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Article

An Iterative Scheme of Arbitrary Odd Order and Its Basins of Attraction for Nonlinear Systems

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Abstract: In this paper, we propose a fifth-order scheme for solving systems of nonlinear equations. The convergence analysis of the proposed technique is discussed. The proposed method is generalized and extended to be of any odd order of the form $2n-1$. The scheme is composed of three steps, of which the first two steps are based on the two-step Homeier's method with cubic convergence, and the last is a Newton step with an appropriate approximation for the derivative. Every iteration of the presented method requires the evaluation of two functions, two Fréchet derivatives, and three matrix inversions. A comparison between the efficiency index and the computational efficiency index of the presented scheme with existing methods is performed. The basins of attraction of the proposed scheme illustrated and compared to other schemes of the same order. Different test problems including large systems of equations are considered to compare the performance of the proposed method according to other methods of the same order. As an application, we apply the new scheme to some real-life problems, including the mixed Hammerstein integral equation and Burgers' equation. Comparisons and examples show that the presented method is efficient and comparable to the existing techniques of the same order.

Keywords: System of nonlinear equations; root finding method; iterative method; order of convergence; Burgers' equation

1 Introduction

One popular research area in mathematics is to find the solution $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)^t$ of the system of nonlinear equation $F(X) = 0$, where $F(X) = (f_1(x), f_2(x), \ldots, f_n(x))^t$, and $X = (x_1, x_2, \ldots x_n)^t \in \mathbb{R}^n$. This type of problems occurs in many applied sciences like engineering, physics, biology and chemistry. Many researchers developed iterative methods for solving this kind of systems using different techniques. The most popular iterative method for solving system of nonlinear equations is the well-known Newton's method which has second order of convergence [1]. To improve the order of convergence and increase the accuracy of the solution obtained, many researchers tried to improve Newton's method. Some authors used different forms and modifications based on Adomian decomposition technique for solving systems of nonlinear equations, see for instance [2–6]. Another way to improve some schemes for systems of nonlinear equation is by using homotopy analysis method and homotopy perturbation method, see for example [7,8]. Grau-Sánchez et al. [9] used the harmonic mean of the derivative to improve an iterative scheme for solving systems of nonlinear equations. By applying some quadrature formulas, some researchers implement their techniques to solve systems of nonlinear equations, for instance [10–12]. Also, some derivative-free schemes for systems of nonlinear equations were proposed, see for example [13–15] and the references therein. One of the well-known modifications of Newton method is Jarratt method of order four. Cordero et al. [16] extended Jarratt method to solve systems of nonlinear equations preserving the same order of convergence. Many variants of Jarratt type methods have been developed, see for example [17–19] and the references therein. Many other different orders of convergence schemes for nonlinear modules can be found in the literature, see for example [20,21] and the references therein.

Some techniques to improve the order of convergence of the iterative schemes for systems of nonlinear equations have been proposed, for instance, see [22,23]. In general, obtaining a higher-order iterative method is not the only important thing; as the computational and the time cost are crucial issue also. So, establishing a high order iterative method based on low computational and time cost is very important.

In this paper, we develop a new multi-step scheme of arbitrary odd order for nonlinear equations. The proposed method can be used in the multidimensional case preserving the same order. The convergence analysis of the new scheme is discussed. Several examples are given to show the efficiency of the generalized method and its comparison with other iterative schemes of the same order. To confirm the applicability of the new technique, we apply the new technique to some real-life problems.

2 The Proposed Method

In this section we will derive the proposed technique for nonlinear modules. We begin by writing the function $f(x)$ as:

$$f(x) = \int_{x_n}^{x} f'(t)dt + f(x_n). \tag{1}$$

As we want $f(x) = 0$, and by using midpoint quadrature formula and writing the equation as an iterative scheme, one gets

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(\frac{x_n + x_{n+1}}{2})}. \tag{2}$$

Now, to write the iterative scheme (2) in explicit form, replace $f'\left(\frac{x_n + x_{n+1}}{2}\right)$ by $f'\left(\frac{x_n + x_{n+1}^*}{2}\right)$ where x_{n+1}^* is the Newton step. So, scheme (2) becomes:

$$\begin{cases} y_n = x_n - \dfrac{1}{2}\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \dfrac{f(x_n)}{f'(y_n)}. \end{cases} \tag{3}$$

The iterative method (3) was proposed by Frontini and Sormani [10]. The multidimensional case of scheme (3) was discussed by Homeier [24] and can be written as:

$$\begin{cases} Y_n = X_n - \dfrac{1}{2}F'(X_n)^{-1}F(X_n), \\ X_{n+1} = X_n - F'(Y_n)^{-1}F(X_n), \end{cases} \tag{4}$$

where $F'(X_n)^{-1}$ and $F'(Y_n)^{-1}$ are the inverse of the first Fréchet derivative of $F(X_n)$ and $F(Y_n)$ respectively. Scheme (4) is of third-order of convergence, and requires at each iteration the evaluation of one function, two Fréchet derivatives and two matrix inversions. In order to increase the convergence order and the computational efficiency of scheme (4) Sharma et al. [20] proposed a new scheme of the fifth-order of convergence by adding one step to scheme (4):

$$\begin{cases} Y_n = X_n - \dfrac{1}{2}F'(X_n)^{-1}F(X_n), \\ W_n = X_n - F'(Y_n)^{-1}F(X_n), \\ X_{n+1} = W_n - (2F'(Y_n)^{-1} - F'(X_n)^{-1})F(W_n). \end{cases} \tag{5}$$

Per iteration, scheme (5) requires the evaluations of two functions, two Fréchet derivatives and two matrix inversions.

Now, to derive the new scheme for solving systems of nonlinear equations, we start by composing scheme (3) to additional Newton step, that is:

$$\begin{cases} y_n = x_n - \dfrac{1}{2}\dfrac{f(x_n)}{f'(x_n)}, \\ w_n = x_n - \dfrac{f(x_n)}{f'(y_n)}, \\ x_{n+1} = w_n - \dfrac{f(w_n)}{f'(w_n)}. \end{cases} \tag{6}$$

Now, to reduce number of functional evaluations at each iteration, we will use divided difference approximation to write the derivative $f'(w_n)$ using some already computed functions from the previous steps. To do that, one can write

$$f'(y_n) = \frac{f(y_n) - f(x_n)}{y_n - x_n} = f[y_n, x_n] \approx f'(x_n), \tag{7}$$

in the same manner, we have

$$f'(y_n) = \frac{f(y_n) - f(w_n)}{y_n - w_n} = f[y_n, w_n] \approx f'(w_n), \tag{8}$$

by adding (7) and (8), one easily can conclude that

$$f'(w_n) \approx 2f'(y_n) - f'(x_n). \tag{9}$$

Now, if we substitute (9) in (6), then we will have a new scheme for solving nonlinear equations:

$$\begin{cases} y_n = x_n - \dfrac{1}{2}\dfrac{f(x_n)}{f'(x_n)}, \\ w_n = x_n - \dfrac{f(x_n)}{f'(y_n)}, \\ x_{n+1} = w_n - \dfrac{f(w_n)}{2f'(y_n) - f'(x_n)}. \end{cases} \tag{10}$$

To generalize scheme (10) to the multidimensional case to solve systems of nonlinear modules, the scheme becomes:

$$\begin{cases} Y_n = X_n - \frac{1}{2}F'(X_n)^{-1}F(X_n), \\ W_n = X_n - F'(Y_n)^{-1}F(X_n), \\ X_{n+1} = W_n - (2F'(Y_n) - F'(X_n))^{-1}F(W_n). \end{cases} \tag{11}$$

Scheme (11) requires at each iteration the evaluation of two functions, two Fréchet derivatives and three matrix inversions. The proposed scheme is of fifth-order of convergence as we will see in the next section.

If we repeat using the same idea of the derivation of scheme (11), we can write a general scheme for solving system of nonlinear equations, and this is the main motivation of our work. The general scheme can be written as:

$$\begin{cases} X_{1,n} = X_{0,n} - \frac{1}{2}F'(X_{0,n})^{-1}F(X_{0,n}), \\ X_{2,n} = X_{0,n} - F'(X_{1,n})^{-1}F(X_{0,n}), \\ X_{3,n} = X_{2,n} - (2F'(X_{1,n}) - F'(X_{0,n}))^{-1}F(X_{2,n}), \\ \vdots \\ X_{q,n} = X_{q-1,n} - (2F'(X_{1,n}) - F'(X_{0,n}))^{-1}F(X_{q-1,n}), \\ \vdots \\ X_{m,n} = X_{m-1,n} - (2F'(X_{1,n}) - F'(X_{0,n}))^{-1}F(X_{m-1,n}). \end{cases} \tag{12}$$

Per iteration, scheme (12) requires the evaluation of $m-1$ functions, two Fréchet derivatives and three matrix inversions. We will prove in the next section that scheme (12) is of order $2m-1$ for any integer $m \geq 3$.

3 Order of Convergence

We will discuss in this section the order of convergence of the proposed schemes (11) and (12). Assume for the next theorems that $C_j = \frac{1}{j!}F'(\alpha)^{-1}F^{(j)}(\alpha), j \geq 2$, and $e_n = X_n - \alpha$.

Theorem 1 Let α be the solution of the system $\mathrm{F(X)} = 0$ where $\mathrm{F} : \mathrm{D} \subseteq \mathrm{R^n} \to \mathrm{R^n}$ be a sufficiently differentiable function on a neighborhood D of α. Suppose that $\mathrm{F'(X)}$ is continuous and nonsingular in α. If $\mathrm{X_0} \in \mathrm{D}$ is an initial approximation which is close enough to α, then the sequence $\{\mathrm{X_n}\}_{\mathrm{n}\geq 0}$ obtained by scheme (11) converges to the root α, and the order of convergence equals 5, with asymptotic equation $\mathrm{e_{k+1}} = \frac{1}{8}\left(4\mathrm{C_2^2} - 3\mathrm{C_3}\right)\left(4\mathrm{C_2^2} - \mathrm{C_3}\right)\mathrm{e_k^5}$.

Proof. By using the Taylor expansion of $F(X_n)$ we can write $F(X_n) = F'(\alpha)\left(e_n + C_2e_n^2 + C_3e_n^3 + C_3e_n^3 + C_4e_n^4 + \ldots\right)$. Now, we use the following Mathematica code to show the convergence order of scheme (11) for $m = 3$:

In[1]:= $\mathrm{F[e\_]} := \mathrm{dF}[\alpha](\mathrm{e} + \mathrm{C_2e^2} + \mathrm{C_3e^3} + \mathrm{C_4e^4})$;

In[2]:= $\mathrm{y} = \mathrm{e} - \mathrm{Series}\left[\frac{1}{2}(\mathrm{F'[e]})^{-1}\mathrm{F[e]}, \{\mathrm{e}, 0, 2\}\right]$;

In[3]:= $\mathrm{w} = \mathrm{e} - \mathrm{Series}\left[(\mathrm{F'[y]})^{-1}\mathrm{F[e]}, \{\mathrm{e}, 0, 2\}\right]$;

In[4]:= $e_{\mathrm{n+1}} = \mathrm{w} - (2\mathrm{F'[y]} - \mathrm{F'[e]})^{-1}\mathrm{F[w]}$ //FullSimplify

Out[4]:= $\left(2\mathrm{C_2^4} - 2\mathrm{C_2^2C_3} + \frac{3\mathrm{C_3^2}}{8}\right)\mathrm{e^5} + \mathrm{O[e^6]}$.

the code shows that we have $e_{n+1} = \left(2C_2^4 - 2C_2^2C_3 + \frac{3C_3^2}{8}\right)e_n^5$, which can be written as:

$e_{n+1} = \frac{1}{8}\left(4C_2^2 - 3C_3\right)\left(4C_2^2 - C_3\right)e_n^5$.

By this, we show that scheme (11) is at least of fifth-order of convergence.

Now, we want to discuss the order of convergence of the generalized scheme given by (12).

Theorem 2 Let α be the solution of the system $\mathrm{F(X)} = 0$ where $\mathrm{F} : \mathrm{D} \subseteq \mathrm{R^n} \to \mathrm{R^n}$ be a sufficiently differentiable function on a neighborhood D of α. Suppose that $\mathrm{F}'(\mathrm{X})$ is continuous and nonsingular in α. If $\mathrm{X_0} \in \mathrm{D}$ is an initial approximation which is close enough to α, then the sequence $\{\mathrm{X_n}\}_{\mathrm{n}\geq 0}$ obtained by scheme (12) converges to the root α, and the order of convergence equals $2\mathrm{m} - 1$, for any integer $\mathrm{m} \geq 3$, with asymptotic equation of the form $\mathrm{e_{k+1}} = \frac{1}{2^{\mathrm{m}}}\left(4\mathrm{C}_2^2 - 3\mathrm{C}_3\right)^{\mathrm{m}-2}\left(4\mathrm{C}_2^2 - \mathrm{C}_3\right)\mathrm{e}_{\mathrm{k}}^{2\mathrm{m}-1}$.

Proof. We will use the mathematical induction to prove the convergence order of scheme (12).

Firstly, we will prove that scheme (12) is convergent for $m = 3$, and the convergence order satisfies $2(3) - 1 = 5$. Note that for $m = 3$, scheme (12) reduces to scheme (11) which we have been proved that it has the fifth-order of convergence in the previous theorem. Now, to complete the proof using the mathematical induction, suppose that scheme (12) is true and converges for all $m \leq r$ for some positive $r > 3$ and satisfy the given asymptotic equation. We need to show that the scheme converges for $m = r + 1$, and satisfy the given asymptotic equation. To do so, consider the following code of Mathematica:

In[1]:= $\mathrm{F[e\_]{:=}\,dF[\alpha](e + C_2e^2 + C_3e^3 + C_4e^4)}$;

In[2]:= $\mathrm{y = e - Series}\left[\frac{1}{2}\left(\mathrm{F}'[\mathrm{e}]\right)^{-1}\mathrm{F[e]}, \{\mathrm{e}, 0, 6\}\right]$;

In[3]:= $\mathrm{w = e - Series}\left[\left(\mathrm{F}'[\mathrm{y}]\right)^{-1}\mathrm{F[e]}, \{\mathrm{e}, 0, 6\}\right]$;

In[4]:= x[m_]:=x[m] $= \frac{1}{2^m}\left(4\mathrm{C}_2^2 - 3\mathrm{C}_3\right)^{m-2}\left(4\mathrm{C}_2^2 - \mathrm{C}_3\right)\mathrm{e}_n^{2m-1}$ //FullSimplify

In[5]:= $e_{\mathrm{n}+1} = \mathrm{x[r]} - \left(2\mathrm{F}'[\mathrm{y}] - \mathrm{F}'[\mathrm{e}]\right)^{-1}\mathrm{F[x[r]]}$ //FullSimplify

Out[5]:= $\frac{1}{2^{r+1}}\left(4\mathrm{C}_2^2 - 3\mathrm{C}_3\right)^{r-2}\left(4\mathrm{C}_2^2 - \mathrm{C}_3\right)\mathrm{e}_n^{2r-1}$

Hence, this shows that for $m = r + 1$, we have $e_{k+1} = \frac{1}{2^{r+1}}\left(4C_2^2 - 3C_3\right)^{r-1}\left(4C_2^2 - C_3\right)e_k^{2r+1}$.

4 Computational Efficiency

In this section, we compare the efficiency index of our proposed method with other methods in the literature. Commonly in the literature, the efficiency index $EI = p^{\frac{1}{d}}$ is used, where p is the order of convergence of the iterative scheme, and d is the number of functions needed to be found per iteration in the iterative scheme. Another common index that can be used in the comparison between iterative scheme is the computational efficiency index $CEI = p^{\frac{1}{d+op}}$, where op is the number of operations per iteration in the iterative scheme. The evaluation of any scalar function is considered as an operation.

To find the number of functions required to be found per iteration in an iterative scheme, the following rules applied: Any computation of $F(X)$ needs n evaluations of scalar functions. Any computation of the Jacobian $F'(X)$ needs n^2 evaluations of scalar functions. Also, the floating points for obtaining the LU factorization are $\frac{2}{3}n^3$, and to solve the triangular system we need n^2 floating points operations. Finally, n^2 operations required to find a matrix-vector multiplication, and n^3 operations needed to find a matrix-matrix multiplication.

We compare the efficiency index and the computational efficiency index for the proposed method (PM) (11) to the following iterative schemes:

- The third-order Frontini-Sormani method (FS) [10] given by (4).
- The fifth order scheme (CHMT) proposed by Cordero et al. [23], given by

$$\begin{cases} Y_n = X_n - F'(X_n)^{-1}F(X_n), \\ W_n = X_n - 2(F'(Y_n) + F'(X_N))^{-1}F(X_n), \\ X_{n+1} = W_n - F'(Y_n)^{-1}F(W_n). \end{cases} \tag{13}$$

- The fifth order scheme (MMK) proposed by Waseem et al. [4], which is given by:

$$\begin{cases} Y_n = X_n - F'(X_n)^{-1}F(X_n), \\ W_n = Y_n - F'(X_n)^{-1}F(Y_n), \\ Z_n = W_n - F'(X_n)^{-1}F(W_n), \\ X_{n+1} = Z_n - F'(X_n)^{-1}F(Z_n). \end{cases} \tag{14}$$

- The fifth-order iterative scheme (SG) presented by Sharma et al. [20], which is defined by scheme (5).

A comparison of the number of functional evaluations of the selected iterative schemes is illustrated in Tab. 1. Also, the computational efficiency indices of the selected schemes are compared (for $n = 2, 3, 4, 5, 10, 20, 50$), see Fig. 1. Note that the proposed scheme does not attain the best efficiency in this comparison, especially for small n. We will see in the next two sections that this issue does not affect the scheme negatively when applied to some numerical tests.

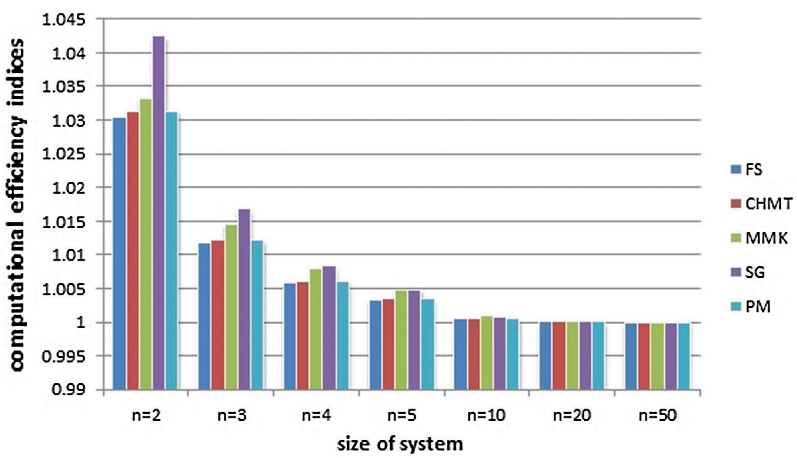

Figure 1: Computational efficiency indices for different sizes of system

Table 1: Comparisons of required functional evaluations per iteration

	FS	CHMT	MMK	SG	PM $_5$
Order of convergence	3	5	5	5	5
Number of functional evaluations	$n + 2n^2$	$2n + 2n^2$	$4n + n^2$	$2n + 2n^2$	$2n + 2n^2$
Efficiency index	$\frac{1}{3^{n+2n^2}}$	$5^{\frac{1}{2n+2n^2}}$	$5^{\frac{1}{4n+n^2}}$	$5^{\frac{1}{2n+2n^2}}$	$5^{\frac{1}{2n+2n^2}}$
Computational efficiency index	$3^{\frac{1}{\frac{4}{3}n^3+6n^2+n}}$	$5^{\frac{1}{2n^3+8n^2+2n}}$	$5^{\frac{1}{\frac{2}{3}n^3+9n^2+4n}}$	$5^{\frac{1}{\frac{4}{3}n^3+6n^2+2n}}$	$5^{\frac{1}{2n^3+8n^2+2n}}$

Fig. 2 illustrates the efficiency indices for the selected methods. Note that CHMT, SG and our proposed method have the same efficiency indices. However, this does not guarantee that they have the same behavior, accuracy and computational time cost.

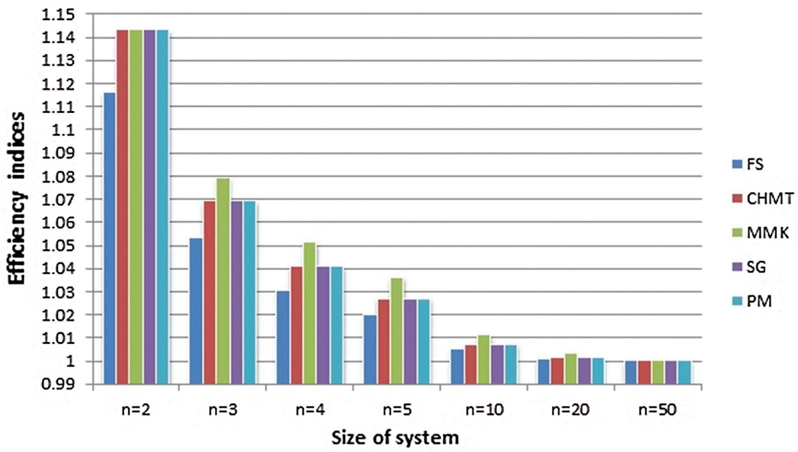

Figure 2: Efficiency indices for different values of n

5 Basins of Attraction

The concept of basins of attraction is a method to show how different starting points affect the behavior of the function. In this way, we can compare different root-finding schemes depending on the convergence area of the basins of attraction. In this sense, the iterative scheme is better if it has a larger area of convergence. Here, we mean by the area of convergence, the number of convergent points to a root α of $f(x)$ in a selected range.

To check the stability and the area of convergence of our proposed method, we select the case $m = 3$ of scheme (12). We denote the proposed method by PM_5. For comparison, we compare PM_5 with the following schemes of the same order of convergence: The scheme CHMT given by Cordero et al. (13), the scheme SG proposed by Sharma and Gupta (5), and the scheme MMK presented by Waseem et al. (14). We choose three test examples to visualize the basins of attraction. All examples are polynomials with roots of multiplicity one. The test polynomials are

- $P_1(z) = z^3 - z$, with roots $z = 0, \pm 1$.
- $P_2(z) = z^4 - 1$, with roots $z = \pm i, \pm 1$.
- $P_3(z) = z^5 + 2z - 1$, with roots $z = -0.945068 \pm 0.854518i$, 0.486389, $0.701874 \pm 0.879697i$

A 4×4 region is centered at the origin to cover all the zeros of the selected polynomials. The step size selected is 0.01; thus, $401 \times 401 = 160801$ points in a uniform grid are selected as initial point for the iterative schemes to generate the basins of attraction. The exact roots were assigned as black dots on the graph. If the scheme needs less number of iterations to converge to a specific root, then the region of that roots appears darker. The convergence criterion selected is a tolerance of 10^{-3} with a maximum of 100 iterations. All calculations have been performed on Intel Xeon CPU-E5-2690 0@2.90 GHz with 32 GB RAM, using Microsoft Windows 10, 64 bit based on X64-based processor. Mathematica 9 has been used to generate all graphs and computations. The dynamics of the four test problems are shown in Figs. 3–5 respectively.

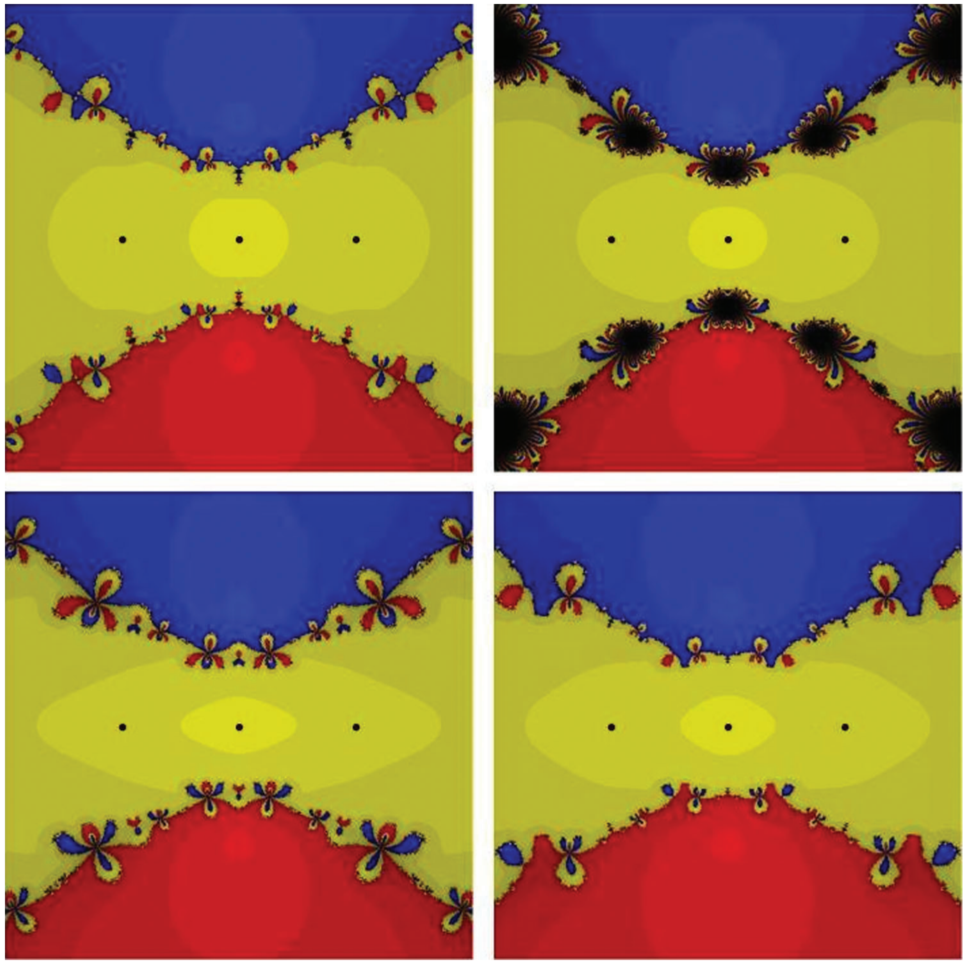

Figure 3: Basins of attraction of $P_1(z) = z^3 - z$. The top row from left to right: CHMT and MMK. The bottom row from left to right: SG and PM_5

Basins of attraction of PM_5 shows that the proposed method is comparable to other methods of the same order, with an area of convergence which is larger or the same as the areas of convergence of the other methods used in the comparison.

6 Numerical Tests and Applications

In this part, we consider some numerical problems to clarify the computational efficiency and convergence behavior of the proposed scheme. All calculations have been performed using 4000 significant digits on Mathematics 9. For comparisons, we find the number of iterations n needed to satisfy the stopping criterion $\| X_n - X_{n-1} \| + \| F(X_n) \| < 10^{-150}$ for each selected method. Also, we use the approximated computational order of convergence for each iterative scheme, which can be found by

$$\mathrm{ACOC} \approx \frac{\ln(\| (X_{n+1} - x_n) \| / \| (X_n - X_{n-1}) \|)}{\ln(\| (X_n - X_{n-1}) \| / \| (X_{n-1} - X_{n-2}) \|)}.$$

Finally, we compare for the selected schemes the distance between two consecutive iterations $\| X_n - X_{n-1} \|$ and the value of $\| F(X_n) \|$ for $n = 1, 2, 3$.

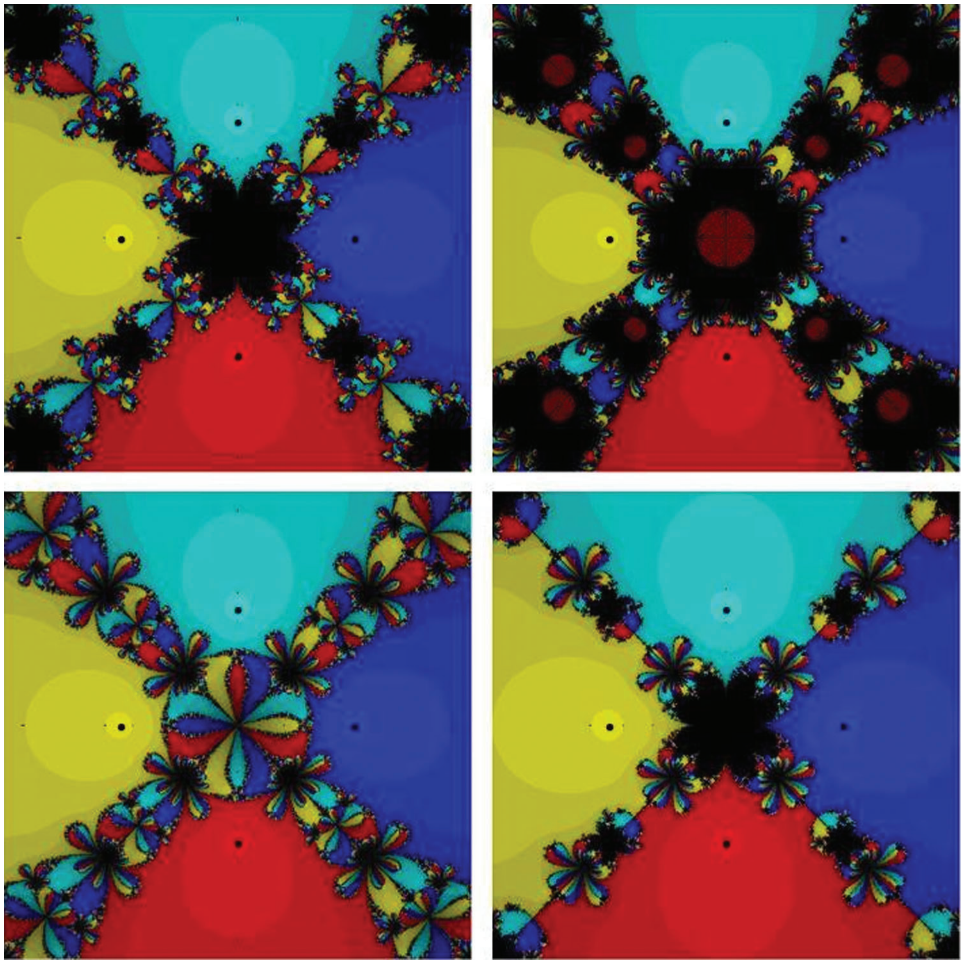

Figure 4: Basins of attraction of $P_2(z) = z^4 - 1$. The top row from left to right: CHMT and MMK. The bottom row from left to right: SG and PM_5

To be consistent in the comparison, we compare the proposed scheme PM_5 defined by scheme (12) for $m = 3$, to the original method which we derived from, that is, FS scheme given by (4). Also, we use the following fifth-order iterative schemes in the comparison: $CHMT_5$ method defined by scheme (13), MMK method defined by (14), and SG method defined by (5). To test the efficiency of the extension of our proposed scheme to higher orders schemes, we compare the proposed scheme PM_7 of seventh-order given by scheme (12) for $m = 4$, to the extension of $CHMT_5$ to the seventh-order scheme $CHMT_7$ given by:

$$\begin{cases} Y_n = X_n - F'(X_n)^{-1}F(X_n), \\ W_n = X_n - 2(F'(Y_n) + F'(X_N))^{-1}F(X_n), \\ Z_n = W_n - F'(Y_n)^{-1}F(W_n), \\ X_{n+1} = Z_n - F'(Y_n)^{-1}F(Z_n). \end{cases} \tag{15}$$

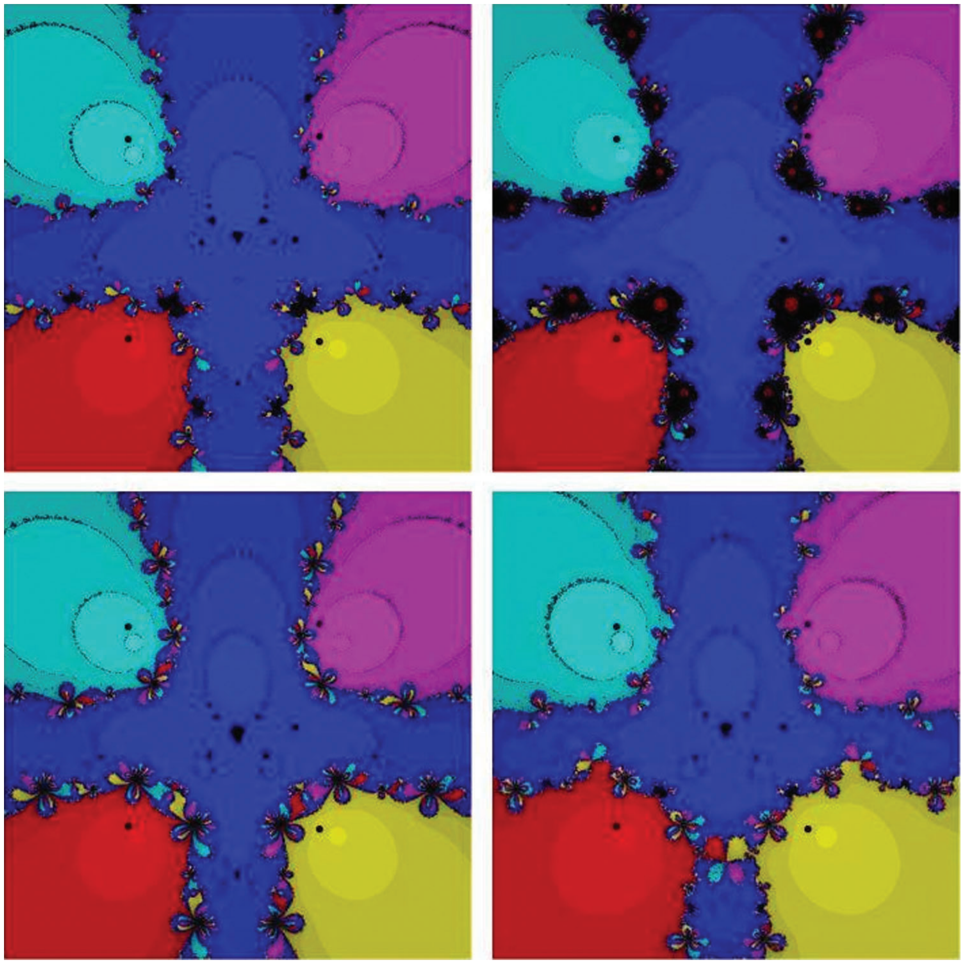

Figure 5: Basins of attraction of $P_3(z) = z^5 + 2z - 1$. The top row from left to right: CHMT and MMK. The bottom row from left to right: SG and PM5

6.1 Numerical Tests

To be not selective in our examples, we choose most test problems from the same papers which contain the schemes used in the comparisons, see [4,20,23]. Also, we choose two distinct initial guesses for all problems to test the validity and the applicability of iterative schemes. We consider the following test problems and applications:

Example 1 Consider the following system of two nonlinear equations:

$$\begin{cases} \mathrm{x} + 1 - \mathrm{e}^{\mathrm{y}} = 0, \\ \mathrm{x} + \cos(\mathrm{y}) - 2 = 0, \end{cases}$$

with initial guesses $\mathrm{X}_0 = \{0, 0\}^{\mathrm{t}}$ and $\mathrm{X}_0 = \{2, 2\}^{\mathrm{t}}$. The exact solution of this problem is $\alpha = \{1.3401918575555883401\ldots, 0.8502329164169513268\ldots\}^{\mathrm{t}}$.

Example 2 Consider the following system of three nonlinear equations:

$$\begin{cases} \cos(y) - \sin(x) = 0, \\ z^x - \dfrac{1}{y} = 0, \\ e^x - z^2 = 0. \end{cases}$$

We consider as an initial solution $X_0 = \{1, 1, 2\}^t$ and $X_0 = \{1, \frac{1}{2}, 1\}^t$. The exact solution of this problem is $\alpha = \{0.909569494520044883 81\ldots,\ 0.661226832274851735 42\ldots,\ 1.575834143906999036 1\ldots\}^t$.

Example 3 Consider the following system

$$\begin{cases} x_i x_{i+1} - 1 = 0, & i = 1, 2, \ldots, n-1, \\ x_i x_1 - 1 = 0, & i = n. \end{cases}$$

For odd n, the exact zeros of F(X) are $\alpha = \overbrace{\{1, 1, \ldots, 1\}}^{49-\text{times}}{}^t$ and $\alpha = \overbrace{\{-1, -1, \ldots, -1\}}^{49-\text{times}}{}^t$. For $n = 49$, we select as an initial guess $X_0 = \overbrace{\{2, 2, \ldots, 2\}}^{49-\text{times}}{}^t$ and $X_0 = \overbrace{\left\{-\frac{1}{2}, -\frac{1}{2}, \ldots, -\frac{1}{2}\right\}}^{49-\text{times}}{}^t$.

Example 4 Consider the nonlinear boundary value problem:

$y'' + y^3 = 0, y(0) = 0, y(1) = 1.$

Assume the following partitioning for the interval $[0, 1]$:

$$u_0 = 0 < u_1 < u_2 < \cdots < u_m < u_{m+1} = 1, u_{j+1} = u_j + h,$$

where $h = \dfrac{1}{m+1}$ is the step size, m is the system size. Let $y_i = y(u_i)$ for $i = 0, 1, 2, \ldots, m+1$. We use the finite difference method to solve the problem, in which the second derivative y'' will be replaced by the central difference $y'' \approx \dfrac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$, $i = 1, 2, \ldots, m$. By this, we obtain $m \times m$ system given by:

$y_{i-1} - 2y_i + y_{i+1} + h^2 y_i^3 = 0, i = 1, 2, \ldots, m.$

We solve this system for $m = 10$ by selecting $X_0 = \overbrace{\{-1, -1, \ldots, -1\}}^{10-\text{times}}{}^t$, and $X_0 = \overbrace{\left\{\frac{5}{3}, \frac{5}{3}, \ldots, \frac{5}{3}\right\}}^{10-\text{times}}{}^t$ as initial guesses. The exact solution for this problem is $\alpha = \{0.680945648372\ldots,\ 1.359281828740\ldots,\ 2.016862032948\ldots,\ 2.606640128407\ldots,\ 3.050046273378\ldots,\ 3.258957241540\ldots,\ 3.181812502482\ldots,\ 2.838449171715\ldots,\ 2.306087498753\ldots,\ 1.672371573489\ldots\}^t$.

Tabs. 2–5 show that our proposed methods PM_5 and PM_7 are efficient with a good performance and comparable to the other methods of the same order. The proposed methods converge to the desired solution either by less number of iterations based on the convergence criterion (Examples 1 and 2), or by the same number of iterations needed to satisfy the convergence criterion with more accurate answers (Examples 3 and 4).

Table 2: Comparisons between different methods for Example **1**

Method	X_0	n	$\parallel X_1 - X_0 \parallel$	$\parallel X_2 - X_1 \parallel$	$\parallel X_3 - X_2 \parallel$	$\parallel F(X_1) \parallel$	$\parallel F(X_2) \parallel$	$\parallel F(X_3) \parallel$	ACOC
FS	$\{0,0\}t$	6	1.65	0.07	2.56E-8	0.088	6.94E-8	8.41E-24	3
	$\{2,2\}t$	7	1.01	0.315	0.00378	0.495	0.00914	3.03E-8	3
$CHMT_5$	$\{0,0\}t$	5	1.56	0.0341	1.08E-8	0.0738	2.46E-8	7.92E-41	5
	$\{2,2\}t$	6	1.19	0.135	4.75E-6	0.218	0.0000108	1.31E-27	5
MMK	$\{0,0\}t$	5	1.60	0.0547	3.28E-8	0.140	7.64E-8	5.33E-38	5
	$\{2,2\}t$	6	1.10	0.226	0.0000968	0.313	0.000198	1.02E-20	5
SG	$\{0,0\}t$	5	1.58	0.00283	1.82E-17	0.00257	5.04E-17	3.43E-85	5
	$\{2,2\}t$	6	1.19	0.135	3.33E-6	0.224	9.23E-6	1.12E-28	5
PM_5	$\{0,0\}t$	5	1.58	0.00215	7.37E-17	0.00125	1.28E-16	2.66E-86	5
	$\{2,2\}t$	5	1.27	0.0607	8.03E-8	0.152	1.46E-7	1.34E-40	5
$CHMT_7$	$\{0,0\}t$	5	1.58	0.00581	2.55E-17	0.0122	5.82E-17	2.12E-117	7
	$\{2,2\}t$	5	1.27	0.0576	1.18E-10	0.0952	2.68E-10	9.39E-71	7
PM_7	$\{0,0\}t$	4	1.59	0.000406	3.75E-29	0.000238	6.50E-29	2.13E-207	7
	$\{2,2\}t$	5	1.31	0.0195	7.78E-15	0.0452	1.38E-14	7.36E-107	7

Table 3: Comparisons between different methods for Example **2**

Method	X_0	n	$\parallel X_1 - X_0 \parallel$	$\parallel X_2 - X_1 \parallel$	$\parallel X_3 - X_2 \parallel$	$\parallel F(X_1) \parallel$	$\parallel F(X_2) \parallel$	$\parallel F(X_3) \parallel$	ACOC
FS	$\{1,1,2\}t$	8	0.790	0.471	0.0420	0.149	0.0372	0.000122	3
	$\{1,\frac{1}{2},1\}t$	7	0.561	0.0703	0.00105	0.235	0.000902	1.87E-9	3
$CHMT_5$	$\{1,1,2\}t$	6	0.555	0.0638	0.0000172	0.0606	0.0000114	151E-23	5
	$\{1,\frac{1}{2},1\}t$	5	0.599	0.0147	4.79E-8	0.0370	3.32E-8	2.79E-36	5
MMK	$\{1,1,2\}t$	***Div.***	–	–	–	–	–	–	–
	$\{1,\frac{1}{2},1\}t$	6	0.500	0.109	9.72E-6	0.364	0.0000259	2.89E-26	5
SG	$\{1,1,2\}t$	8	1.63	0.334	1.09	0.398	0.343	0.231	5
	$\{1,\frac{1}{2},1\}t$	6	0.581	0.0289	7.18E-7	0.106	5.47E-7	1.81E-30	5
PM_5	$\{1,1,2\}t$	5	0.553	0.0303	4.98E-8	0.0355	3.51E-8	3.01E-37	5
	$\{1,\frac{1}{2},1\}t$	5	0.600	0.0195	4.77E-8	0.0397	4.17E-8	1.04E-36	5
$CHMT_7$	$\{1,1,2\}t$	5	0.554	0.0607	3.99E-7	0.0453	2.69E-7	7.66E-44	7
	$\{1,\frac{1}{2},1\}t$	5	0.605	0.00794	1.82E-13	0.00769	1.21E-13	3.02E-88	7
PM_7	$\{1,1,2\}t$	5	0.549	0.0188	1.83E-11	0.0189	1.62E-11	5.49E-75	7
	$\{1,\frac{1}{2},1\}t$	5	0.606	0.0112	1.00E-13	0.00905	9.65E-14	8.98E-91	7

Table 4: Comparisons between different methods for Example **3**

Method	X_0	n	$\parallel X_1 - X_0 \parallel$	$\parallel X_2 - X_1 \parallel$	$\parallel X_3 - X_2 \parallel$	$\parallel F(X_1) \parallel$	$\parallel F(X_2) \parallel$	$\parallel F(X_3) \parallel$	ACOC
FS	$\{2,2,\ldots,2\}t$	7	6.46	0.538	0.000711	1.12	0.00142	3.67E-12	3
	$\{-\frac{1}{2},-\frac{1}{2},\ldots,-\frac{1}{2}\}t$	7	3.00	0.499	0.000711	0.964	0.00142	3.67E-12	3
$CHMT_5$	$\{2,2,\ldots,2\}t$	5	6.91	0.0911	3.16E-10	0.183	6.31E-10	3.26E-52	5
	$\{-\frac{1}{2},-\frac{1}{2},\ldots,-\frac{1}{2}\}t$	5	3.39	0.114	1.06E-9	0.227	2.12E-9	1.41E-49	5
MMK	$\{2,2,\ldots,2\}t$	5	6.82	0.176	3.19E-8	0.357	6.37E-8	1.37E-41	5
	$\{-\frac{1}{2},-\frac{1}{2},\ldots,-\frac{1}{2}\}t$	6	1.67	1.81	0.0151	3.18	0.0302	3.32E-13	5
SG	$\{2,2,\ldots,2\}t$	5	6.87	0.130	3.61E-9	0.262	7.22E-9	1.28E-46	5
	$\{-\frac{1}{2},-\frac{1}{2},\ldots,-\frac{1}{2}\}t$	5	3.14	0.362	7.75E-7	0.706	1.55E-6	5.84E-35	5
PM_5	$\{2,2,\ldots,2\}t$	5	6.91	0.0911	3.16E-10	0.183	6.31E-10	3.26E-52	5
	$\{-\frac{1}{2},-\frac{1}{2},\ldots,-\frac{1}{2}\}t$	5	3.39	0.114	1.06E-9	0.227	2.12E-9	1.41E-49	5
$CHMT_7$	$\{2,2,\ldots,2\}t$	5	6.98	0.0178	2.92E-19	0.0355	5.84E-19	1.93E-136	7
	$\{-\frac{1}{2},-\frac{1}{2},\ldots,-\frac{1}{2}\}t$	5	3.48	0.0236	2.20E-18	0.0471	4.39E-18	2.62E-130	7
PM_7	$\{2,2,\ldots,2\}t$	5	6.98	0.0178	2.92E-19	0.0355	5.84E-19	1.93E-136	7
	$\{-\frac{1}{2},-\frac{1}{2},\ldots,-\frac{1}{2}\}t$	5	3.48	0.0236	2.20E-18	0.0471	4.39E-18	2.62E-130	7

Table 5: Comparisons between different methods for Example **4**

Method	X_0	n	$\parallel X_1 - X_0 \parallel$	$\parallel X_2 - X_1 \parallel$	$\parallel X_3 - X_2 \parallel$	$\parallel F(X_1) \parallel$	$\parallel F(X_2) \parallel$	$\parallel F(X_3) \parallel$	ACOC
FS	$\{-1,-1,\ldots,-1\}t$	7	4.55	0.349	0.000285	0.0288	0.0000219	1.56E-14	3
	$\{\frac{5}{3},\frac{5}{3},\ldots,\frac{5}{3}\}t$	7	2.96	0.825	0.00726	0.0751	0.000552	2.56E-10	3
$CHMT_5$	$\{-1,-1,\ldots,-1\}t$	5	4.59	0.315	9.42E-8	0.0246	7.40E-9	1.72E-41	5
	$\{\frac{5}{3},\frac{5}{3},\ldots,\frac{5}{3}\}t$	6	2.50	0.873	0.00667	0.106	0.000845	9.74E-15	5
MMK	$\{-1,-1,\ldots,-1\}t$	5	4.86	0.0205	4.64E-13	0.00160	3.59E-14	3.02E-67	5
	$\{\frac{5}{3},\frac{5}{3},\ldots,\frac{5}{3}\}t$	5	4.05	0.455	6.91E-9	0.146	1.34E-9	6.52E-47	5
SG	$\{-1,-1,\ldots,-1\}t$	5	4.73	0.166	3.65E-8	0.0124	2.82E-9	1.85E-42	5
	$\{\frac{5}{3},\frac{5}{3},\ldots,\frac{5}{3}\}t$	8	1.55	14.3	4.39	0.160	0.725	0.163	5
PM_5	$\{-1,-1,\ldots,-1\}t$	5	4.79	0.101	2.66E-9	0.00765	2.06E-10	2.5E-48	5
	$\{\frac{5}{3},\frac{5}{3},\ldots,\frac{5}{3}\}t$	5	3.42	0.333	1.40E-6	0.0272	1.08E-7	1.21E-34	5
$CHMT_7$	$\{-1,-1,\ldots,-1\}t$	5	5.01	0.139	2.81E-13	0.0106	2.23E-14	3.08E-96	7
	$\{\frac{5}{3},\frac{5}{3},\ldots,\frac{5}{3}\}t$	5	3.00	0.327	2.52E-7	0.0533	3.16E-8	4.63E-51	7
PM_7	$\{-1,-1,\ldots,-1\}t$	5	4.85	0.0305	2.11E-16	0.00229	1.63E-17	1.36E-116	7
	$\{\frac{5}{3},\frac{5}{3},\ldots,\frac{5}{3}\}t$	5	3.61	0.125	4.57E-12	0.0106	3.54E-13	3.10E-86	7

6.2 Applications

To check the applicability of the proposed scheme on real-life problems, we apply it on the mixed Hammerstein integral equation and Burgers' equation.

Problem 1 Consider the mixed Hammerstein integral equation:

$$\mathrm{x(s)} = 1 + \frac{1}{5}\int_0^1 \mathrm{G(s,t)x(t)^3dt},$$

such that $\mathrm{x} \in \mathrm{C}[0,1]$, and $\mathrm{s,t} \in [0,1]$, and the kernel $\mathrm{G(s,t)}$ is given by

$$\mathrm{G(s,t)} = \begin{cases} \mathrm{(1-s)t}, & \mathrm{t \leq s}, \\ \mathrm{(1-t)s}, & \mathrm{s \leq t}. \end{cases}$$

The integral equation is transformed into a finite-dimensional problem using the Gauss-Legender quadrature formula given by

$$\int_0^1 f(t)dt \approx \sum_{j=1}^{8} \omega_j f\left(t_j\right),$$

where the abscissas t_j and the weights ω_j are determined for $n = 8$ by the Gauss–Legendre quadrature formula. If we set $x(t_i) = x_i$, for $i = 1, 2, \ldots, 8$, then we obtain the following system of nonlinear equations

$$x_i - 1 - \frac{1}{5}\sum_{j=1}^{8} a_{ij}x_j^3 = 0, j = 1, 2, \ldots, 8,$$

where

$$a_{ij} = \begin{cases} \omega_j t_j (1 - t_i), & j \leq i, \\ \omega_j t_i \left(1 - t_j\right), & i < j, \end{cases}$$

where the abscissas t_j and the weights ω_j are known and presented in Tab. 6 for $m = 8$. The initial solutions considered are $X_0 = \{0,0,0,0,0,0,0,0\}^t$ and $X_0 = \{2,2,2,2,2,2,2,2\}^t$. The exact solution of this problem is $\alpha = \{1.002096245031\ldots, 1.009900316187\ldots, 1.019726960993\ldots, 1.026435743030\ldots, 1.026435743030\ldots, 1.019726960993\ldots, 1.009900316187\ldots, 1.002096245031\ldots\}^t$.

Table 6: Abscissas and weights and of Gauss-Legendre quadrature formula for ***m*** = ***8***

j	***tj***	ω***j***
1	0.019855071751231884158219565715263504 78…	0.050614268145188129576265677154981 09…
2	0.101666761293186630204223031762084781 58…	0.111190517226687235272177997213120 44…
2	0.237233795041835507091130475405376825 47…	0.156853322938943643668981100993300 65…
4	0.408282678752175097530261928819908009 66…	0.181341891689180991482575224638597 8060…
5	0.591717321247824902469738071180091990 33…	0.181341891689180991482575224638597 8060…
6	0.762766204958164492908869524594623174 52…	0.156853322938943643668981100993300 65…
7	0.898333238706813369795776968237915218 41…	0.111190517226687235272177997213120 44…
8	0.980144928248768115841780434284736495 21…	0.050614268145188129576265677154981 09…

Comparisons in Tab. 7 show that the proposed schemes have a good functioning and comparable to the other iterative methods of the same order.

Table 7: Comparisons between different methods for Problem **1**

Method	X_0	n	$\parallel X_1 - X_0 \parallel$	$\parallel X_2 - X_1 \parallel$	$\parallel X_3 - X_2 \parallel$	$\parallel F(X_1) \parallel$	$\parallel F(X_2) \parallel$	$\parallel F(X_3) \parallel$	ACOC
FS	$\{0,0,\ldots,0\}t$	5	2.86	0.0140	6.64E-9	0.0654	3.10E-8	3.63E-27	3
	$\{2,2,\ldots,2\}t$	5	2.75	0.0422	1.91E-7	0.197	8.94E-7	8.71E-23	3
$CHMT_5$	$\{0,0,\ldots,0\}t$	4	2.87	0.0000848	1.79E-26	0.000396	8.34E-26	0	5
	$\{2,2,\ldots,2\}t$	4	2.79	0.0000881	2.10E-26	0.000412	9.83E-26	0	5
MMK	$\{0,0,\ldots,0\}t$	4	2.87	0.0000132	4.19E-30	0.0000615	1.96E-29	0	5
	$\{2,2,\ldots,2\}t$	4	2.78	0.00442	1.80E-17	0.0207	8.40E-17	0	5
SG	$\{0,0,\ldots,0\}t$	4	2.87	0.000490	9.53E-22	0.00229	4.45E-21	0	5
	$\{2,2,\ldots,2\}t$	4	2.78	0.00417	4.35E-17	0.0195	2.03E-16	0	5
PM_5	$\{0,0,\ldots,0\}t$	4	2.87	0.000484	7.34E-22	0.00226	3.43E-21	0	5
	$\{2,2,\ldots,2\}t$	4	2.79	0.00243	2.40E-18	0.0113	1.12E-17	0	5
$CHMT_7$	$\{0,0,\ldots,0\}t$	3	2.87	2.72E-7	0	1.27E-6	0	0	7
	$\{2,2,\ldots,2\}t$	3	2.79	1.56E-6	0	7.31E-6	0	0	7
PM_7	$\{0,0,\ldots,0\}t$	3	2.87	0.0000170	0	0.0000796	0	0	7
	$\{2,2,\ldots,2\}t$	3	2.79	0.000137	0	0.000642	1.00E-33	0	7

Problem 2 Consider the following Burgers' equation selected from [25]:

$$\begin{cases} \mathrm{u_t + uu_x = Du_{xx}} \\ \mathrm{u(x,0)} = \dfrac{2\mathrm{D}\beta\pi\sin(\pi \mathrm{x})}{\alpha + \beta\cos(\pi \mathrm{x})}, \quad 0 \le \mathrm{x} \le 1 \\ \mathrm{u(0,t)} = 0, \quad \mathrm{t} \ge 0 \\ \mathrm{u(1,t)} = 0, \quad \mathrm{t} \ge 0. \end{cases}$$

We use discretization to solve this problem. Let $h = \dfrac{b-a}{N}$, and $k = \dfrac{T}{M}$ be the spatial and temporal step sizes respectively, where N and M are numbers of subintervals in x and t directions respectively. Therefore, for this problem we select points (x_i, t_j) from a grid of domain $[0,1] \times [0,T]$, where, $x_i = 0 + ih$, $i = 0, 1, \ldots, M$ and $t_j = 0 + jk$, $j = 0, 1, \ldots, N$. Let w_{ij} be the approximate solution at (x_i, t_j). By applying the central differences to u_x and u_{xx}, and the backward difference to u_t, we get the following nonlinear system:

$$\frac{w_{ij} - w_{i,j-1}}{k} + w_{ij}\left(\frac{w_{i+1,j} - w_{i-1,j}}{2h}\right) = \frac{D}{h^2}\left(w_{i+1,j} - 2w_{ij} + w_{i-1,j}\right).$$

If we let $\sigma = \dfrac{Dk}{h^2}$, and $z_i = w_{ij}$, then we are trying to solve the following system of equations:

$F_i(z_1, z_2, \ldots, z_M) = z_i + \dfrac{k}{2h} z_i(z_{i+1} - z_{i-1}) - \sigma(z_{i+1} - 2z_i + z_{i-1}) - w_{i,j-1} = 0.$

For the unknowns $z_1, z_2, \ldots, z_M$. The last term $w_{i,j-1}$ is known from the previous time step. The first and the last equations in the system can be replaced by using the given boundary conditions. In our problem, the first and the last equations in the system are

$F_1 = z_1,$

$F_M = z_M.$

So, we have M equations with M unknowns.

We find the approximate solution of the problem using the proposed method PM_5 at $x = 0.5$ and $t = 0.2$. To check the effect of the temporal step sizes on the solution, we select different values for k, which means a different number of steps to reach the wanted time. Consider for our problem that the diffusion coefficient $D = 0.05$, $\alpha = 5$, and $\beta = 4$. The exact solution of the given Burgers' equation is given by:

$$u(x,t) = \frac{2D\beta\pi e^{-D\pi^2 t}\sin(\pi x)}{\alpha + \beta e^{-D\pi^2 t}\cos(\pi x)}.$$

The exact solution for this problem is $u(0.5, 0.2) = 0.2277071734\ldots$. Based on spatial step size equals $\frac{1}{10}$, we choose $X_0 = \overbrace{\{0.6, 0.6, \cdots, 0.6\}}^{10-\text{times}}$. Tab. 8 illustrates the numerical results of this problem. The effect of the selected k is clear. The results become better whenever we have a smaller temporal step size. Based on that, we will compare our proposed schemes to the other schemes for $h = 0.1$ and $k = 0.01$. We compare the approximate solutions at $x = 0.5$ and $t = 0.2$ for $n = 3$ and $n = 4$, that is X_3 and X_4. Also, we find the norms of the functions $F(X_3)$ and $F(X_3)$. Finally, we find the norm of the difference between the two consecutive iterations $X_4 - X_3$ for each selected method. Comparisons results are shown in Tab. 9. It is clear that the proposed schemes perform in a good way, and in general, give results which are better than the other selected schemes.

Table 8: Numerical results for Problem 2

h	k	n	$u(0.5, 0.2)$	$w(0.5, 0.2)$	Error	$\Vert F(X_n) \Vert$	$\Vert X_4 - X_3 \Vert$
0.1	0.04	3	0.227707...	0.227014...	0.000693...	1.89E-165	1.59E-165
0.1	0.04	4	0.227707...	0.227014...	0.000693...	7.06E-829	
0.1	0.02	3	0.227707...	0.226946...	0.000761...	1.10E-195	2.00E-195
0.1	0.02	4	0.227707...	0.226946...	0.000761...	1.37E-980	
0.1	0.01	3	0.227707...	0.226918...	0.000789...	6.10E-229	1.77E-228
0.1	0.01	4	0.227707...	0.226918...	0.000789...	1.00E-1147	

Table 9: Comparisons between different methods for Problem 2

Method	$X_3(0.5, 0.2)$	$X_4(0.5, 0.2)$	$\Vert F(X_3) \Vert$	$\Vert F(X_4) \Vert$	$\Vert X_4 - X_3 \Vert$
FS	0.2269178...	0.2269178...	5.51E-52	8.23E-158	1.82E-51
$CHMT_5$	0.2269178...	0.2269178...	6.10E-229	1.00E-1147	1.39E-228
MMK	0.2269178...	0.2269178...	1.60E-213	3.80E-1070	5.38E-213
SG	0.2269178...	0.2269178...	4.12E-226	2.74E-1133	1.18E-225
PM_5	0.2269178...	0.2269178...	6.10E-229	1.00E-1147	1.77E-228
$CHMT_7$	0.2269178...	0.2269178...	4.51E-628	0	2.56E-627
PM_7	0.2269178...	0.2269178...	4.51E-628	0	2.56E-627

7 Conclusion

In this study, we have proposed an iterative scheme for systems of nonlinear equations of fifth-order of convergence. We have improved the proposed scheme to a generalized scheme of arbitrary odd order. The proposed method is based on Frontini–Sormani iterative method and developed using additional step with the usage of first derivative approximation. The software Mathematica has been used to show the order of convergence of the proposed method. Different comparisons were used to compare our proposed scheme to the other schemes of the same order, including the efficiency index, computational efficiency index, basins of attractions and several numerical problems. Comparisons show that the efficiency index and the computational efficiency index need not be proper tools for the efficiency of the iterative scheme. As an application, we test the proposed method on the mixed Hammerstein integral equation and Burgers' equation. Comparisons show that the proposed scheme is of excellent performance and overall, it is comparable to the other iterative techniques used in the comparisons regarding the convergence speed, accuracy and the area of convergence in the basins of attraction.

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