tilde{\mathbf{s}}_a, \tilde{n}_t) = \sqrt{\sum_{j=1}^{m} (\tilde{\mathrm{q}}_{a_j} \cdot \mathbf{\Phi}_j^{\sigma})^2} \tag{18}$$

In the previous equation $(\tilde{\mathbf{s}}_a)$ is the matrix $(w \times 6)$ of the alternating equivalent values of the stress state components, that represents the result of the combination for all the elements. $\mathbf{\Phi}_j^{\sigma}$ is the $(w \times 6)$ stress modal matrix for the *j-th* mode. This combination is commonly used, it is extremely fast and implemented in many FE structural calculation environments.

The distribution of alternating equivalent values of the stress state components is the result of the combination. The distribution is associated to the same cycles numerousness (reference cycles number $\tilde{n}_t$). This allows to rapidly evaluate the safety factor between the obtained stress and the strength stress value corresponding to the reference cycles number.

Another problem is to compare, in multiaxial stress conditions, the alternating stress spectrum obtainable from the “true” synthesis, that leads to a uniaxial stress ($s_B(t)$ o $G_\sigma(\omega)$ [Braccesi, Cianetti, Lori and Pioli (2008), Braccesi, Cianetti and Landi (2005)]), with the six components spectrum, obtainable from the proposed modal combination process (18). Because of both *Preumont*’s criterion and *Braccesi*’s one follows *von Mises* stress approach (as the same authors assert in [Pitoiset and Preumont (2000), Braccesi, Cianetti, Lori and Pioli (2008), Braccesi, Cianetti and Landi (2005)]), the relationship between the “true” result, obtained from the uniaxial stress synthesis ($s_B(t)$ o $G_\sigma(\omega)$), and the representation of the stress state, obtained from the proposed combination $\tilde{\mathbf{s}}_{vm_a}$ ($w\times 1$), synthesized by *von Mises stress* operator *VM* is clear.

$$(\tilde{\mathbf{s}}_{vm_a})_i = VM\left(\sqrt{\sum_{j=1}^{m}(\tilde{\mathrm{q}}_{a_j}\cdot\mathbf{\Phi}_j^\sigma)^2}\right) \tag{19}$$

When the vector $\tilde{\mathbf{s}}_{vm_a}$ is known, it is very easy to evaluate the damage through equation (16).

The result of the present research activity is a method which demonstrates that in a model of *w* elements (i.e. 200.000) (see paragraph 4) and *m lagrangian coordinates* (i.e. 84) the standard procedure forces to extract *w* time histories or *w* stress tensor PSD matrices and then to synthesize *w* time histories or *w* uniaxial equivalent PSD functions in order to perform, for each one of them, *w* cycles counting operations in time or frequency domain, and to obtain *w* load spectra from which to evaluate the damage (first part of equation 9). Instead, the proposed method needs to extract just *m* time histories or PSD functions with consequent *m* cycles counting operations in time or in frequency domain, obtaining *m* load spectra. From these load spectra, through equation (14), it is possible to obtain the *m* combination factors, by assuming a reference cycles number. When these factors are known, with a single linear combination operation (18), in a similar way of the time domain stress recovery operation for a single load step, the alternating equivalent stress state is obtainable, expressed in terms of *von Mises* stress. The *i-th* element damage could be obtained starting from equation (16). The advantage (already highlighted for the limit case) in terms of computational time and data storage proves to be even more relevant.

Thus it is possible to define a safety factors distribution CS_{s_i}, expressed, for a single element, in terms of stress, through an easy ratio represented in the following equation:

$$CS_{s_i} = \alpha \cdot \tilde{n}_t^{\beta} / \tilde{\mathbf{s}}_{a_i} \tag{20}$$

In lower flow chart of figure 2, a synthetic block diagram of the proposed method is shown.

3.3 Speeding up of the proposed or reference method

In order to speed up the evaluation of the stress and/or damage and /or safety coefficient maps it is fundamental to know how many modes take part to the stress state and to select them. Generally the most part of the structural response is provided by the first normal modes, even if the type and characteristics of the inputs influence modes participation. The following criterion is proposed:

$$(\tilde{s}_{vm})_j = \tilde{\mathrm{q}}_{a_j} \cdot max\left(VM\left(\mathbf{\Phi}_j^{\sigma}\right)\right) \tag{21}$$

$(\tilde{s}_{vm})_j$ is the maximum contribute to the *von Mises* stress of the *j*-th mode, *VM* represents the *von Mises* operator and *max* means the evaluation of the maximum value extended to all the elements for the considered mode.

For each mode the maximum value of the *von Mises* stress is found and combined with the equivalent value $\tilde{\mathrm{q}}_{a_j}$ of the relative lagrangian coordinate. The result $(\tilde{s}_{vm})_j$ is the maximum contribute of that mode to the maximum stress response.

Repeating this operation for each mode it is possible to obtain a vector $\tilde{\mathbf{s}}_{vm}$ of $(m \times 1)$ dimensions that allows to select and chose the subset of z modes or *lagrangian coordinates* to be used to accomplish the combination operation for the reference method (equations (1) and (3)) or for the proposed one (18). This subset is always less numerous than the modes set interested by the dynamic analysis frequency range. This selection makes still faster the component fatigue behaviour evaluation under random loads.

4 Method validation

The proposed evaluation method has been verified by using as a test case a complex multibody model of a military shelter. In particular, the fatigue behaviour of one of its “smart” legs, used for the system stabilization, has been analyzed. This mechanical and electric device is made of AISI 304 steel and its fatigue behaviour had to be verified for transportation load condition i.e. under random loads applied for a time of 4 hours and 30 minutes and defined by power spectral density functions (PSD) expressed in terms of acceleration.

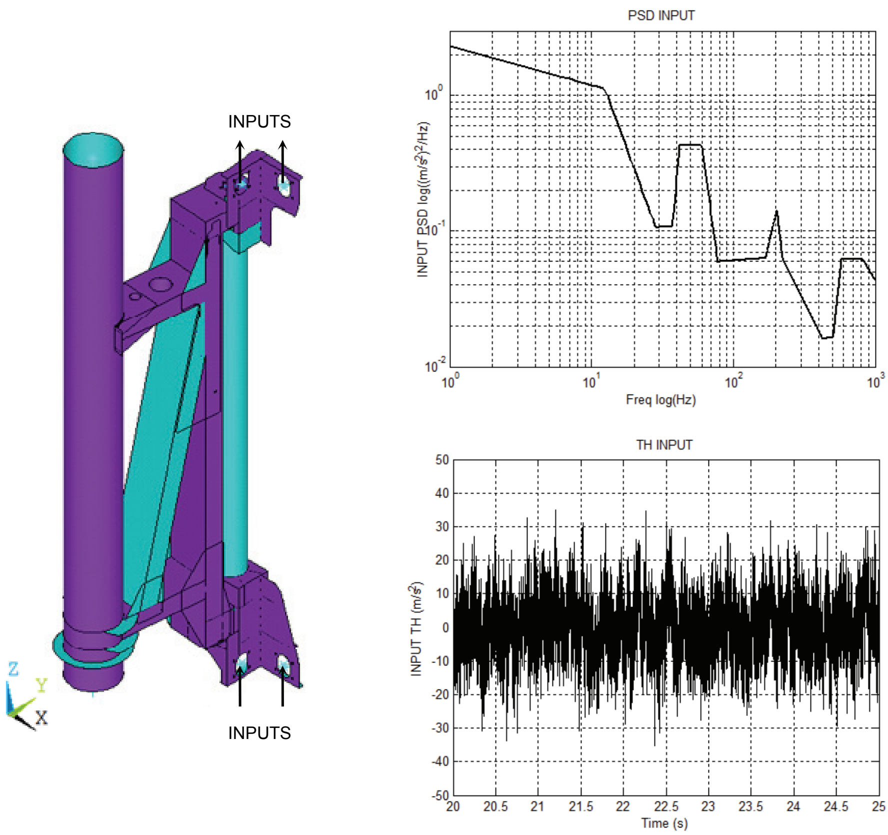

Figure 3: FE model and load conditions both in time and frequency domain. Input PSD function and a 5 second window of the associated input time history

The FE model (realized and analyzed in FEM/FEA Altair Hypermesh/Radioss environment) is characterized by 215.000 *shell* finite elements. In figure 3 the FE model is shown. In the same figure it is possible to see the application points of the accelerometric inputs that, in this test, are considered only along vertical direction. Even if the inputs are two, the loads are hypothesized completely correlated and considerable as a single input, characterized by a single power spectral density function $\mathbf{G}_{in}(\omega)$ (1×1), represented in figure 3, and defined in a frequency range from 1 to 1000 Hz.

The verification of the method goodness was conducted both in time and frequency domain, by comparing the proposed method and its results versus the results obtainable through the standard calculation, based on FEA approach.

A modal model, characterized by 84 modes (84 *lagrangian coordinates*) in the frequency range of interest, was built to represent the considered component. The fatigue strength curve adopted for the verification is characterized by α and β respectively equal to 2332.49 MPa and $-$ 0.2367.

The calculation in frequency domain was the first step. By subjecting the modal model to an imposed acceleration load condition (*base motion*), expressed through the previously described spectrum, the $\mathbf{G}_q(\omega)$ (84×84) *lagrangian coordinates* PSD matrix has been obtained.

For the reference calculation, the hints of paragraph 3.1 had to be done for each element (215.000 times !): the recovery of PSD stress matrix $\mathbf{S}_i(\omega)$, the generation of PSD function $\mathrm{G}_\sigma(\omega)$ (*Preumont*), the obtaining of load spectrum $(\mathbf{s}_a, \mathbf{n})_i$ (*Dirlik*) and the evaluation of the "true" damage D_i^t (*Palmgren-Miner* and *Corten-Dolan*). Then, for each element (215.000 times !), the equivalent stress was evaluated both by considering the counted cycles total number $\bar{n}_t$, obtained by *Dirlik's* method, ($\bar{\sigma}_{a_i}$) and by considering a reference cycles number $\tilde{n}_t$ constant for all elements and equal to 1×10^6 cycles ($\tilde{\sigma}_{a_i}$). This $\tilde{n}_t$ value is very close to the most of $\bar{n}_{t_i}$values, that is of cycles number effective "counted" for each element. The whole standard evaluation of component fatigue behaviour have been done by using a freeware software (*Fatigue*) built and developed in C++ environment by the author et al. in previous research activities [Braccesi, Cianetti, Lori and Pioli (2005)].

The use of the proposed procedure needed to obtain the load spectrum $(\mathbf{q}_a, \mathbf{n})_j$ (*Dirlik*) of each *lagrangian coordinate* (84 times!) starting from the corresponding diagonal term of $\mathbf{G}_q(\omega)$ matrix [Braccesi, Cianetti and Landi (2005)]. By choosing the above reference cycles number $\tilde{n}_t$ equal to 1×10^6 cycles, the *lagrangian coordinate* equivalent value $\tilde{\mathrm{q}}_{a_j}$ has been determined for each mode (84 times!). Through the modal combination *SRSS* the alternating equivalent stress state maps $\tilde{\mathbf{s}}_a$ has been obtained, by using the $\tilde{\mathrm{q}}_{a_j}$ vector, for each component of the stress tensor and, in particular, the map of ideal *von Mises* stress $\tilde{\mathbf{s}}_{vm_a}$ was evaluated. Through equation number (16) the "modal" damage map D_i has been obtained.

In figure 4 the main results and comparisons expressed in terms of damage and alternating equivalent stress are shown. In the left column the comparisons between the damage results of the two methods are shown (D_i^t vs. D_i). The damage values versus the element identification number (ID), the damage cumulative function and the damage distribution (histogram) are represented. In the right columns the same representations for the alternating equivalent stress ($\tilde{\sigma}_{a_i}$ vs. $(\tilde{\mathrm{s}}_{vm_a})_i$) are shown. The comparison attests the model goodness. It is appreciable the punctual correspondence among the damage and alternating equivalent stress values obtained through the two methods for each considered element as well as among the damage/stress distributions or among the damage/stress cumulatives.

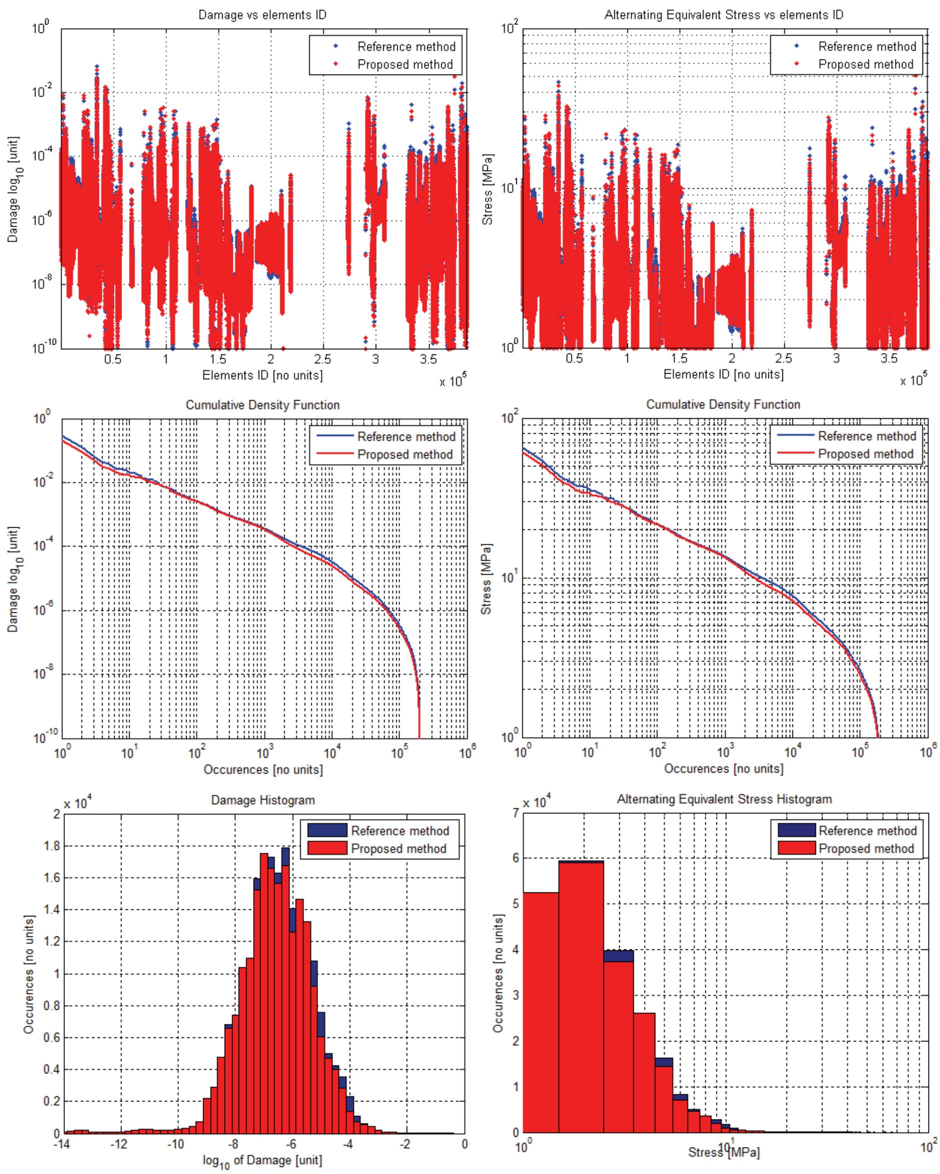

Figure 4: Results comparison in terms of damage D_i^t vs. D_i (left) and alternating equivalent stress $\tilde{\sigma}_{a_i}$ vs. $(\tilde{s}_{vm_a})_i$ (right). Frequency domain analysis. All elements.

The most stresses and damaged element is no. 374013. For this element the reference method obtains 64.5 MPa as alternating equivalent stress and a damage equal to 0.262; the proposed method obtains 62.8 MPa as alternating equivalent stress and a damage equal to 0.234. This is an more than acceptable result for an evaluation method of the preliminary design phase.

Table 1: Results summary in terms of CPU time

Analysis type	Reference method	Proposed method
Frequency domain analysis 84 modes 215.000 elements 0÷1000 Hz (sampled at 1 Hz)	32 [hours]	12 [minutes]
Time domain analysis 84 modes 70.000 elements 150 s (sampled at 1/2000 s)	48 [hours]	16 [minutes]

What is more clear about the usability of the proposed method is its computational speed. The post processing computational times, starting from the *lagrangian coordinates* evaluation, are the following: for the standard method 32 hours, for the proposed method 12 minutes (tab.1) !

In figures 5 and 6, a comparison between the maps of alternating equivalent stress and of damage, obtained by the two methods (reference and proposed ones), is shown. Both in figure 5 and 6, in the upper row, maps obtained by reference method are shown on all component and on two principal elements subsets. In the lower row, maps obtained by proposed method are shown. All the comparisons, in terms of maps, show the excellent agreement between the reference results and those obtained by the proposed methodology.

About the time domain analysis an input time history of 150 *s* (Fig. 3), sampled at a frequency of 2000 Hz, has been considered. It was reconstructed by the previous input PSD function (Fig. 3). Obviously this analysis has no meaning for the component fatigue verification because the load conditions should be kept for 4 hours and 30 minutes. Instead, it has meaning as verification of the proposed methodology in time domain. The procedure was applied to the modal model response time histories and compared with the reference approach results. For the reference model the previously described steps are still valid with the exception of the uniaxial stress evaluation, done through *Braccesi*'s approach, and its counting, that was done by *RainFlow counting*. Also for the proposed method there are no relevant differences

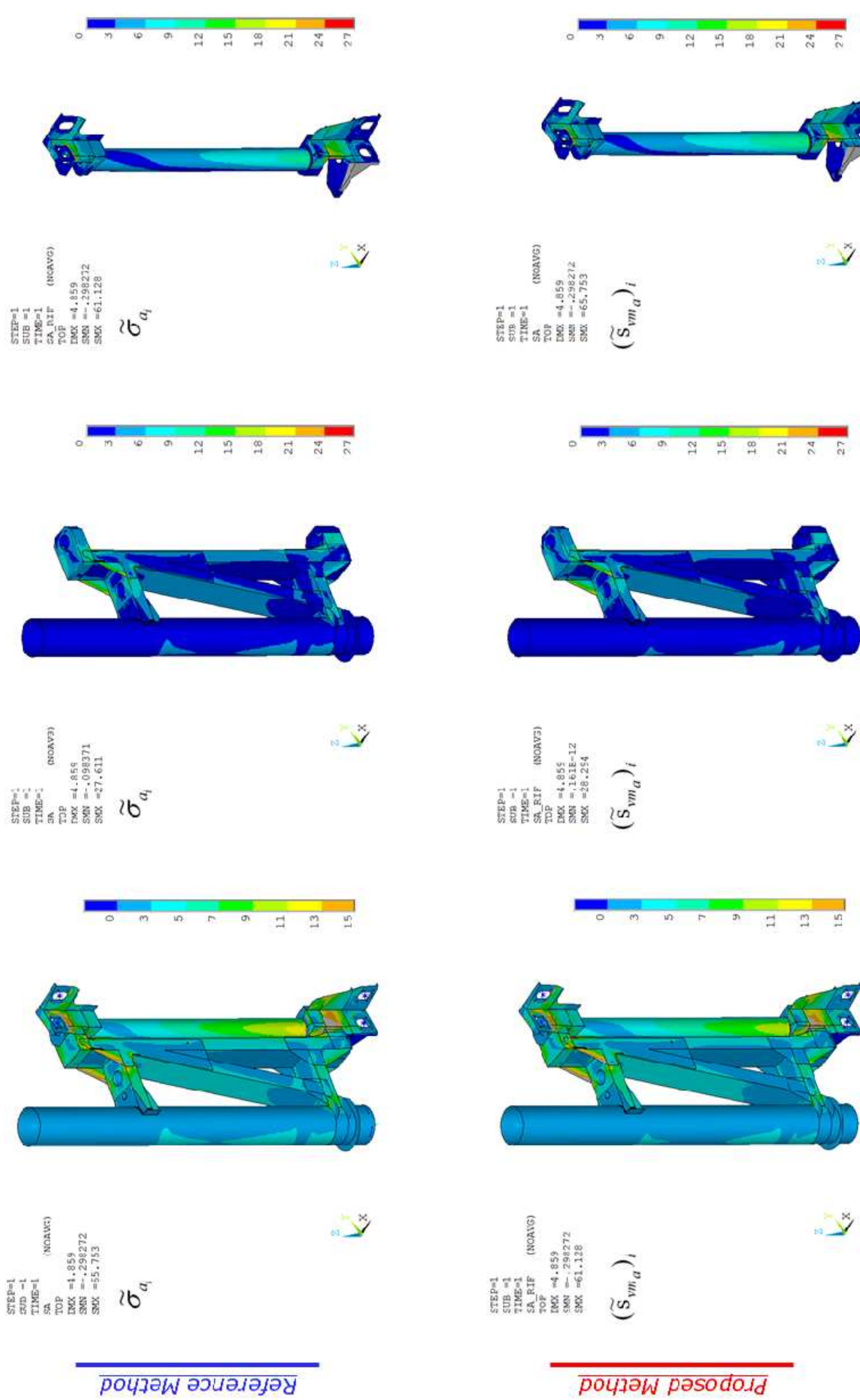

Figure 5: Comparison between alternating equivalent stress maps. Reference method (upper row) vs. Proposed method (lower row). Frequency analysis results

with the exception of the *lagrangian coordinates* cycle counting method that was a *RainFlow* approach.

Because of the short length of the analysis, a reference cycles number $\tilde{n}_t$ less than previous one and equal to 1×10^4 has been chosen. This $\tilde{n}_t$ value is very close to the most of $\bar{n}_{t_i}$ values, that is of cycles number effective "counted" for each element. In figure 7, a results report, similar to the one obtained through the frequency domain analysis, is shown. In order to reduce as much as possible the time domain analysis computational burden, an elements subset, equal to 1/3 (70.000 elements) of the total set, has been considered during the post processing process. The analysis results and the relative comments are comparable to the ones obtained for the frequency domain analysis. The correspondences among the damage and stress punctual values, element by element, and among the damage/stress distributions and the relative damage/stress cumulatives are excellent.

The most stresses and damaged element is no. 374013. For this element the reference method obtains 64.2 MPa as alternating equivalent stress and a damage equal to 0.00257; the proposed method obtains 63.1 MPa as alternating equivalent stress and a damage equal to 0.00239.

What is clear in this case too is the computational speed of the proposed method. The *RainFlow* counting penalizes even more the reference method respect the proposed one. The computational times of the only post processing phase, i.e. starting from the lagrangian coordinates evaluation, are: 48 hours for the standard method and 16 minutes for the proposed method (tab.1) !

The comparison between the maps of alternating equivalent stress and damage, obtained by the two methods (reference and proposed ones) for time domain analyses, is not reported in the paper, but all the comparisons, in terms of maps, show an excellent agreement between the reference and proposed results, just like those obtained for frequency domain analysis.

In figures from 8 to 10 a better representation of some of the obtained results is shown.

As concerns the reference method, in figure 8 the results relative to element no. 374013 are shown. In the left column frequency domain results are shown. In the right column time domain ones are illustrated. In the upper row PSD function and time history (a 5 second window) of the uniaxial stresses $s_B(t)$ and $\mathrm{G}_\sigma(\omega)$ are shown. The mid row shows the PDF function, obtained by *Dirlik* approach, and the *RainFlow* matrix (*from-to*). In the lower row the cumulatives are compared. In these last graphs, the equivalent values $\tilde{\sigma}_{a_i}$ of the stress state, obtained with the reference cycle number $\tilde{n}_t$ values (1×10^6 and 1×10^4 cycles), are shown together with the fatigue strength curve of the material.

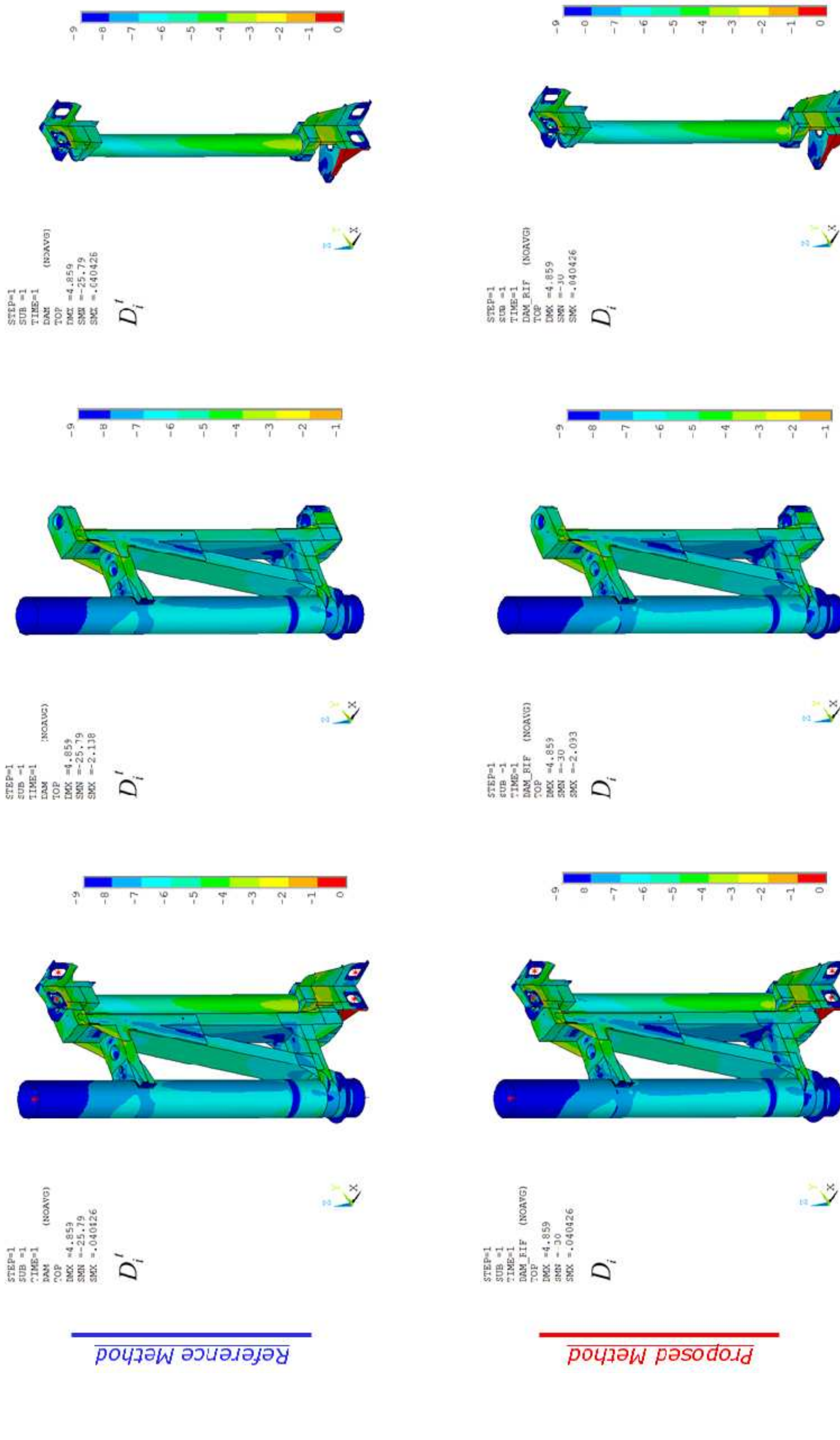

Figure 6: Comparison between damage maps ($\log_{10}$ values). Reference method (upper row) vs. Proposed method (lower row). Frequency analysis results

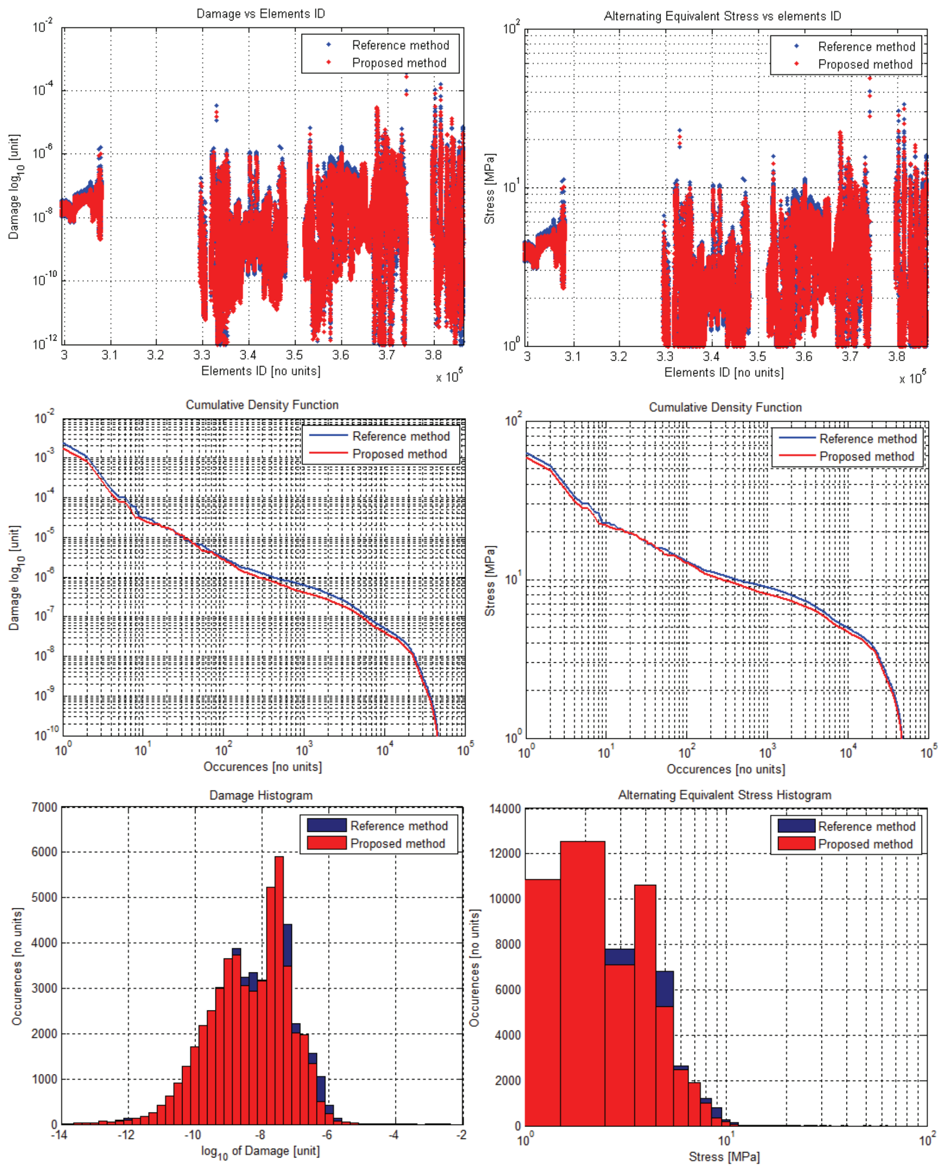

Figure 7: Results comparison in terms of damage D_i^t vs. D_i (left) and alternating equivalent stress $\tilde{\sigma}_{a_i}$ vs. $(\tilde{s}_{vm_a})_i$ (right). Time domain analysis. Subset of elements.

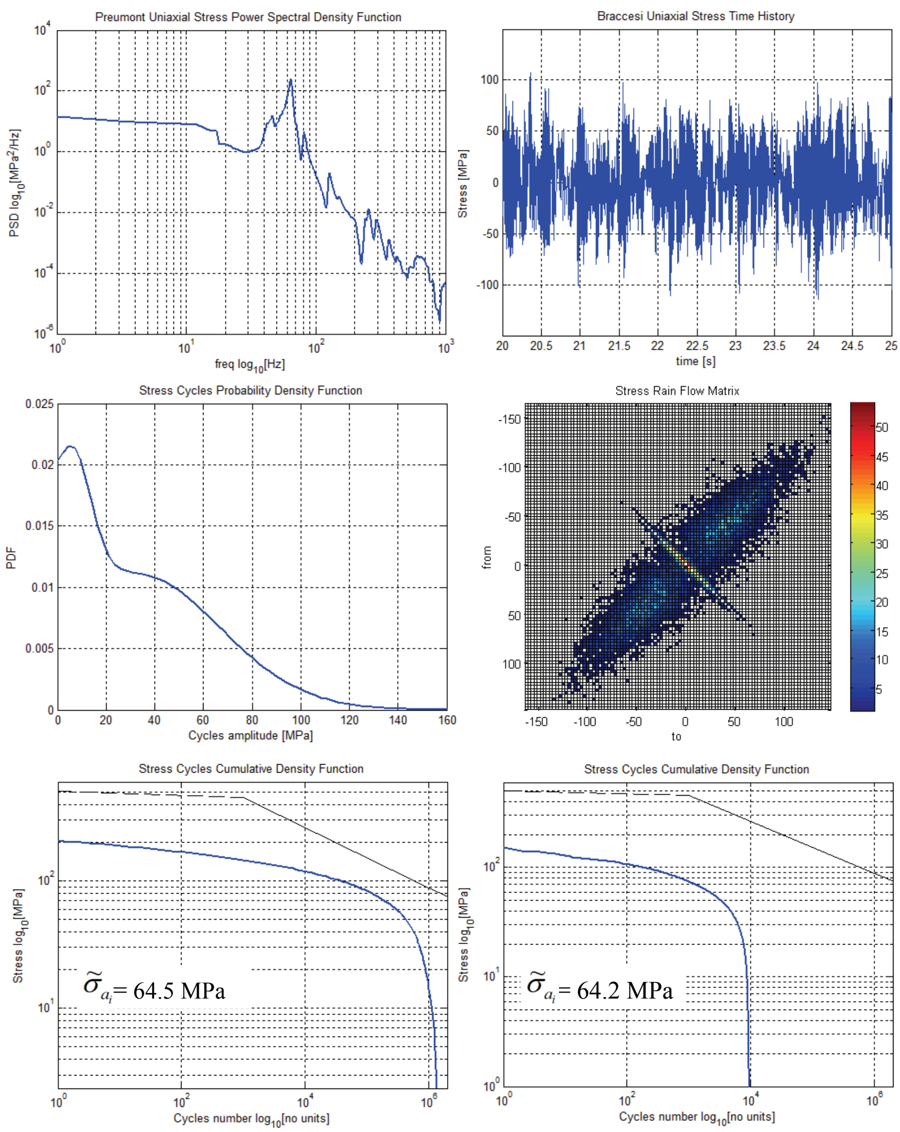

Figure 8: Reference method. Some results relative to stress state of element no.374013 are shown. Equivalent stress $\tilde{\sigma}_{a_i}$ values, obtained with the reference cycle number $\tilde{n}_t$ values ($1 \cdot 10^6$ and $1 \cdot 10^4$ cycles), are shown.

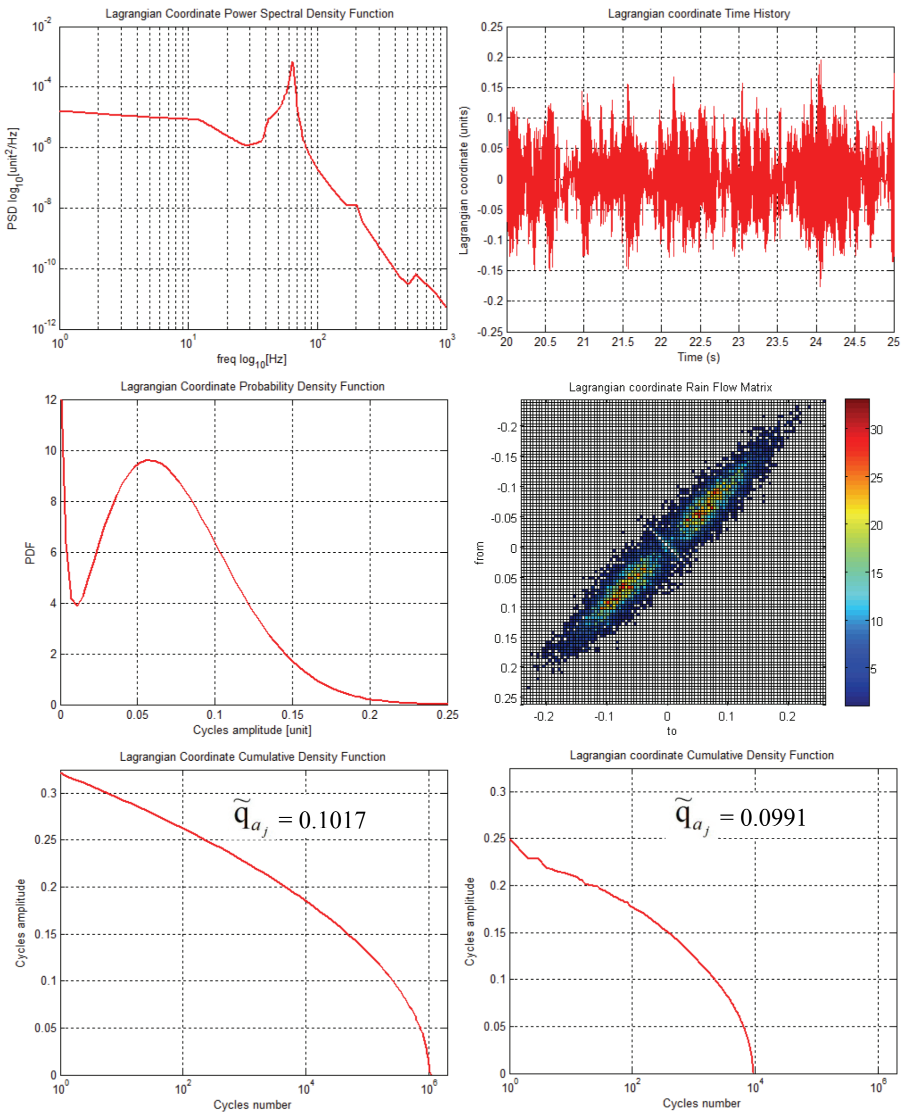

Figure 9: Proposed method. Some results relative to *lagrangian coordinate* no.5 are shown. Equivalent values $\tilde{q}_{a_j}$, obtained with the reference cycle number $\tilde{n}_t$ values ($1 \cdot 10^6$ and $1 \cdot 10^4$ cycles), are shown.

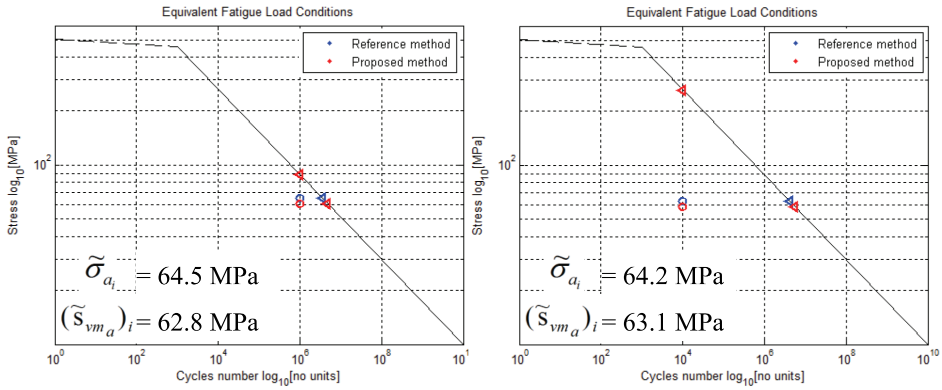

Figure 10: Element no. 374013. Comparison between equivalent load conditions obtained by proposed $(\tilde{s}_{vm_a})_i$ and reference $\tilde{\sigma}_{a_i}$ methods. Left graph shows the equivalent load conditions (circles) obtained in frequency domain. Right graph shows the equivalent load conditions obtained in time domain.

As concerns the proposed method, in figure 9 the results relative to *lagrangian coordinate* no.5 are shown. In the left column frequency domain results are shown. In the right column time domain ones are illustrated. In the upper row *lagrangian coordinate* PSD function and time history (a 5 second window) are shown. The mid row shows its PDF function, obtained by *Dirlik* approach, and the *RainFlow* matrix (*from-to*). In the lower row the *lagrangian coordinate* cumulatives are compared. In these last graphs the equivalent values $\tilde{q}_{a_j}$ of the modal coordinate, obtained with the reference cycle number $\tilde{n}_t$ values (1×10^6 and 1×10^4 cycles), are shown.

In figure 10 the equivalent fatigue load conditions, obtained, for the max damaged element (no. 374013), by using the two methods, are compared. Left graph shows the equivalent load conditions (circles) obtained in frequency domain. Right graph shows the equivalent load conditions obtained in time domain. The load conditions are compared with the fatigue strength curve; the limit strength values of cycles and stresses, relative to the equivalent conditions, are indicated by triangles.

Finally, in figure 11, the trends of lagrangian coordinate equivalent values $\tilde{q}_{a_j}$, of the maximum values of *von Mises* modal stress for each mode and of the maximum contribute of each mode to the maximum stress state $(\tilde{s}_{vm})_j$ are shown. They are those obtained in the frequency domain analysis and they are used for the modal combination ($\tilde{q}_{a_j}$) and for the partial verification ($(\tilde{s}_{vm})_j$) of the speeding up approach represented by equation (21). From the graphs analysis, especially for the one relative to the modes contribute on maximum stress, it is possible to observe

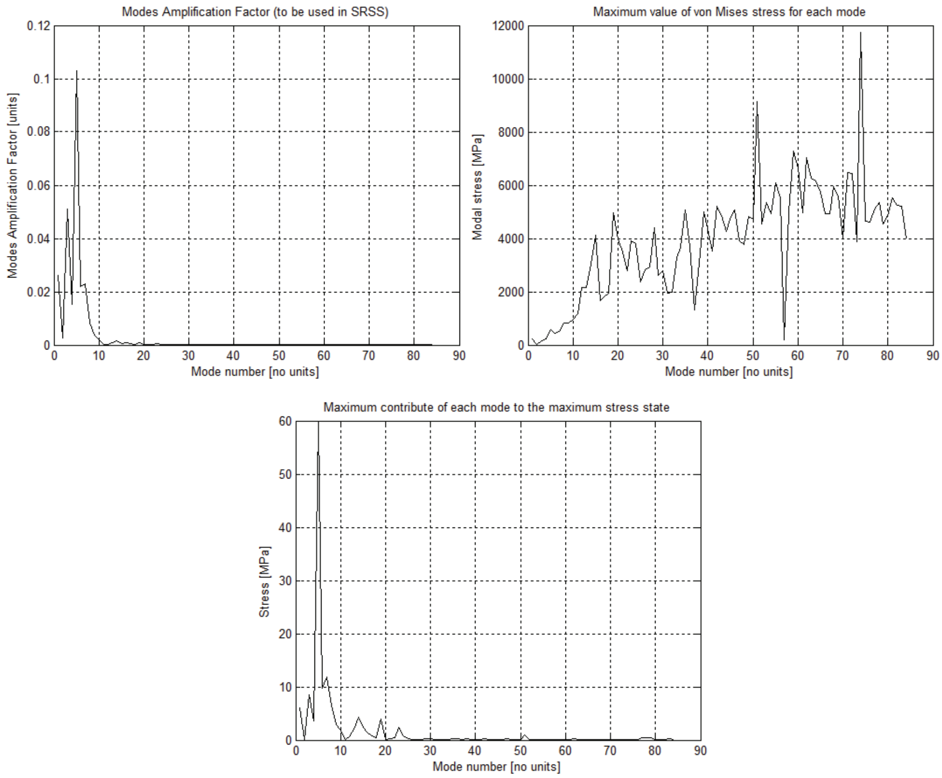

Figure 11: Trends of the modal participation factors $\tilde{\mathbf{q}}_a$ (upper left), of the maximum value of the von Mises stress for each mode $max\left(VM\left(\mathbf{\Phi}_j^\sigma\right)\right)$ (upper right) and of the maximum contribute of each mode to the maximum stress state $\tilde{\mathbf{s}}_{vm}$ (lower)

how just some modes take part to the stress state condition, in particular the mode ID no. 5. In particular, it is interesting to show how much the speeding up procedure obtains a result close to that obtainable by the proposed and reference methods. The maximum contribute of only mode no.5 obtained by (21) is about 60 MPa.

To verify the goodness of the speeding up procedure an ulterior test, not reported in the paper, has been conducted on an elements sample (elements subset among the more stressed ones). It has demonstrated how, to consider just the first 10 normal modes, modifies both damage and alternating equivalent stress values by a quantity less than the 0.02 %. This assumption could be an ulterior development for the proposed method speed.

5 Conclusions

In the present paper an innovative method for damage and alternating equivalent stress evaluation of a component subjected to random loads is shown. The method is based on the component modal modelling and on the availability of the component dynamic response, expressed in terms of *lagrangian coordinates.* The method, useful for a fatigue behaviour evaluation both in frequency and in time domain, has been validated by using an industrial test case and by comparing its results with ones obtained through a well-known and verified procedure. The comparison demonstrates the proposed method goodness in terms of results agreement (excellent) but especially in terms of computational burden that is drastically reduced by its use.

References

Bendat S, Piersol G. (2000): *Random data. Analysis and measurement procedures*, Wiley Series in Probability and Statistics, Third Edition.

Bishop N.W.M., Sherratt F. (2000): *Finite element based fatigue calculations*, NAFEMS Ltd.

Braccesi C., Cianetti F. (2011): Random Loads Fatigue and dynamic simulation: a new procedure to evaluate the behaviour of non-linear systems, *Int. J. of Structural Durability & Health Monitoring*, Vol. 7, No. 2, pp. 83-118.

Braccesi C., Cianetti F., Silvioni L. (2010): Analysis of mission profiles for military vehicles. Definition and validation of instruments for the synthesis of equivalent load conditions, *Int. J. Vehicle Structures & Systems*, Vol. 2, No. 3-4, pp. 127-138.

Braccesi C., Cianetti F., Lori G., Pioli D. (2009): The frequency domain in virtual fatigue estimation of nonlinear systems: the problem of non-Gaussian states of stress, *Int. Journal of Fatigue*, Vol. 31/4, pp. 766-775.

Braccesi C., Cianetti F., Lori G., Pioli D. (2008): An equivalent uniaxial stress process for fatigue life estimation of mechanical components under multiaxial stress conditions, *Int. Journal of Fatigue*, Vol. 30/8, pp. 1479-1497.

Braccesi C., Cianetti F., Lori G., Pioli D. (2005): Fatigue behaviour analysis of mechanical components subject to random bimodal stress process: frequency domain approach, *Int. Journal of Fatigue,* Vol. 27, pp. 335-345.

Braccesi C., Cianetti F., Lori G., Pioli D. (2005): A frequency method for fatigue life estimation of mechanical components under bimodal random stress process, *Int. Journal of Structural Integrity & Durability*, Vol. 1, No. 4, pp. 277-290.

Braccesi C., Cianetti F., Landi L. (2005): Random loads fatigue. The use of spectral methods within multibody simulation, *Proceedings of IDETC/CIE ASME 2005, 20th Biennial Conference on Mechanical Vibration and Noise (VIB)*, Long Beach (CA), USA.

Braccesi C., Cianetti F. (2005): A procedure for the virtual evaluation of the stress state of mechanical systems and components for automotive industry: development and experimental validation, *Proc. of The Institution of Mechanical Engineers Part D - Journal of Automobile Engineering*, Vol. 219, pp. 633-643.

Braccesi C., Cianetti F., Lori G., Pioli D. (2005): L'approccio nel dominio della frequenza alla valutazione del comportamento a fatica di componenti meccanici soggetti a sollecitazioni di tipo random, *Proocedings of IGF Workshop: "Progettazione a fatica in presenza di multiassialità tensionali. Problemi teorici e risvolti applicativi"*, pp. 57-80, ISBN 88-86281-97-8, Ferrara, Italy.

Braccesi C., Cianetti F., Lori G., Pioli D. (2005): Qualifica virtuale di attuatori di uso aeronautico: prove di vibrazione e di durata, *Proceedings of 24th Italian National Conference AIAS*, Milano, Italy.

Braccesi C., Cianetti F. (2004): Development of selection methodologies and procedures of the modal set for the generation of flexible body models for multibody simulation, *Proc. of The Institution of Mechanical Engineers Part K - Journal of Multi-Body Dynamics*, Vol. 218, No. 1, pp. 19-30.

Collins J.A. (1992*), Failure of materials in mechanical design*, John Wiley & Sons.

Corten H.T., Dolan T.J. (1956): Cumulative Fatigue Damage, *International Conference on fatigue of metals*, London.

Craig R.R., Bampton M.C.C. (1968): Coupling of substructures for dynamic analyses, *AIAA Journal*, Vol. 6, No. 7, pp. 1313-1319.

Dirlik T. (1985): *Application of computer in fatigue analysis*, PhD Thesis, University of Warwick, Warwick, England.

Gupta A.K., Chen D.C. (1983): A simple method of combining modal responses,
Haibach E. (2002), *Endurance Strength, Methods and data for calculation of components*, Vol. 2, Springer, Berlin.

Harris C., Piersol A. (2002): *Harris' Shock and Vibration Handbook*, 5th Edition, Mc Graw-Hill.

Lori G. (2005): *Frequency methods for the fatigue design of mechanical components subjected to random loads*, PhD Thesis in Mechanical Engineering, University of Perugia, Perugia, Italy.

Murakami Y. (1992): *The Rainflow method in fatigue*, Butterworth-Heinenmann.

Ogata K. (2002): *Modern control engineering*, Fifth Edition, Prentice Hall.

Pioli D. (2005): *Development of methods for the fatigue design of mechanical systems*, PhD Thesis in Mechanical Engineering, University of Perugia, Perugia, Italy.

Pitoiset X., Preumont A. (2000): Spectral methods for multiaxial random fatigue analysis of metallic structures, *Int. Journal of Fatigue*, Vol. 22, pp.541-550.

Rychlik I. (1987): A new definition on the rainflow cycle counting method, *Int. Journal of Fatigue*, Vol.9, No.2, pp.119-121.

Schiehlen W. (1997): Multibody Systems Dynamics: Roots and perspective, *Multibody system dynamics*, Vol.1, pp.149-188.

Shabana A.A. (1998): *Dynamics of multibody systems*, John Wiley & Sons.

Shabana A.A. (1997): Flexible Multibody Dynamics: Review of past and recent developments, *Multibody system dynamics*, Vol.1, pp.189-222.

Socie D.F., Marquis G.B. (2000): *Multiaxial Fatigue*, SAE International.

Susmel L. (2002): *Fatigue design under complex stress states* (*La progettazione a fatica in presenza di stati complessi di sollecitazione*), PhD Thesis in Mechanical Engineering, University of Padova, Padova, Italy.

You B.R., Lee S.B. (1996): A critical review on multiaxial fatigue assessments of metals, *Int. Journal of Fatigue*, Vol. 18, No. 4, pp. 235-244.