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Article

Synchronization Phenomena Investigation of a New Nonlinear Dynamical System 4D by Gardano's and Lyapunov's Methods

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Abstract: Synchronization is one of the most important characteristics of dynamic systems. For this paper, the authors obtained results for the nonlinear systems controller for the custom Synchronization of two 4D systems. The findings have allowed authors to develop two analytical approaches using the second Lyapunov (Lyp) method and the Gardano method. Since the Gardano method does not involve the development of special positive Lyp functions, it is very efficient and convenient to achieve excessive system SYCR phenomena. Error is overcome by using Gardano and overcoming some problems in Lyp. Thus we get a great investigation into the convergence of error dynamics, the authors in this paper are interested in giving numerical simulations of the proposed model to clarify the results and check them, an important aspect that will be studied is Synchronization Complete hybrid SYCR and anti-Synchronization, by making use of the Lyapunov expansion analysis, a proposed control method is developed to determine the actual. The basic idea in the proposed way is to receive the evolution of between two methods. Finally, the present model has been applied and showing in a new attractor, and the obtained results are compared with other approximate results, and the nearly good coincidence was obtained.

Keywords: Chaos; Lu model; anti-synchronization; hybrid synchronization; Gardano's method; nonlinear dynamical system

1 Introduction

Chaotic systems with real state variables are boing found and studied with increased attention in several aspects of nonlinear dynamical systems, the first physical and mathematical model of a chaotic system is the system of Lorenz, which only includes real variables discovered in 1963 and opens the way to other chaos systems such as the system of Chen, Lu's system (2002), Liu's system (2004), and the system of the Pan system (2009). Each system has a 3-D of differential equations and just one positive Lyapunov exponent [1–5].

Exponent Lyapunov and nonlinear dynamic systems attractor play an important and actively involved role in classifying these systems and have attracted increasing interest in engineering application and different scientific research as, encrypting [6–9], engineering [10–13], and nonlinear circuits [14]. One important application in the field of engineering is secure communication i.e., messages crazy by simple chaotic processes like these are not always secure. The suggestion is that the higher dimensional hyperchaotic systems can be used to solve this problem, thereby the randomness and unpredictability [15–20].

Rössler performed the first 4-D hyperchaotic system with real variables in 1979 with two positive exponents of Lyapunov and discovered a further 4-D and 5-D hyperchaotic system with three positive exponents of Lyapunov [21–25] and some other systems. In contrast to low dimensions, dynamic systems with higher dimensions are effective and interesting [17–19].

Several papers on the subject today are dedicated to studying the new hyperchaotic systems in higher dimensions (Dimension 5) [16]. But most of these research focuses on 4-D and 5-D systems only, while a few kinds of researches are available in 6 dimensional nonlinear dynamical systems [17–20]. The 4D system consists of ten-term operators with four parameters and different features which include the Lyapunov exponents of balances and stability. The thrilling attractor is one of the latest dynamic systems classifications, with recent research separating attractors into self-enthusiastic or secret ones. Within the following paragraphs, the results of this work are summarized.

- The synchronization of similar 4-D hyperchaotic systems are studied and is then theoretically introduced as an Engineering application to detect error dynamics between each and its stable communication.
- Non-linear stability-based control methods in Lyapunov, Gardano approach design the various controllers of synchronization phenomena.
- By comparing the results of the Lyapunov method with the Gardano method, the best fitting controllers are found.

2 The Description of the Problem and Our Solution

We use the second Lyapunov method and the methods of Gardano, where we infer that Lyapunov functions as a certain constructive tool as:

$$V(e) = \frac{1}{2}\sum_{i=1}^{n} e_i^2 = e^T P e,$$

$$P = diag(1/2, 1/2, \ldots, 1/2) \tag{1}$$

There P is a regular function and the Lyapunov function derivatives

$$\dot{V}(e) = \sum_{i=1}^{n} e_i \dot{e}_i = -e^T Q e \tag{2}$$

Be certainly negative, i.e., Q is a positive, square matrix. Nevertheless, if the Q matrix is defined as a negative, we will change the P matrix to ensure that a given Q matrix is obtained.

While in Gardano [1–4], the distinguishing formula of Eq. (3) is considered in the 4D hyperchaotic method

$$(\lambda + B_4)\left(\lambda^3 + B_1\lambda^2 + B_2\lambda + B_3\right) = 0 \quad (3)$$

Let

$$g = B_3 - \frac{1}{3}B_1B_2 + \frac{2}{27}{B_1}^3 \quad (4)$$

$$\Delta = {B_3}^2 + \frac{4}{27}{B_2}^3 - \frac{2}{3}B_1B_2B_3 - \frac{1}{27}{B_1}^2{B_2}^2 + \frac{4}{27}{B_1}^3B_3 \quad (5)$$

This approach allows one to find the roots of the cubic equation (Eq. (3)) on the basis of Δ as:

- If $\Delta = 0$, then there are three roots in the second term of Eq. (3), but one is multiple:

$$\lambda_1 = -2\sqrt[3]{\frac{g}{2}} - \frac{B_1}{3}$$

$$\lambda_{2,3} = \sqrt[3]{\frac{g}{2}} - \frac{B_1}{3} \quad (6)$$

- When $\Delta < 0$, then three separate root terms have been described by the second term (3):

$$\lambda_{i+1} = \sqrt[6]{16(g^2 - \Delta)}\, cos\frac{cos^{-1}\frac{-g}{\sqrt{g^2-\Delta}} + 2\pi i}{3} - \frac{B_1}{3}, \quad i = 0, 1, 2. \quad (7)$$

- When $\Delta > 0$, then the second term of Eq. (3) has one real root and two complexes conjugate roots with imaginary sections which do not vanish:

$$\begin{cases} \lambda_1 = \sqrt[3]{\frac{-g-\sqrt{\Delta}}{2}} + \sqrt[3]{\frac{-g+\sqrt{\Delta}}{2}} - \frac{B_1}{3} \\ \lambda_2 = -\frac{1}{2}\left(\sqrt[3]{\frac{-g-\sqrt{\Delta}}{2}} + \sqrt[3]{\frac{-g+\sqrt{\Delta}}{2}}\right) - \frac{B_1}{3} + i\frac{\sqrt{3}}{2}\left(\sqrt[3]{\frac{-g-\sqrt{\Delta}}{2}} - \sqrt[3]{\frac{-g+\sqrt{\Delta}}{2}}\right) \\ \lambda_3 = -\frac{1}{2}\left(\sqrt[3]{\frac{-g-\sqrt{\Delta}}{2}} + \sqrt[3]{\frac{-g+\sqrt{\Delta}}{2}}\right) - \frac{B_1}{3} - i\frac{\sqrt{3}}{2}\left(\sqrt[3]{\frac{-g-\sqrt{\Delta}}{2}} - \sqrt[3]{\frac{-g+\sqrt{\Delta}}{2}}\right) \end{cases} \quad (8)$$

Here, to construct all roots with negative real parts, not choosing an appropriate nonlinear controller U like the Lyapunov method is essential.

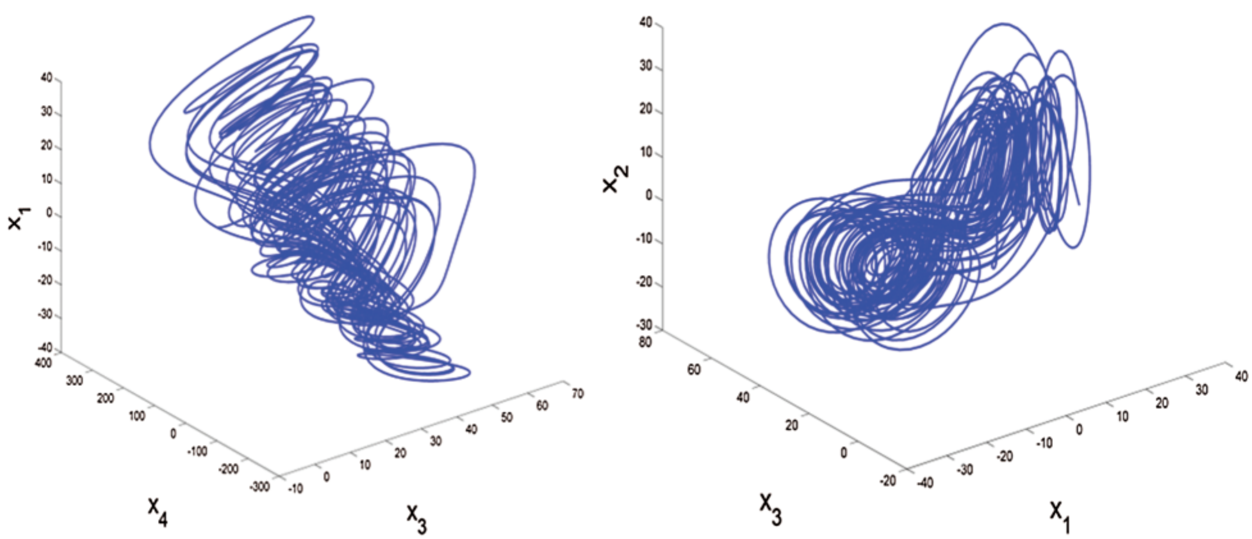
Briefly, this final point poses three fundamental questions. First, does the Lyapunov method always succeed? Second, is the Gardano method better? Thirdly, how can these two approaches be distinguished? This paper starts with two ways of answering these questions.

3 System Portrayal

The Lorenz system was one of the most commonly studied 3-D chaotic systems. By adding a linear feedback controller, the original design was changed into a 4-D and 5-D hyperchaotic design. The new 4-D hyperchaotic system that contains is designed three positive Lyapunov Exponents $LE_1 = 0.94613$, $LE_2 = 0.28714$, $LE_3 = 0.0047625$, and one negative Lyapunov Exponents $LE_4 = -12.4021$. The 4-D system which is described by the following mathematical form:

$$\begin{cases} \dot{x}_1 = a\,(x_2 - x_1) + x_4 \\ \dot{x}_2 = cx_1 - x_1x_3 - x_2 \\ \dot{x}_3 = -bx_3 + x_1x_2 \\ \dot{x}_4 = hx_4 - x_1x_3 \end{cases} \tag{9}$$

where the real state variables are x_1, x_2, x_3, x_4, and a, b, c, h, the all positive real parameters are equal $(10, 8/3, 34, 2.5)$, and this system is rich of dynamical properties. Figs. 1a and 1b show the 3-D attractor of the system (9), while Figs. 2a and 2b display the 2-D attractor of the structure (9).



(a) (b)

Figure 1: 3-D attractor of the system (9) in the (a) $(\boldsymbol{x_1}, \boldsymbol{x_3}, \boldsymbol{x_4})$ space; (b) $(\boldsymbol{x_1}, \boldsymbol{x_3}, \boldsymbol{x_2})$ space

3.1 Lyapunov Exponents and Dimensions

The numerical simulation of a = 10, b = 8/3, c = 34, h = 2, 5 was performed based on Wolf Algorithm and MATLAB software. The initial value (15, 8, −1, −2) of the system (9) was hyperchaotic, with three positive exponents of Lyapunov, i.e., $LE_1 = 0.94613$, $LE_2 = 0.28714$, $LE_3 = 0.0047625$.

The exponents of the plot of Lyapunov are shown in Fig. 3.

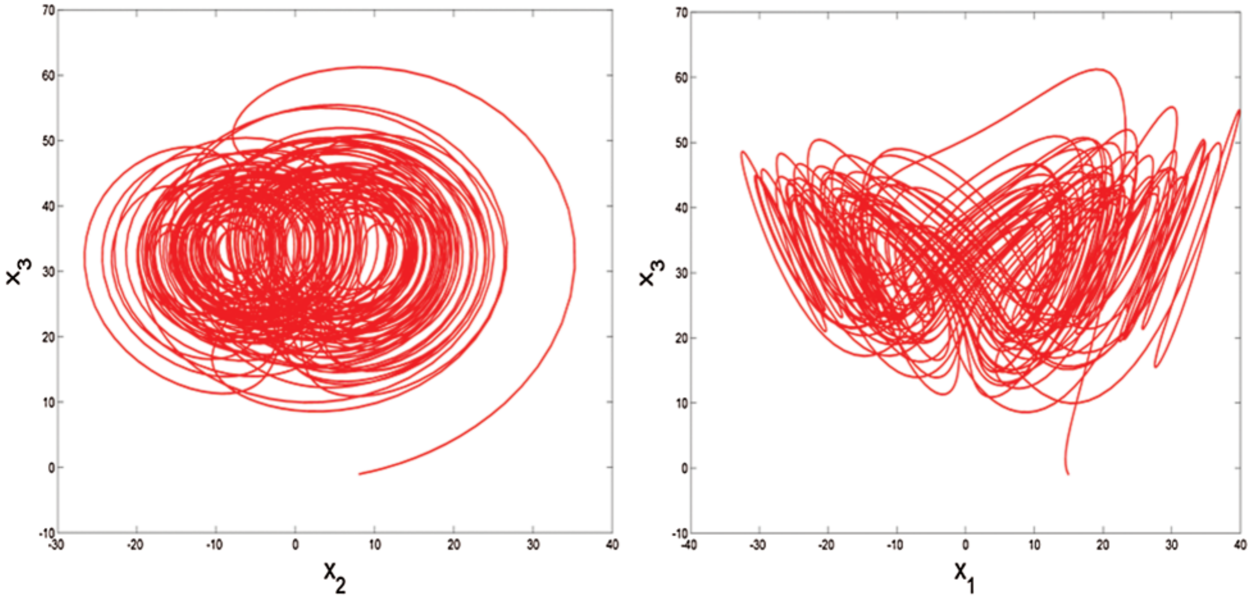


Figure 2: 2-D attractor of the system (9) in the (a) (x_2, x_3) plane; (b) (x_1, x_3) plane

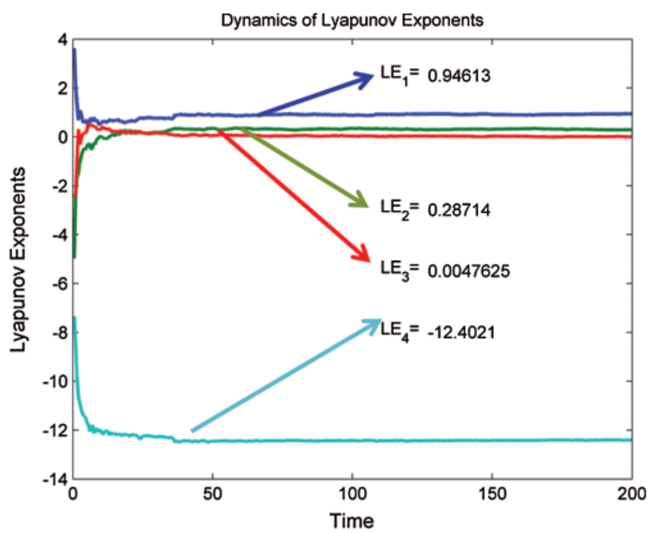


Figure 3: 4-D hyperchaotic system exponents of Lyapunov

Dimensions of Lyapunov are found as:

$$D_{LE} = j + \frac{1}{|LE_{j+1}|} \sum_{i=1}^{j} LE_i = 5 + \frac{LE_1 + LE_2 + LE_3}{|LE_4|} = 4.9002$$

4 Synchronization Phenomena

4.1 The New Lorenz 4-D Hyperchaotic Systems are Synchronized

In this section one of the main applications of secure communication engineering is considered theoretical studies and numerical simulations. Therefore, the first system (*called drive system*) represents the image or message information to be transmitted and, while the second system represents noise following this information, it ensures that it is not penetrated. The second system (*called response system*) assumes the machine (9) is a drive mechanism and is writable as,

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix}=\begin{bmatrix}-a & a & 0 & 1\\ c & -1 & 0 & 0\\ 0 & 0 & -b & 0\\ 0 & 0 & 0 & h\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}+\begin{bmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-x_1x_3\\ x_1x_2\\ -x_1x_3\end{bmatrix} \tag{10}$$

$$A=\begin{bmatrix}-a & a & 0 & 1\\ c & -1 & 0 & 0\\ 0 & 0 & -b & 0\\ 0 & 0 & 0 & h\end{bmatrix},\quad B=\begin{bmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix},\quad C=\begin{bmatrix}-x_1x_3\\ x_1x_2\\ -x_1x_3\end{bmatrix}$$

A and BC represents parameters matrix and nonlinear part of the system (9) respectively.

While the response system is as follows:

$$\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\\ \dot{y}_3\\ \dot{y}_4\end{bmatrix}=A\begin{bmatrix}y_1\\ y_2\\ y_3\\ y_4\end{bmatrix}+\left(B\begin{bmatrix}-y_1y_3\\ y_1y_2\\ -y_1y_3\end{bmatrix}+\begin{bmatrix}u_1\\ u_2\\ u_3\\ u_4\end{bmatrix}\right) \tag{11}$$

and let $U=[u_1,u_2,u_3,u_4]^T$ is the nonlinear controller to be designed.

The synchronization error dynamics between the 4-D hyperchaotic system (10) and system (11) is defined as $e_i=y_i-x_i$, $i=1,2,3,4$ and satisfied that, $\lim\limits_{t\to\infty} e_i=0$.

The dynamics of the error are defined as follows:

$$\begin{bmatrix}\dot{e}_1\\ \dot{e}_2\\ \dot{e}_3\\ \dot{e}_4\end{bmatrix}=A_1\begin{bmatrix}e_1\\ e_2\\ e_3\\ e_4\end{bmatrix}+\left(BD+\begin{bmatrix}u_1\\ u_2\\ u_3\\ u_4\end{bmatrix}\right),\quad D=\begin{bmatrix}-e_1e_3-x_3e_1-x_1e_3\\ e_1e_2+x_2e_1+x_1e_2\\ -e_1e_3-x_3e_1-x_1e_3\end{bmatrix}$$

i.e.,

$$\begin{cases}\dot{e}_1=ae_2-ae_1+e_4+u_1\\ \dot{e}_2=\mathrm{c}e_1-e_2-e_1e_3-x_3e_1-x_1e_3+u_2\\ \dot{e}_3=-be_3+e_1e_2+x_2e_1+x_1e_2+u_3\\ \dot{e}_4=he_4-e_1e_3-x_3e_1-x_1e_3+u_4\end{cases} \tag{12}$$

The matrix A_1 equal to A, i.e., $A_1 = A$ for identical systems. But, in non-identical systems (different) the matrix $A_1 \neq A$.

Now, by designing several controllers based on Lyapunov and Gardano methods we will try to control the error system (12), and compare them. Here arises the problem of the two methods, which is the better method? Our questions are answered in the following theorems.

Theorem 1. If the regulator U of structure (12) is scheme as the following:

$$\begin{cases} u_1 = -\mathrm{c}e_2 + x_3e_2 - x_2e_3 + x_3e_4 \\ u_2 = -ae_1 \\ u_3 = x_1e_4 + e_1e_4 \\ u_4 = -e_1 - 2he_4 \end{cases} \tag{13}$$

The system (11) can then be tracked with two methods by system (10).

Proof. Replace the error dynamics (12) mechanism over control, we receive:

$$\begin{cases} \dot{e}_1 = ae_2 - ae_1 + e_4 - \mathrm{c}e_2 + x_3e_2 - x_2e_3 + x_3e_4 \\ \dot{e}_2 = \mathrm{c}e_1 - e_2 - e_1e_3 - x_3e_1 - x_1e_3 - ae_1 \\ \dot{e}_3 = -be_3 + e_1e_2 + x_2e_1 + x_1e_2 + x_1e_4 + e_1e_4 \\ \dot{e}_4 = -he_4 - e_1e_3 - x_3e_1 - x_1e_3 - e_1 \end{cases} \tag{14}$$

We are now building a positively defined Lyapunov candidate, based on the **Lyapunov method**,

$$V(e) = e^T Pe = \frac{1}{2}e_1^2 + \frac{1}{2}e_2^2 + \frac{1}{2}e_3^2 + \frac{1}{2}e_4^2 \tag{15}$$

where P defines as in Eq. (1), Lyapunov's derivative $V(e)$ function is time-related as:

$$\dot{V} = e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 + e_4\dot{e}_4$$

$$\dot{V} = e_1(ae_2 - ae_1 + e_4 - \mathrm{c}e_2 + x_3e_2 - x_2e_3 + x_3e_4) + e_2(\mathrm{c}e_1 - e_2 - e_1e_3 - x_3e_1 - x_1e_3 - ae_1) \\ + e_3(-be_3 + e_1e_2 + x_2e_1 + x_1e_2 + x_1e_4 + e_1e_4) + e_4(-he_4 - e_1e_3 - x_3e_1 - x_1e_3 - e_1)$$

$$\dot{V} = -ae_1^2 - e_2^2 - be_3^2 - he_4^2 = -e^T Qe \tag{16}$$

where $Q = diag(a, 1, b, h)$, so $Q > 0$. Consequently, $\dot{V}(e_i)$ is negative definite on R^4. The nonlinear controller is suitable and the complete synchronization is achieved.

The characteristic equation between system (12) and control (13) is defined in **Gardano method**, and according to Eq. (3),

$$(\lambda + p)\left(\lambda^3 + B_1\lambda^2 + B_2\lambda + B_3\right) = 0$$

where

$$\begin{cases} B_1 = a + h + 1 \\ B_2 = 1 + a + h + a^2 + c^2 - 2ac + ah \\ B_3 = ah + a^2h + c^2h - 2ach + 1 \end{cases} \tag{17}$$

For simplified, we substitute the value of constants as $a=10$, $b=8/3$, $c=34$, $h=2.5$, in the above equation can be rewritten as:

$$(\lambda+(8/3))\left(\lambda^3+13.5\lambda^2+614.5\lambda+1466\right)=0 \tag{18}$$

Therefore, we have $g=-1117$, $\Delta=2.6403\times 10^7$. Since $\Delta>0$ than the roots are calculate according to Eq. (8) as:

$$\begin{cases}\lambda_1=\sqrt[3]{-2010.692480}+\sqrt[3]{3127.692480}-4.5\xrightarrow{yields}\lambda_1=-2.4973\\ \lambda_2=-\frac{1}{2}(2.003)-4.5+i\frac{\sqrt{3}}{2}(-27.2459)\xrightarrow{yields}\lambda_2=-5.5013-23.5957i\\ \lambda_3=-\frac{1}{2}(2.003)-4.5-i\frac{\sqrt{3}}{2}(-27.2459)\xrightarrow{yields}\lambda_2=-5.5013+23.5957i\\ \lambda_4=-8/3\end{cases} \tag{19}$$

Of course, all roots with negative actual parts are successfully synced with system (11) and system (10), therefore the Gardano method is efficient. The evidence is complete. Such tests are numerically checked in Figs. 4 and 5. Where we take the drive system and the response system's initial values are $(15,8,-1,-2)$ and $(-15,-10,16,8)$ respectively.

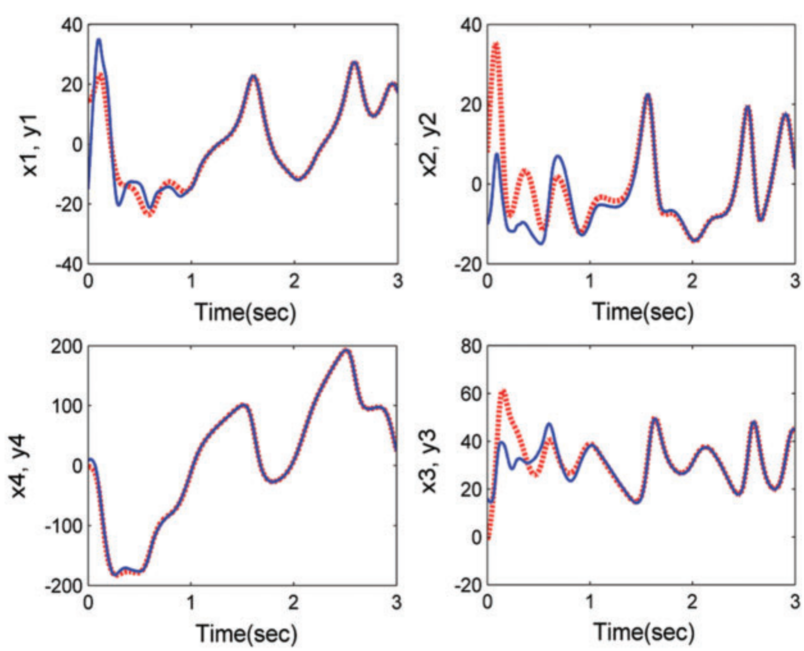


Figure 4: Anti-synchronization between systems (11) and (10) with control (13)

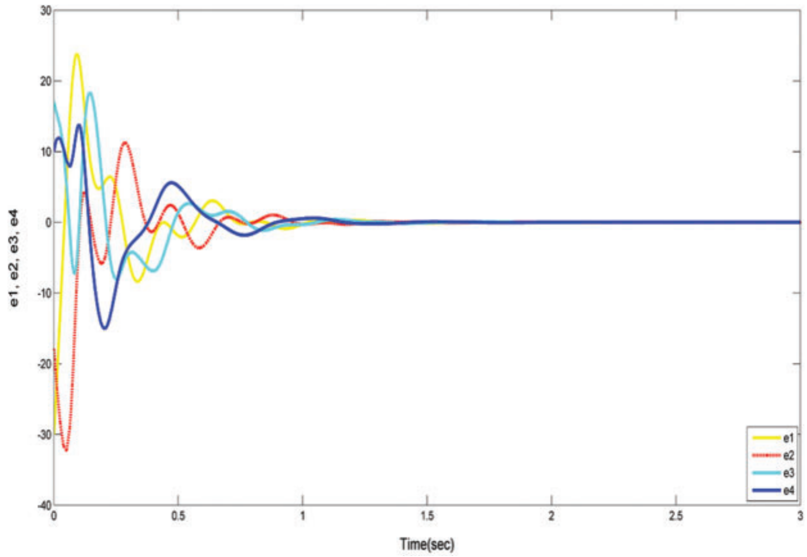


Figure 5: The convergence of system (12) with controllers (13)

4.2 Anti-Synchronization

The non-linear control approach, which uses two theoretical methods, offers an anti-synchronization between two related highly hyperchaotic systems. To stop collisions, it involves two systems; the first (called drive systems) reflects the first, and the second (called the responses system). This mechanism is the second train, which is used to ensure that there is no collision with the first train. The second is used to prevent collisions. The first train will have a second system. Suppose that the system (10) is the drive system and the reaction system (11).

The 4D hyperchaotic system (10) and system (11) are described as anti-synchronization error dynamics as $e_i = y_i + x_i$, $i = 1, 2, 3, 4$ and satisfied that, $\lim_{t\to\infty} e_i = 0$.

The error dynamics is calculated as the following:

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \\ \dot{e}_4 \end{bmatrix} = A_1 \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} + \left(BD + \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \right), \quad D = \begin{bmatrix} -y_1e_3 - x_3e_1 + 2y_1x_3 \\ e_1e_2 - x_2e_1 - x_1e_2 + 2x_1x_2 \\ -y_1e_3 - x_3e_1 + 2y_1x_3 \end{bmatrix}$$

i.e.,

$$\begin{cases} \dot{e}_1 = ae_2 - ae_1 + e_4 + u_1 \\ \dot{e}_2 = \mathrm{c}e_1 - e_2 - y_1e_3 - x_3e_1 + 2y_1x_3 + u_2 \\ \dot{e}_3 = -be_3 + e_1e_2 - x_2e_1 - x_1e_2 + 2x_1x_2 + u_3 \\ \dot{e}_4 = he_4 - y_1e_3 - x_3e_1 + 2y_1x_3 + u_4 \end{cases} \tag{20}$$

Theorem 2. For system (15) with nonlinear control $U = [u_1, u_2, u_3, u_4]^T$ such that

$$\begin{cases} u_1 = x_3 e_2 + x_2 e_3 + x_3 e_4 \\ u_2 = -e_1(c+a) + y_1 e_3 - 2y_1 x_3 + x_1 e_3 \\ u_3 = -e_1 e_2 - 2x_1 x_2 + y_1 e_4 \\ u_4 = -e_1 - 2he_4 - 2y_1 x_3 \end{cases} \tag{21}$$

Proof. From the above control (21) with the error system (20), we get:

$$\begin{cases} \dot{e}_1 = ae_2 - ae_1 + e_4 + x_3 e_2 + x_2 e_3 + x_3 e_4 \\ \dot{e}_2 = -e_2 - x_3 e_1 - ae_1 + x_1 e_3 \\ \dot{e}_3 = -be_3 - x_2 e_1 - x_1 e_2 + y_1 e_4 \\ \dot{e}_4 = -he_4 - y_1 e_3 - x_3 e_1 - e_1 \end{cases} \tag{22}$$

Now, based on the **Lyapunov method:**

The derivative of the Lyapunov function $V(e)$ with respect to time is

$$\dot{V} = e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 + e_4\dot{e}_4$$

$$\dot{V} = \boldsymbol{e_1}(ae_2 - ae_1 + e_4 + x_3 e_2 + x_2 e_3 + x_3 e_4) + \boldsymbol{e_2}(-e_2 - x_3 e_1 - ae_1 + x_1 e_3) + \boldsymbol{e_3}(-be_3 - x_2 e_1 - x_1 e_2 + y_1 e_4) + \boldsymbol{e_4}(-he_4 - y_1 e_3 - x_3 e_1 - e_1)$$

$$\dot{V} = -ae_1^2 - e_2^2 - be_3^2 - he_4^2 = -e^T Qe \tag{23}$$

where $Q = diag(a, 1, b, h)$, so $Q > 0$. Consequently, $\dot{V}(e_i)$ is negative definite on R^4. The nonlinear controller is suitable and the anti-synchronization is achieved.

In **Gardano method**

$$\begin{cases} B_1 = a + h + 1 \\ B_2 = 1 + a + h + \mathrm{a}^2 + ah \\ B_3 = ah + \mathrm{a}^2 h + 1 \end{cases} \tag{24}$$

After substituting the values of the constants (a,b,c,h), we get

$$(\lambda + (8/3))\left(\lambda^3 + 13.5\lambda^2 + 138.5\lambda + 276\right) = 0$$

Therefore, we have $g = -165$, $\Delta = 96855.1644$. Since $\Delta > 0$ than the roots are calculate according to Eq. (8) as:

$$\begin{cases} \lambda_1 = \sqrt[3]{-73.10781180} + \sqrt[3]{238.1078118} - 4.5 \overset{yields}{\rightarrow} \lambda_1 = -2.4833 \\ \lambda_2 = -\frac{1}{2}(2.017) - 4.5 + i\frac{\sqrt{3}}{2}(-10.379) \overset{yields}{\rightarrow} \lambda_2 = -5.5083 - 8.9889i \\ \lambda_3 = -\frac{1}{2}(2.017) - 4.5 - i\frac{\sqrt{3}}{2}(-10.379) \overset{yields}{\rightarrow} \lambda_2 = -5.5083 + 8.9889i \\ \lambda_4 = -8/3 \end{cases} \tag{25}$$

Of course, all roots with negative actual parts are successfully synced with the system (11) and system (10), therefore the Gardano method is efficient. The evidence is complete. These results are numerically checked in Figs. 6 and 7. Where the drive system and the response system initial values are used $(15, 8, -1, -2)$ and $(-15, -10, 16, 8)$ respectively.

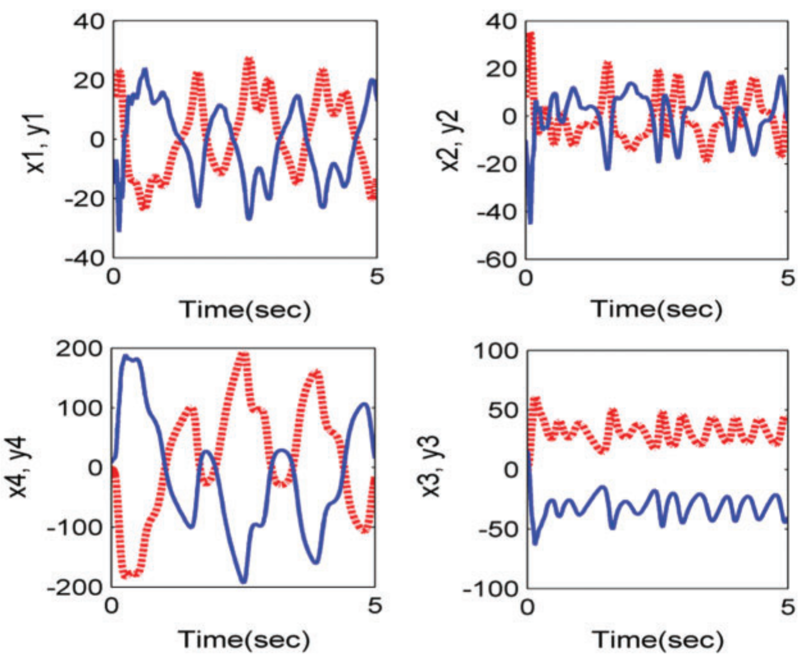


Figure 6: Anti-synchronization between systems (11) and (10) with control (13)

4.3 Hybrid Synchronization

Hybrid synchronization is a mixture of the previous two phenomena (*Complete synchronization and Anti-synchronization*). There are thus two appropriate systems, one system (*Called drive system*), the other system (*Called response system*). There are two systems. Assume that the drive system is (10) and the reaction system is (11):

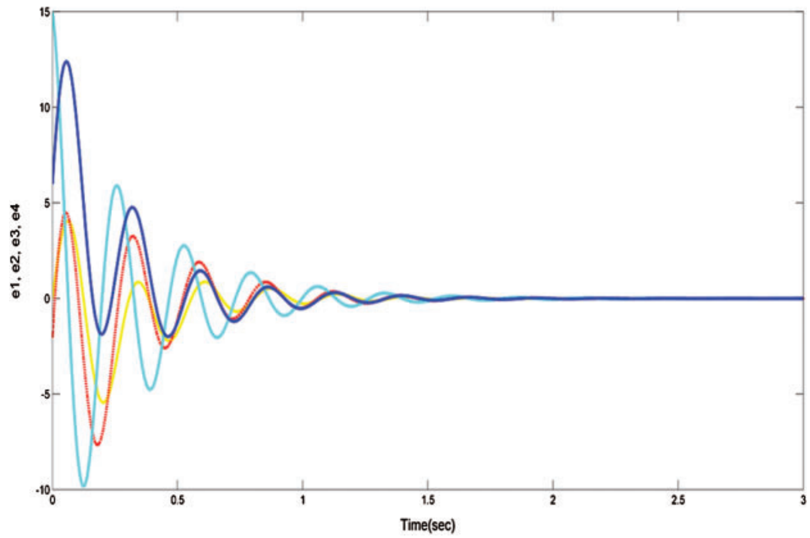


Figure 7: The convergence of system (20) with controllers (21)

The hybrid synchronization error dynamics is defined as $e_i = y_i - x_i$, $e_j = y_j + x_j$, $i = 1, 3$; $j = 2, 4$, and satisfied that , $\lim_{t\to\infty} e_i = 0$.

The error dynamics is calculated as the following:

$$\begin{bmatrix}\dot{e}_1\\ \dot{e}_2\\ \dot{e}_3\\ \dot{e}_4\end{bmatrix} = A_1\begin{bmatrix}e_1\\ e_2\\ e_3\\ e_4\end{bmatrix} + \left(BD + \begin{bmatrix}u_1\\ u_2\\ u_3\\ u_4\end{bmatrix}\right), \quad D = \begin{bmatrix}-y_1e_3 - x_3e_1 + 2y_1x_3\\ e_1e_2 - x_2e_1 - x_1e_2 + 2x_1x_2\\ -y_1e_3 - x_3e_1 + 2y_1x_3\end{bmatrix}$$

i.e.,

$$\begin{cases}\dot{e}_1 = ae_2 - ae_1 + e_4 + u_1\\ \dot{e}_2 = ce_1 - e_2 - y_1e_3 - x_3e_1 + 2y_1x_3 + u_2\\ \dot{e}_3 = -be_3 + e_1e_2 - x_2e_1 - x_1e_2 + 2x_1x_2 + u_3\\ \dot{e}_4 = he_4 - y_1e_3 - x_3e_1 + 2y_1x_3 + u_4\end{cases} \tag{26}$$

Theorem 3. If the nonlinear control U of error dynamical system (6) is designed as the following:

$$\begin{cases}u_1 = -e_2(a+c) + 2ax_2 + 2x_4 - x_3e_2 + x_2e_3 - x_3e_4\\ u_2 = -2cx_1 + 2y_1x_3 - x_1e_3\\ u_3 = y_1e_2 - e_1e_2 + 2x_1x_2 + y_1e_4\\ u_4 = -e_1 - 3he_4 + 2y_1x_3\end{cases} \tag{27}$$

Proof. Rewrite system (26) with control (27) as follows:

$$\begin{cases} \dot{e}_1 = -ae_1 + e_4 - ce_2 - x_3e_2 + x_2e_3 - x_3e_4 \\ \dot{e}_2 = \mathrm{c}e_1 - e_2 - y_1e_3 + x_3e_1 - x_1e_3 \\ \dot{e}_3 = -be_3 - x_2e_1 + x_1e_2 + y_1e_2 + y_1e_4 \\ \dot{e}_4 = -2he_4 - y_1e_3 + x_3e_1 - e_1 \end{cases} \tag{28}$$

Now, based on the **Lyapunov method:**

The derivative of the Lyapunov function $V(e)$ with respect to time is

$$\dot{V} = e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 + e_4\dot{e}_4$$

$$\dot{V} = \boldsymbol{e_1}(-ae_1 + e_4 - ce_2 - x_3e_2 + x_2e_3 - x_3e_4) + \boldsymbol{e_2}(\mathrm{c}e_1 - e_2 - y_1e_3 + x_3e_1 - x_1e_3) \\ + \boldsymbol{e_3}(-be_3 - x_2e_1 + x_1e_2 + y_1e_2 + y_1e_4) + \boldsymbol{e_4}(-he_4 - y_1e_3 + x_3e_1 - e_1)$$

$$\dot{V} = -ae_1^2 - e_2^2 - be_3^2 - he_4^2 = -e^TQe \tag{29}$$

where $Q = diag(a, 1, b, h)$, so $Q > 0$. Consequently, $\dot{V}(e_i)$ is negative definite on R^4. The nonlinear controller is suitable and the anti-synchronization is achieved.

In **Gardano method:**

$$\begin{cases} B_1 = a + 2h + 1 \\ B_2 = 1 + a + 2h + \mathrm{c}^2 + 2ah \\ B_3 = 2ah + 2\mathrm{c}^2h + 1 \end{cases} \tag{30}$$

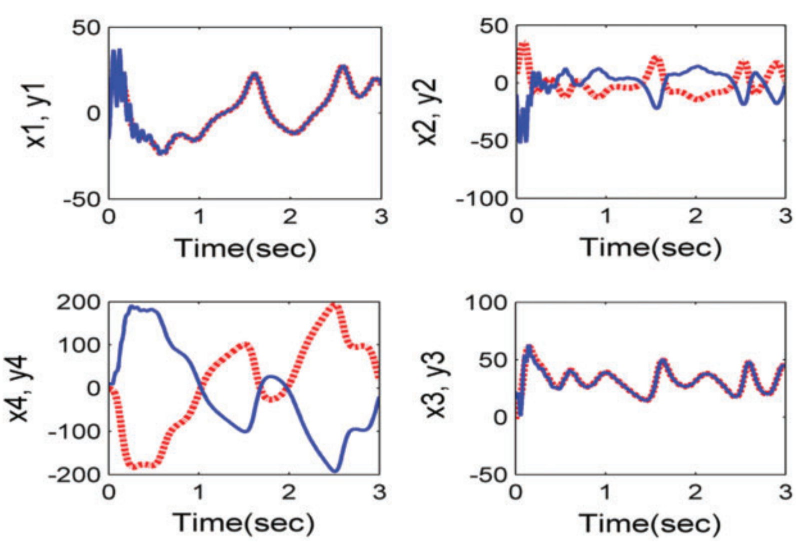


Figure 8: Anti-synchronization between systems (11) and (10) with control (13)

After substituting the values of the constants (a, b, c, h), we get

$$(\lambda + (8/3))\left(\lambda^3 + 16\lambda^2 + 1222\lambda + 5831\right) = 0$$

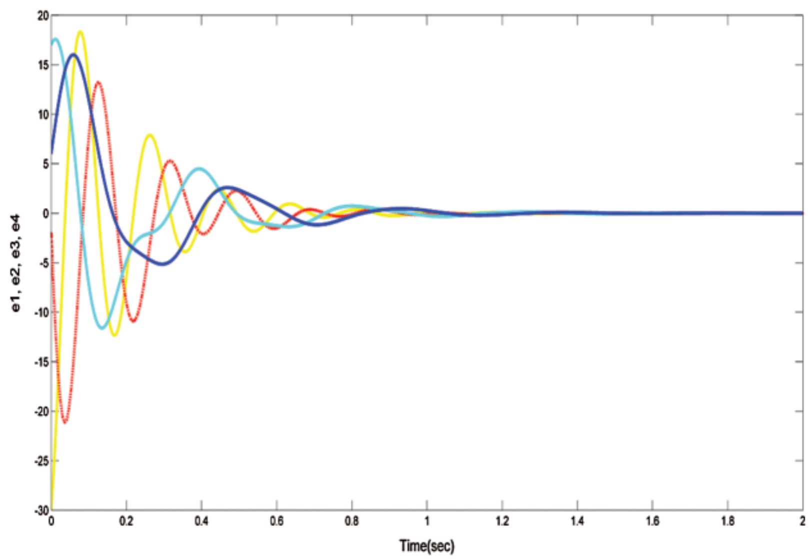


Figure 9: The convergence of system (26) with controllers (27)

Tab.1 shows the variance amongst second method of Lyapunov and Cardano's method.

Table 1: The variance amongst second method of Lyapunov and Cardano's method

S. No.	Second method of Lyapunov	Cardano's method
1.	Essentially, a quadratic function is created.	Based on the origins.
2.	Achieving conditions: A quadratically appositive function is a negative derivative.	Conditions: All roots with a very negative component.
3.	You need to often adjust this feature.	No modification required.
4.	Agreements with structures via co-factor systems (Lyapunov function)	Directly (no co-factor) processes structures.
5.	In a while, crashed.	Still effective.
6.	You did not have to find the solution.	The solution needs to be found.
7.	Dig in the late 19th century.	Discovered at the beginning of the 16th century

The structure of the subject of synchronization phenomena in two methods is shown in Fig. 10.

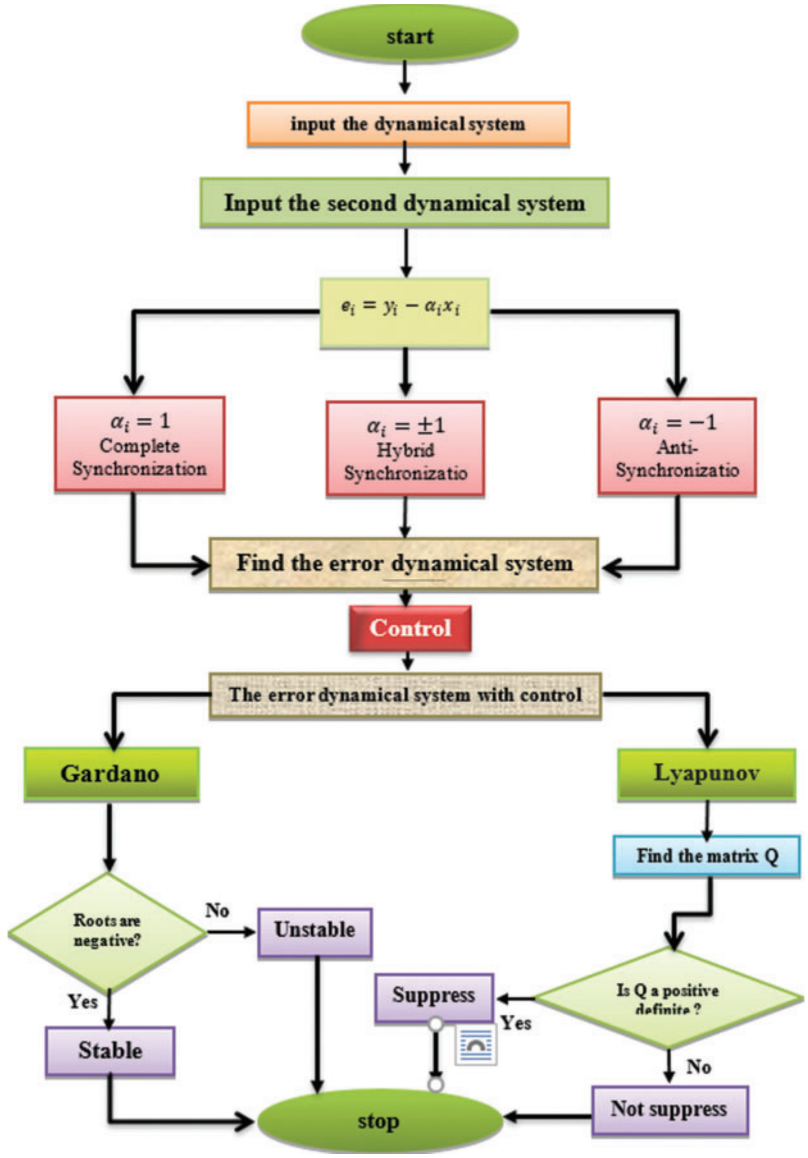


Figure 10: A diagram showing the structure of the subject of synchronization phenomena in two methods

Therefore, we have $g = -382.926$, $\Delta = 2.177149231 \times 10^8$. Since $\Delta > 0$ than the roots are calculate according to Eq. (8) as:

$$\begin{cases} \lambda_1 = \sqrt[3]{-7186.120027} + \sqrt[3]{7569.045953} - 5.3 \overset{yields}{\rightarrow} \lambda_1 = -4.9965 \\ \lambda_2 = -\dfrac{1}{2}(0.337) - 5.3 + i\dfrac{\sqrt{3}}{2}(-38.932) \overset{yields}{\rightarrow} \lambda_2 = -5.4684 - 33.7157i \\ \lambda_3 = -\dfrac{1}{2}(0.337) - 5.3 - i\dfrac{\sqrt{3}}{2}(-38.932) \overset{yields}{\rightarrow} \lambda_2 = -5.4684 + 33.7157i \\ \lambda_4 = -8/3 \end{cases} \tag{31}$$

In addition, all roots with negative actual parts are successfully synced with system (11) and system (10), therefore the Gardano method is efficient. The evidence is complete. Such tests are numerically checked in Figs. 8 and 9. Where the drive system and the response system initial values are used $(15, 8, -1, -2)$ and $(-15, -10, 16, 8)$ respectively.

Thus, all questions in this section are answered in these theorems, and the following table indicates that the Cardano method is stronger than the Lyapunova method.

5 Conclusions

The second method Lyapunov and the Cardano method are based on nonlinear models and two theoretical approaches. We have been trying to grasp the discrepancies in each process and how to achieve synchronization? Within this article, two identical 4D hyperchaotic systems deal with the synchronization phenomena. What is the best method? This paper, therefore, answers all these questions in the Cardano method and makes notice that a supporting function, like the Lyapunov method, should not be created or modified. The Cardano process is better than the Lyapunov procedure. The computational simulation was used to describe the same findings.

The following scheme shows the topics of research:

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